A NOTE ON THE HOMOTOPY-COMMUTATIVITY OF SUSPENSIONS

C. S. Hoo

(received March 12, 1967)

Introduction. Let A and X be spaces. Then as is well-known, $[\Sigma A, X]$ is a group where Σ denotes the suspension. We wish to find conditions on A which will imply that this group is abelian for all spaces X, that is, ΣA is homotopy-commutative. This is equivalent to saying that conil $A \leq 1$ (see [2] for definition). Our results contain relations between conil A and the generalised Whitehead product of [1]. We work in the category of complexes with base points.

§ 1. Let A, B be spaces. We can consider the following as a cofibration: $A \vee B \xrightarrow{j} A \times B \xrightarrow{q} A \wedge B$, where j is the inclusion and q is the projection. Then there exists a map $p : \Sigma(A \times B) \to \Sigma(A \vee B)$ such that $p(\Sigma j) \simeq 1_{\Sigma(A \vee B)}$. In fact, if we define the maps $p_1, p_2 : A \times B \to A \vee B$ by $p_1(a, b) = (a, *)$, $p_2(a, b) = (*, b)$, then we can take p to be the map $\nabla (\Sigma p_1 \vee \Sigma p_2)\varphi'$ where $\varphi' : \Sigma(A \times B) \to \Sigma(A \times B) \vee \Sigma(A \times B)$ is the suspension structure, and ∇ is the folding map. This implies that $(\Sigma j)^{\#} : [\Sigma(A \times B), X] \to [\Sigma(A \vee B), X]$ is an epimorphism, and hence from the exact sequence of the cofibration j, that $(\Sigma q)^{\#} : [\Sigma(A \wedge B), X] \to [\Sigma(A \times B), X]$ is a monomorphism.

Let $i_1 : \Sigma A \to \Sigma A \vee \Sigma B$, $i_2 : \Sigma B \to \Sigma A \vee \Sigma B$ be the inclusions. Then following Arkowitz [1], we can define the generalised Whitehead product (GWP) $[i_1, i_2] \in [\Sigma(A \wedge B), \Sigma A \vee \Sigma B]$. This GWP is represented by a map $\tilde{k} : \Sigma(A \wedge B) \to \Sigma A \vee \Sigma B$ such that $[\tilde{k}(\Sigma q)]$ = commutator of

Canad. Math. Bull. vol. 10, no. 5, 1967.

https://doi.org/10.4153/CMB-1967-066-7 Published online by Cambridge University Press

$[i_1(\Sigma p_A)]$ and $[i_2(\Sigma p_B)]$ where $p_A : A \times B \to A$, $p_B : A \times B \to B$ are the projections. Let $H : \Sigma A \vee \Sigma B \to \Sigma(A \vee B)$ be the homeomorphism given by $H(k_A(a, t), *) = k_{A \vee B}(a, *, t)$, $H(*, k_B(b, t)) = k_{A \vee B}(*, b, t)$ where $k_A : A \times I \to \Sigma A$, $k_B : B \times I \to \Sigma B$, $k_{A \vee B} : (A \vee B) \times I \to \Sigma(A \vee B)$ are the projections. It is easily verified that $Hi_1(\Sigma p_A) = \Sigma p_1$ and $Hi_2(\Sigma p_B) = \Sigma p_2$. Hence $[H\tilde{k}(\Sigma q)] = H_{\#}[\tilde{k}(\Sigma q)] = [\Sigma p_1] + [\Sigma p_2] - ([\Sigma p_2] + [\Sigma p_1]) = [\nabla(\Sigma p_1 \vee \Sigma p_2)\varphi'] + [\nabla(\Sigma p_2 \vee \Sigma p_1)\varphi'\mu']$ where $\mu' : \Sigma(A \times B) \to \Sigma(A \times B)$ is the inverse for the suspension structure. Now let $\varphi_1' : \Sigma(A \vee B) \to \Sigma(A \vee B) \vee \Sigma(A \vee B)$, $\mu_1' : \Sigma(A \vee B) \to \Sigma(A \vee B)$ be the suspension structure on $A \vee B$. Then it is again easily verified that $\nabla(\Sigma p_2 \vee \Sigma p_1)\varphi'\mu' = \mu_1'\nabla(\Sigma p_1 \vee \Sigma p_2)\varphi' = \mu_1' p$. Thus we have

LEMMA 1. $[H\tilde{k}(\Sigma q)] = [p] + [\mu_1' p]$ where $[\tilde{k}] = [i_1, i_2]$.

Since $H_{\#}$ is an isomorphism and $(\Sigma q)^{\#}$ is a monomorphism, we have

LEMMA 2. $[i_1, i_2] = 0$ if and only if $p\mu' \simeq \mu_1' p$.

Now let $\alpha \in [\Sigma A, X]$, $\beta \in [\Sigma B, X]$ be represented by maps $f : \Sigma A \to X$, $g : \Sigma B \to X$, respectively. Then it is easily seen that $[\alpha, \beta] = (\nabla(f \vee g))_{\#}[i_1, i_2]$. Thus we have

LEMMA 3. If $p\mu' \simeq \mu_1' p$, then $[\alpha, \beta] = 0$ for all $\alpha \in [\Sigma A, X]$, $\beta \in [\Sigma B, X]$.

Let us now consider the case $A = B$. Let $\Delta : A \to A \times A$ be the diagonal map. If $\alpha, \beta \in [\Sigma A, X]$, it is shown in [1] that $(\Sigma\Delta)^{\#}(\Sigma q)^{\#}[\alpha, \beta] = (\alpha, \beta)$ = the commutator of α, β. Thus we have

THEOREM 1. If $p\mu' \simeq \mu_1' p : \Sigma(A \times A) \to \Sigma(A \vee A)$, then conil $A \leq 1$.

https://doi.org/10.4153/CMB-1967-066-7 Published online by Cambridge University Press

REMARKS. By Theorem 1, if $p\mu' \simeq \mu_1' p$, then conil $A \leq 1$. Let us now consider briefly when this condition $p\mu' \simeq \mu_1' p$ will hold. We observe that if $p : \Sigma(A \times A) \to \Sigma(A \vee A)$ is an H' map with respect to the suspension structures, that is $\varphi_1' p \simeq (p \vee p)\varphi'$, then we will also have $p\mu' \simeq \mu_1' p$, and hence conil $A \leq 1$. In particular if $[p]$ belongs to the image of the suspension $\Sigma_{\#} : [A \times A, A \vee A] \to [\Sigma(A \times A), \Sigma(A \vee A)]$ then conil $A \leq 1$. Following [3], let us consider the adjoint $[\bar{p}] \in [A \times A, \Omega\Sigma(A \vee A)]$. Then consider the fibration $\theta : \Omega\Sigma(A \vee A) \to \Omega(\Sigma(A \vee A) \flat \Sigma(A \vee A))$ where the fibre is the co-projective plane associated with the H' space $(\Sigma(A \vee A), \varphi_1', \mu_1')$ and $\Sigma(A \vee A) \flat \Sigma(A \vee A)$ denotes the flat product. Then $\theta_{\#}([\bar{p}]) \in [A \times A, \Omega(\Sigma(A \vee A) \flat \Sigma(A \vee A))]$ is called the Hopf invariant of $[\bar{p}]$. In [3], we have

THEOREM 2. p is an H' map with respect to the suspension structures on $\Sigma(A \times A)$, $\Sigma(A \vee A)$ if and only if the Hopf invariant $\theta_{\#}([\bar{p}]) = 0$.

REFERENCES

1. M. Arkowitz, The generalised Whitehead product, Pac. J. Math. 12 (1962) 7-23.

2. I. Berstein and T. Ganea, Homotopical nilpotency. Ill. J. Math. 5 (1961) 99-130.

3. I. Berstein and P.J. Hilton, On suspensions and comultiplications. Topology 2 (1963) 73-82.

4. T. Ganea, On some numerical homotopy invariants, Proc. International Congress of Mathematicians (1962) 467-472.

5. T. Ganea, P.J. Hilton and F.P. Peterson, On the homotopy-commutativity of loop-spaces and suspensions Topology 1 (1962) 133-141.

https://doi.org/10.4153/CMB-1967-066-7 Published online by Cambridge University Press

6. P. J. Hilton, Homotopy theory and duality, Gordon and Breach (New York) 1963.

University of Alberta
Edmonton, Alberta

https://doi.org/10.4153/CMB-1967-066-7 Published online by Cambridge University Press