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A QUANTITATIVE ESTIMATE ON FIXED-POINTS OF COMPOSITE MEROMORPHIC FUNCTIONS

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ABSTRACT. Let $f(z)$ be a transcendental meromorphic function of finite order, $g(z)$ a transcendental entire function of finite lower order and let $\alpha(z)$ be a non-constant meromorphic function with $T(r,\alpha) = S(r,g)$. As an extension of the main result of [7], we prove that

$$T(r,g) = o\Big(\bar{N}\Big(r, \frac{1}{f(g)-\alpha}\Big)\Big), \quad r \in J,$$

where J has a positive lower logarithmic density.

1. **Introduction and main results.** Let $f(z)$ be a transcendental meromorphic function and $g(z)$ be a transcendental entire function. A point z_0 at which $f(z_0) = z_0$ is called a *fixed-point* of $f(z)$. First, let us assume that the reader is familiar with Nevanlinna theory of meromorphic functions and its standard notations. Throughout, we denote by $\rho(f)$, $\lambda(f)$, and $\sigma(f)$, respectively, the order and the lower order of $f(z)$, and the convergence exponent for its zeros, and by $S(r,f)$ the quantity such that $S(r,f) = o\big(T(r,f)\big)$ as $r \notin E$, $r \longrightarrow \infty$, where E denotes a set of r with finite linear measure, not necessarily the same at each occurrence, and $T(r,f)$ is the Nevanlinna characteristic of $f(z)$. As usual, $N(r, 1/f)$ denotes the counting function for the zeros of $f(z)$ and $\bar{N}(r, 1/f)$ for the distinct zeros in the sense of Nevanlinna.

The present author and Yang [11], [12] presented some quantitative measures on the number of zeros of $f\big(g(z)\big) - P(z)$, in terms of the growth of $f(z)$ and $g(z)$, in the case where $f(z)$ and $g(z)$ are entire, transcendental and $P(z)$ is a non-constant polynomial. In addition, assuming that $\rho\big(f(g)\big) < \infty$ and $P(z)$ is allowed to be a non-constant rational function, an excellent estimate was established in Langley [8], *i.e.*,

$$N\Big(r, \frac{1}{f(g)-P}\Big) \neq o\Big(T\big(r, f(g)\big)\Big).$$

For the case when $f(z)$ is meromorphic and transcendental, the existence of infinitely many zeros of $f\big(g(z)\big) - Q(z)$ was proved in [3], provided that $f(g)$ is of finite order and Q is a non-constant rational function. Following this, an extension of the latter case was made in [7] and actually, it is shown there that the exponent of convergence σ for the zeros of $f\big(g(z)\big) - \alpha(z)$ satisfies $\sigma \geq \lambda(g)$, provided that $f(z)$ is a meromorphic function

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of finite order, $g(z)$ is a transcendental entire function of finite lower order $\lambda(g)$ and $\alpha(z)$ a non-constant meromorphic function such that $\rho(\alpha) < \lambda(g)$. For the general case, *i.e.*, for any transcendental meromorphic function $f(z)$, entire function $g(z)$ and non-constant rational function $Q(z)$, Bergweiler [1] recently verified that $f\big(g(z)\big) - Q(z)$ has infinitely many zeros and the further result that if $f(z)$ has at least two poles and $Q(\infty) = \infty$, then $\sigma \geq \lambda(g)$. The main purpose of the paper is to prove the following:

THEOREM 1. *Let $f(z)$ be transcendental meromorphic in the complex plane, $g(z)$ transcendental entire, and let $\alpha(z)$ be a non-constant meromorphic function such that $T(r,\alpha) = S(r,g)$. Assume that $\rho(f) < \infty$ and $\lambda(g) < \infty$. Then there exists a set J of r with positive lower logarithmic density such that*

$$\lim_{\substack{r\to\infty \\ r\in J}} \frac{\bar{N}\Big(r, 1/\big(f(g)-\alpha\big)\Big)}{T(r,g)} = \infty. \tag{1}$$

If we put a stronger restriction to the growth of $f(z)$, then we can remove the assumption on the finite lower order of $g(z)$ from Theorem 1. Actually, we have the following, as did in Bergweiler [2], for the transcendental entire $f(z)$.

THEOREM 2. *Let $f(z)$, $g(z)$ and $\alpha(z)$ be given as in Theorem 1. Assume, instead, that*

$$\log T(r,f) \leq \frac{\log r}{\phi(\log\log r)}, \quad (r \notin E), \tag{2}$$

where $\phi(x)$ is a positive increasing function and such that

$$\int^{\infty} \frac{dx}{\phi(x)} < \infty.$$

Then (1) is valid, where J has logarithmic density one.

2. **Proofs of Theorems 1 and 2.**

PROOF OF THEOREM 1. Suppose that (1) does not hold, that is, there exists an $A > 0$ and a set I with lower logarithmic density one such that for $r \in I$

$$\bar{N}\left(r, \frac{1}{f(g)-\alpha}\right) < AT(r,g).$$

We can write $f = f_1/f_2$, where f_1 and f_2 are two entire functions with finite order and without common zeros. Set

$$R(z) := f_1(g) - \alpha f_2(g) = f_2(g)\big(f(g)-\alpha\big).$$

Then it is obvious that each zero of $f_1(g) - \alpha f_2(g)$ is either a zero of $f(g) - \alpha$ or a pole of α. This implies that

$$\bar{N}\left(r, \frac{1}{R}\right) < \big(A + o(1)\big)T(r,g), \quad (r \in I). \tag{3}$$

Since $g(z)$ is of finite lower order, by a result of Hayman [6, Lemma 4], there exists a subset J of I with positive lower logarithmic density such that for $r \in J$, $T(3r,g) \leq BT(r,g)$, where B is a sufficiently large and positive number. A result of Ninno-Suita [9] implies the following estimate

$$\begin{aligned} T(r,R) &\leq T\big(r,f_1(g)\big) + T\big(r,f_2(g)\big) + S(r,g) \\ &\leq 2T\big(M(r,g),f_1\big) + 2T\big(M(r,g),f_2\big) + S(r,g) \\ &\leq M(r,g)^d, \quad (r \notin E), \end{aligned}$$

so that

$$\log T(r,R) \leq d \log M(r,g) \leq 2dT(3r,g) \tag{4}$$

$$\leq 2dBT(r,g), \quad (r \in J \setminus E), \tag{5}$$

where $d > \rho(f)$, since $f(z)$ is of finite order. By the lemma of logarithmic derivative, we can find a positive number K and an unbounded sequence $\{r_j\} \subset J$ such that

$$T(r_j,g') + T(r_j,\alpha) + T(r_j,\alpha') + T\Big(r_j,\frac{R'}{R}\Big) \leq KT(r_j,g).$$

Now differentiating the equality $R = f_1(g) - \alpha f_2(g)$ gives

$$g'f_1'(g) - \alpha g' f_2'(g) - \frac{R'}{R} f_1(g) + \Big(\alpha \frac{R'}{R} - \alpha'\Big) f_2(g) = 0.$$

An application of a theorem of Steinmetz [10] (also see [5]) to the above equation gives the existence of four polynomials P_1, P_2, P_3 and P_4, not all zeros, such that

$$P_1 f_1' + P_2 f_2' + P_3 f_1 + P_4 f_2 = 0.$$

Using the same methods as in [7] implies that f solves the following differential equation

$$f'(az+b) = c_1 + c_2 f + c_3 f^2, \tag{6}$$

where $a(\neq 0)$, b and c_i $(1 \leq i \leq 3)$ are all constants.

Below we treat two cases.

CASE 1: $c_3 = 0$. Then $c_2 \neq 0$. It is obvious from (6) that $c_1 + c_2 f$ has just one zero or pole. And hence we can write $c_1 + c_2 f = Qe^{\beta}$, Q is a non-zero rational function and β is a non-constant entire function. By differentiation, we have immediately

$$c_2 f' = (Q' + Q\beta')e^{\beta} = \Big(\frac{Q'}{Q} + \beta'\Big)(c_2 f + c_1),$$

and further

$$c_2 = \Big(\frac{Q'}{Q} + \beta'\Big)(az+b),$$

so that $\beta' \equiv 0$, which is a contradiction.

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CASE 2: $c_3 \neq 0$. We can write

$$f'(az+b) = c_3(f-\tau)(f-\kappa).$$

When $\tau = \kappa$, it is easy to see that $\big(1/(f-\tau)\big)'$ is rational, so is f, and a contradiction follows.

When $\tau \neq \kappa$, both τ and κ are the Picard exceptional values of $f(z)$. And therefore, we have for a non-zero rational function P and a non-constant entire function γ

$$\frac{f-\tau}{f-\kappa} = Pe^{\gamma}. \tag{7}$$

By differentiation of (7) we have

$$(\tau-\kappa)f' = \Big(\frac{P'}{P} + \gamma'\Big)(f-\tau)(f-\kappa),$$

so that $\gamma' \equiv 0$, which is a contradiction.

Now Theorem 1 follows.

In order to make the proof of Theorem 2 clear, let us first prove the following.

LEMMA 1. *Let $h(z)$ be an entire function with zero order. Then for all sufficiently large r*

$$\log M(r,h) < N(r^2) + n(0)\log r + 1, \tag{8}$$

where $N(r) = N(r, 1/h)$ and $n(r) = n(r, 1/h)$.

Actually, we can write

$$f(z) = z^m \prod_{s=1}^{\infty}\Big(1 - \frac{z}{z_s}\Big), \quad z = re^{i\theta},\ m = n(0),$$

so that

$$\begin{aligned}\log|f(z)| &\le \int_0^{\infty} \log\Big(1+\frac{r}{t}\Big)\,dn(t) + m\log r\\ &= r\Big(\int_0^r + \int_r^{r^2} + \int_{r^2}^{\infty}\Big)\frac{n(t)}{t(t+r)}\,dt + m\log r.\end{aligned}$$

Since $f(z)$ is of zero order, for sufficiently large r we have $n(t) < t^{1/3}$, $r < t$, and hence

$$r\int_{r^2}^{\infty}\frac{n(t)}{t(t+r)}\,dt < r\int_{r^2}^{\infty} t^{-5/3}\,dt < 1.$$

Obviously, we can obtain the following inequality

$$\log|f(z)| \le N(r) + \frac{1}{2}\big(N(r^2) - N(r)\big) + m\log r + 1,$$

which leads to (8).

We need a result of [4].

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LEMMA 2. *Let $g(z)$ be an entire function and $\phi(x)$ a positive increasing function with*

$$\int^{\infty} \frac{dx}{\phi(x)} < \infty.$$

Then there exists a set J with logarithmic density one such that

$$\lim_{\substack{r\to\infty \\ r\in J}} \frac{\log M(r,g)}{T(r,g)\phi\big(\log T(r,g)\big)} = 0.$$

Now we go back to the proof of Theorem 2. Actually, it suffices for the proof of Theorem 2 that we can prove an inequality similar to (5) under the assumption of Theorem 2. First, we can write $f = f_1/f_2$ where $f_1(z)$ and $f_2(z)$ are two entire functions with zero order, since $\rho(f) = 0$. An application of Lemma 1 immediately shows that for $j = 1, 2$,

$$\begin{aligned}\log T(r,f_j) &< \log N\Big(r^2, \frac{1}{f_j}\Big) + O(\log\log r)\\ &< \log T(r^2, f) + O(\log\log r)\\ &< \frac{2\log r}{\phi\big(\log(2\log r)\big)} + O(\log\log r), \quad r \notin E.\end{aligned}$$

Now we can make the following estimation:

$$\begin{aligned}\log T(r,R) &\le \log T\big(M(r,g), f_1\big) + \log T\big(M(r,g), f_2\big) + S(r,g)\\ &< \frac{4\log M(r,g)}{\phi\Big(\log\big(2\log M(r,g)\big)\Big)} + O\big(\log\log M(r,g)\big) + S(r,g)\\ &= o\big(T(r,g)\big), \quad (r \in J).\end{aligned}$$

The latter equality follows from Lemma 2, where J has logarithmic density one.

Theorem 2 follows.

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