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A Remark on Extensions of CR Functions from Hyperplanes

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Abstract. In the characterization of the range of the Radon transform, one encounters the problem of the holomorphic extension of functions defined on $\mathbb{R}^2 \setminus \Delta_{\mathbb{R}}$ (where $\Delta_{\mathbb{R}}$ is the diagonal in $\mathbb{R}^2$) and which extend as "separately holomorphic" functions of their two arguments. In particular, these functions extend in fact to $\mathbb{C}^2 \setminus \Delta_{\mathbb{C}}$ where $\Delta_{\mathbb{C}}$ is the complexification of $\Delta_{\mathbb{R}}$. We take this theorem from the integral geometry and put it in the more natural context of the CR geometry where it accepts an easier proof and a more general statement. In this new setting it becomes a variant of the celebrated "edge of the wedge" theorem of Ajrapetyan and Henkin.

Let $x = (x_1, x_2)$ and $z = (z_1, z_2)$, with $z = x + iy$, be variables in $\mathbb{R}^2$ and $\mathbb{C}^2$, respectively. We will use the notations

$$T = (\mathbb{R} \times \{0\}) \cup (\{0\} \times \mathbb{R}), \quad \dot{T} = T \setminus \{0\}.$$

We consider a C^1-curve without boundary L in $\mathbb{R}^2$ and focus our attention on the CR manifold $(\mathbb{R}^2 + iT) \setminus L$. We note that the set $(\mathbb{R}^2 + iT) \setminus L$ is rather unusual in several complex variables, in the sense that it is neither a manifold with boundary nor a wedge-like domain. In this set there is a distinguished subset $E = \mathbb{R}^2 \setminus L$, which plays the role of edge, and four manifolds issuing from it, namely the ones which are defined by $y_1 \geq 0$, $y_2 \geq 0$, $y_1 \leq 0$ and $y_2 \leq 0$. We denote by Q_j, $j = 1, \dots, 4$, the four quadrants of $\mathbb{R}^2$ with vertex at 0. The aim of this paper is to prove the following extension result.

Theorem 1 *Assume that for any $x \in \mathbb{R}^2 \setminus L$ and a suitable j we have $(x + Q_j) \cap L = \varnothing$. Let $f\colon \mathbb{R}^2 \setminus L \to \mathbb{C}$ be a continuous function which extends, as a separately holomorphic continuous function (*i.e.*, a continuous CR function) to the set $(\mathbb{R}^2 + iT) \setminus L$. Then f extends as a holomorphic function to $\mathbb{C}^2$ unless L is a straight line, in which case f extends to $\mathbb{C}^2 \setminus L^{\mathbb{C}}$, the complement of the complexification of L.*

Before giving the proof, let us make some comments. When L is a line, defined say by $l(x) = 0$, then the function $f(z) = \frac{1}{l(z)}$ exhibits an example of a function which extends to $\mathbb{C}^2 \setminus L^{\mathbb{C}}$ but not to the whole $\mathbb{C}^2$; we are thus in the second instance of the statement of Theorem 1. We point out that the above theorem generalizes results from [1, 5] in the context of the characterization of the range of the exponential Radon transform. In those statements L was assumed to be the diagonal of $\mathbb{R}^2$, which can be easily generalized to any staight line. Our improvement consists in treating the case of general curves L. It is worth noticing that this is the first time that this

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extension problem is treated in the framework of the CR geometry. More precisely, it is reduced to an "edge of the wedge" theorem of the type from [2] in the presence of a singular set such as L. Also, let us notice that an initial holomorphic extension to a neighborhood of $(\mathbb{R}^2 \setminus L) + iT$ is guaranteed by [2]. Our goal is to continue in this process of extension, first reaching the set $(\mathbb{R}^2 \setminus L) + i\mathbb{R}^2$, and then eventually the whole set $\mathbb{C}^2$ or $\mathbb{C}^2 \setminus L^{\mathbb{C}}$ according as L is curved or straight.

Proof of Theorem 1 We begin by noticing that any CR function on $(\mathbb{R}^2 + iT) \setminus L$ extends holomorphically to a neighborhood of $(\mathbb{R}^2 \setminus L) + iT$. In fact, fix any point $x \in \mathbb{R}^2$ outside L. Then locally around x, $(\mathbb{R}^2 + iT) \setminus L$ is the union of the two hyperplanes $y_1 = 0$ and $y_2 = 0$. By the edge of the wedge theorem from [2], f extends as a holomorphic function to the wedges

$$Q_1 = \{y_1 \geq 0, y_2 \geq 0\}, \quad Q_2 = \{y_1 \geq 0 y_2 \leq 0\},$$
$$Q_3 = \{y_1 \leq 0, y_2 \leq 0\}, \quad Q_4 = \{y_1 \leq 0 y_2 \geq 0\}.$$

Then f extends to a full neighborhood of x. We observe next that by [4] the analyticity of f in x propagates along the complex lines $\gamma_1(z_1) = (z_1, x_2)$ and $\gamma_2(z_2) = (x_1, z_2)$. Since all the points of $(\mathbb{R}^2 \setminus L) + iT$ lie on a line of type γ_1 or γ_2, we conclude that f extends to a neighborhood of $(\mathbb{R}^2 \setminus L) + iT$ which we will denote by V.

At this point our plan is to use the continuity principle to gain extendibility to other points of $\mathbb{C}^2$. Now we will be able to extend f to those points z for which there exists a continuous family of analytic discs attached to $(\mathbb{R}^2 + iT) \setminus L$, that is, having their boundaries in $(\mathbb{R}^2 + iT) \setminus L$ starting from an initial disc entirely contained in V and ending up with a disc passing through z. The hard part of this task is to find discs attached to $(\mathbb{R}^2+iT)\setminus L$. Let us recall some standard notations. The symbol Δ denotes the standard disc in $\mathbb{C}$ and A an analytic disc in $\mathbb{C}^2$, that is, an analytic mapping $A(\zeta) = (z_1(\zeta), z_2(\zeta))$ for $\zeta \in \Delta$ which is of class $C^{1,\alpha}$ (*i.e.,* differentiable with α-Hölder continuous derivative up to $\partial\Delta$). We denote by the same notation A both the disc $A(\Delta)$ and its parametrization $\zeta \mapsto A(\zeta)$. The disc A is said to be attached to $(\mathbb{R}^2+iT)\setminus L$ when $A(\partial\Delta) \subset (\mathbb{R}^2+iT)\setminus L$. The set $(\mathbb{R}^2+iT)\setminus L$ is contained in the set defined by $y_1 y_2 = 0$, and hence a disc A is attached to $(\mathbb{R}^2 + iT) \setminus L$ if $y_1(\zeta)y_2(\zeta) = 0\, \forall \zeta \in \partial\Delta$ and if for $y_1(\zeta)$ and $y_2(\zeta)$ simultaneously 0 we have $(x_1(\zeta), x_2(\zeta)) \notin L$. To check this condition it is convenient to look for a represention formula for analytic discs which involves only the imaginary part. Let K be the Cauchy transform, *i.e.,*

$$K(g)(\zeta) = \frac{1}{2\pi i} \int_{\partial\Delta} \frac{g(\tau)}{\tau - \zeta}\, d\tau.$$

Then if we have a holomorphic function $h(\zeta) = u(\zeta) + iv(\zeta)$, it is easily verified that

$$h(\zeta) = 2iK(v)(\zeta) + u(0) - iv(0). \tag{1}$$

Let us point out that (1) gives a holomorphic function h starting from its arbitrary imaginary part v.

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Let $\eta_1(\zeta)$ and $\eta_2(\zeta)$ for $\zeta \in \partial\Delta$ be two positive $C^{1,\alpha}$ functions with unitary mean whose support is contained in $\{\operatorname{Re}\zeta \geq 0\}$ and $\{\operatorname{Re}\zeta \leq 0\}$, respectively, and which are simultaneously 0 only at $\pm i$. (Their choice will be further specified in the course of the proof.) We write $\eta(\zeta) = (\eta_j(\zeta))_j$, and take $x^o = (x^o_1, x^o_2)$ and $y^o = (y^o_1, y^o_2)$ in $\mathbb{R}^2$. The analytic disc

$$A_{x^o, y^o\cdot\eta}(\zeta) = \big(2iK(y^o_j\eta_j)(\zeta) + x^o_j - iy^o_j\big)_{j=1,2}$$

is attached to the set defined by $y_1y_2 = 0$, and its center is $(x^o_j + iy^o_j)_{j=1,2}$. To get the disc attached to $(\mathbb{R}^2 + iT) \setminus L$, we need $A(\zeta) \notin L$ whenever $\operatorname{Im} A(\zeta) = 0$. But we see that the only case when $\operatorname{Im} A(\zeta) = 0$ is when $\eta_1(\zeta)$ and $\eta_2(\zeta)$ are simultaneously 0, that is, for $\zeta = +i$ and $-i$. By (1) we have

$$A(i) = \Big(x^o_j - \frac{1}{2\pi}\int_0^{2\pi} y^o_j\eta_j(e^{i\theta})\frac{\cos(\theta)}{1-\sin(\theta)}\,d\theta\Big)_j,$$
$$A(-i) = \Big(x^o_j + \frac{1}{2\pi}\int_0^{2\pi} y^o_j\eta_j(e^{i\theta})\frac{\cos(\theta)}{1+\sin(\theta)}\,d\theta\Big)_j.$$

We call

$$a = -\frac{1}{2\pi}\int_0^{2\pi}\eta_1(e^{i\theta})\frac{\cos(\theta)}{1-\sin(\theta)}\,d\theta, \quad b = -\frac{1}{2\pi}\int_0^{2\pi}\eta_2(e^{i\theta})\frac{\cos(\theta)}{1-\sin(\theta)}\,d\theta,$$
$$c = \frac{1}{2\pi}\int_0^{2\pi}\eta_1(e^{i\theta})\frac{\cos(\theta)}{1+\sin(\theta)}\,d\theta, \quad d = \frac{1}{2\pi}\int_0^{2\pi}\eta_2(e^{i\theta})\frac{\cos(\theta)}{1+\sin(\theta)}\,d\theta.$$

With our notations we have

$$A(i) = (x^o_1 + y^o_1a, x^o_2 + y^o_2b), \quad A(-i) = (x^o_1 + y^o_1c, x^o_2 + y^o_2d).$$

To carry on our proof we need the following.

Proposition 2 *For every point $z^o = (z^o_1, z^o_2)$ such that $\operatorname{Re} z^o \notin L$, we can choose the functions η_j $j = 1, 2$ in such a way that*

$$A_{x^o, ty^o\cdot\eta}(\pm i) \notin L \quad \text{for any } 0 \leq t \leq 1. \tag{2}$$

Proof We recall that the functions η_1 and η_2 must be chosen with support in the half-circles $\{e^{i\theta} : -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}\}$ and $\{e^{i\theta} : \frac{\pi}{2} \leq \theta \leq \frac{3}{2}\pi\}$, respectively, vanishing only at the points $\pm i$, and with unit mean value. For the rest, we can play freely with the displacement of their masses in order to achieve (2). To this end we will make a choice which depends on x^o and y^o. We assume without loss of generality that it is for the first quadrant Q_1 that we have $(x^o + Q_1) \cap L = \varnothing$. Suppose first, $y^o_1 > 0$ $y^o_2 > 0$. We take any η_2 and choose η_1 with its mass so close to $-i$ that a is small (negative) and c is big (positive) and so that (ay^o_1, by^o_2) and (cy^o_1, dy^o_2) are close to the

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$x_2 \geq 0$ and $x_1 \geq 0$ axes, respectively. We get, therefore, for $0 \leq t \leq 1$,

$$(x_1^o - t y_1^o a, x_2^o - t y_2^o b) \notin L, \tag{3}$$

$$(x_1^o + t y_1^o c, x_2^o + t y_2^o d) \in x^o + Q_1 \subset \mathbb{R}^2 \setminus L. \tag{4}$$

Suppose now $y_1^o > 0$ and $y_2^o < 0$. In this case, we take η_1 and η_2 both with most of their masses at $-i$ so that ay_1^o and by_2^o are small and therefore (3) follows. In this situation cy_1^o and dy_2^o are both positive, and hence (4) also trivially holds. In both the above cases, the discs obtained by the described choices of η_1 and η_2 do not intesect L at either of the two points where they meet $\mathbb{R}^2$, namely $\pm i$, and this is the case for all values of the parameter t for $0 \leq t \leq 1$; thus they are attached to $(\mathbb{R}^2 \setminus L) + iT$. It is clear that all other choices of signs for the y_j can be handled likewise which concludes the proof of the proposition. ∎

End of proof of Theorem 1 It follows easily from Proposition 2 that any CR function f on $(\mathbb{R}^2 + iT) \setminus L$ extends to any $z^o \in (\mathbb{R}^2 \setminus L) + i\mathbb{R}^2$. In fact, for every such z^o with real part $x^o = (x_1^o, x_2^o) \notin L$, we can find a continuous family of analytic discs A, depending on the parameter t with $0 \leq t \leq 1$, attached to $(\mathbb{R}^2 + iT) \setminus L$ and such that for $t = 0$ the disc A reduces to the single point x^o, and for $t = 1$, the center of A reaches the point z^o. Then by applying the continuity principle to the function f, which was already known to extend holomorphically to a neighborhood V of $(\mathbb{R}^2 \setminus L) + iT$, f extends, in fact, to the whole family of discs A for any t in $[0, 1]$, hence in particular to z^o. We shall denote by F the holomorphic extension of f.

Denote now by M the hypersurface $L + i\mathbb{R}^2$ of $\mathbb{C}^2$. We suppose first that M contains no complex curve, that is, it is "minimal" in the sense of Tumanov. This implies that M contains no complex straight line and hence L is not a line; in particular, it is not a line parallel to the axes. In this case the complex lines $z_1 = x_1^o$ or $z_2 = x_2^o$ for $x^o \notin L$ cover the full set $(L + iT) \setminus L$, and hence by the propagation theorem from [4] already used, it extends through M over $L + i\dot{T}$. But according to the theory by Tumanov [6], it also extends by minimality at the points of L. Thus F is holomorphic on the whole $\mathbb{C}^2$.

The other case is when M contains a complex curve, say γ. For any $z^o = x^o + iy^o \in \gamma$, we have $T_{z^o}\gamma = T_{x^o}L + iT_{x^o}L$. Hence γ contains the straight line issued from z^o in direction $iT_{x^o}L$, and therefore γ is in fact the straight complex line $\gamma = z^o + (T_{x^o}L + iT_{x^o}L)$ and L is the real line $L = x^o + T_{x^o}L$. At this point, we switch from the notation γ to $L^{\mathbb{C}}$. We then prove that F extends also to those points in $L + i\mathbb{R}^2$ which are not in $L^{\mathbb{C}}$. We denote by $l(z) = 0$ an equation for $L^{\mathbb{C}}$ and notice that M is foliated by the complex lines $\{z : l(z) = it\}$ for all values of the real parameter t. All these lines meet $\mathbb{R}^2$ outside L except for the line $L^{\mathbb{C}}$ which corresponds to $t = 0$. We also notice that the boundary values of F on M define a CR function on M. We then apply [4] and conclude that the analyticity of F propagates along the above lines L_t for $t \neq 0$ to reach all points in M except for those which belong to $L^{\mathbb{C}}$. This completes the proof of Theorem 1. ∎

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