A. Kodama
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ON BIMEROMORPHIC AUTOMORPHISMS OF HYPERBOLIC COMPLEX SPACES

AKIO KODAMA

Introduction

Let X be a hyperbolic complex space[1] in the sense of S. Kobayashi [2]. We write $\mathrm{Aut}(X)$ (resp. $\mathrm{Bim}(X)$) for the group of all biholomorphic (resp. bimeromorphic) automorphisms of X.

In this note, we shall prove

THEOREM 1. *Let f be a meromorphic mapping from a complex manifold M into a hyperbolic complex space Y. Then f is holomorphic. In particular, we have* $\mathrm{Aut}(X) = \mathrm{Bim}(X)$ *for any hyperbolic complex manifold X.*

In general we have $\mathrm{Aut}(X) \neq \mathrm{Bim}(X)$ for a hyperbolic complex space X with singularities. In fact, we shall show the following

THEOREM 2. *There exists a normal irreducible complete hyperbolic complex space X with* $\mathrm{Aut}(X) \neq \mathrm{Bim}(X)$.

Thus we have obtained a negative answer to Problem E. 5. in [3].

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1. Preliminaries

For later purpose, in this section we shall recall some definitions. A *meromorphic mapping* f from a complex space X into a complex space Y in the sense of Remmert is a set-valued function satisfying the

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(1) In this note, by a complex space we mean a reduced irreducible Hausdorff complex analytic space.

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following conditions:

(i) the restriction $f|_W: W \to Y$ is a holomorphic mapping for some open dense subset W of X;

(ii) the graph $\Gamma_f := \{(x, y) \in X \times Y \mid y \in f(x)\}$ of f is an analytic subset of $X \times Y$ which coincides with the topological closure of the set $\{(x, f(x)) \in X \times Y \mid x \in W\}$ in $X \times Y$;

(iii) the canonical projection $\pi: \Gamma_f \to X$ is proper.

We remark here that the set W in (i) can be chosen in such a way that $X - W$ is an analytic subset of X. Let X, Y and Z be three complex spaces. Then, for given meromorphic mappings $f: X \to Y$ and $g: Y \to Z$, we can define the composed meromorphic mapping $g \circ f: X \to Z$ if the full inverse image of W by f is dense in X, where W is an open dense subset of Y on which g is holomorphic (cf. Whitney [5]). In general we have $g(f(A)) \neq (g \circ f)(A)$ for a subset A of X. We say that X and Y are *bimeromorphically* (resp. *biholomorphically*) *equivalent* if there exist meromorphic (resp. holomorphic) mappings $f: X \to Y$ and $g: Y \to X$ such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In this case, we call f and g *bimeromorphic* (resp. *biholomorphic*) *mappings* and the inverse to each other. Moreover, in the case of $X = Y$ these are called *bimeromorphic* (resp. *biholomorphic*) *automorphisms* of X. A surjective holomorphic mapping $\pi: X \to Y$ is called a *proper modification* of Y with center S if it is proper and the restriction $\pi: X - \pi^{-1}(S) \to Y - S$ is a biholomorphic mapping for some nowhere dense analytic subset S of Y. For any proper modification $\pi: X \to Y$ with center S, its inverse is always meromorphic. More precisely speaking, we can define a meromorphic mapping $\psi: Y \to X$ by using the holomorphic mapping $\pi^{-1}: Y - S \to X - \pi^{-1}(S)$ and it is the inverse of $\pi: X \to Y$.

2. Proof of Theorem 1

As remarked in Preliminaries, there exists an analytic subset A of M such that the restriction $f|_{M-A}: M - A \to Y$ is holomorphic. Putting $g = f|_{M-A}$, we shall prove that g can be extended to a holomorphic mapping.

First we may assume that A is a non-singular complex submanifold of M by the same arguments as in Theorem 4.1, Chap. VI of [2]. Then, since the problem is local, we may further assume that M is a polydisc:

$$D \times D^{m-1} = \{(z, t^1, \cdots, t^{m-1}) \in C^m \,||z| < 1, |t^i| < 1 \ (1 \leqq i \leqq m-1)\}$$

and A is contained in the subset defined by $z = 0$.

For each fixed $t \in D^{m-1}$, we define a holomorphic mapping g_t from the punctured disc D^* into Y by $g_t(z) = g(z, t)$. Once it is shown that $g_t: D^* \to Y$ can be extended to a holomorphic mapping $\tilde{g}_t: D \to Y$ for each $t \in D^{m-1}$, the rest of our proof can be done with exactly the same arguments as in Theorem 4.1., Chap. VI of [2]. Thus we have only to show that g_t is extendable. By a result of Kwack [4], it is enough to show the existence of a sequence of points z_k of D^* converging to the origin such that $g_t(z_k)$ converges to a point p of Y. Now, since $f: M = D \times D^{m-1} \to Y$ is meromorphic, $f(0, t)$ is a compact analytic subset of Y. We take a point p of $f(0, t)$ arbitrarily. Then, by (ii) in section 1, there are points z_k of D^* such that the sequence $\{((z_k, t), f(z_k, t))\}$ converges to the point $((0, t), p)$ of the graph Γ_f of f in $M \times Y$, because the restriction $f|_{D \times \{t\}}: D \times \{t\} \to Y$ is also meromorphic (cf. [5], Corollary 4. H., p. 196). This implies that $\lim_{k \to \infty} z_k = 0$ and $\lim_{k \to \infty} g_t(z_k) = p \in Y$, and hence the proof is completed. q.e.d.

LEMMA 1. *There exist a normal irreducible complex space S, a compact hyperbolic complex manifold T and a meromorphic mapping $f: S \to T$ which is not holomorphic.*

Proof. Let T be a compact projective algebraic manifold which is hyperbolic. Then, as remarked in [2], p. 100, T can be imbedded into some complex projective space $P_n(C)$ in such a way that T is projectively normal, that is, the affine cone $C(T) := \{$all complex lines through the origin o of C^{n+1} representing the points of $T\}$ is a normal complex space. It is clear that $C(T)$ is non-singular except at the origin. Let $\pi: C(T) - \{o\} \to T$ be the restriction of the natural projection $C^{n+1} - \{o\} \to P_n(C)$. Then, obviously π cannot be extended to a holomorphic mapping from $C(T)$ into T. On the other hand, by the theorem of resolution of singularities by Hironaka [1] and an extention theorem by Kwack [4] it is easily verified that π can be extended to a meromorphic mapping $\tilde{\pi}: C(T) \to T$. The triple system $(\tilde{\pi}, C(T), T)$ satisfies our assertion. q.e.d.

LEMMA 2. *There exist normal irreducible complete hyperbolic com-*

plex spaces $\tilde{U}$ *and* $\tilde{V}$ *with bimeromorphic mappings* $s:\tilde{V}\to\tilde{U}$ *and* $t:\tilde{U}\to\tilde{V}$, *where* t *is not holomorphic.*

Proof. Take a triple system (f, S, T) as in Lemma 1. Let Γ_f be the graph of the meromorphic mapping $f: S\to T$ and $\pi:\Gamma_f\to S$ the canonical projection. Let $g: S\to\Gamma_f$ be the inverse meromorphic mapping of the proper modification $\pi:\Gamma_f\to S$. Then there exists a point x_0 of S such that $g(x_0)$ is not a single point, because f is not holomorphic. Take an open neighborhood $\tilde{U}$ of x_0 in S which is complete hyperbolic. Since S is normal, we may assume that $\tilde{U}$ is also irreducible. Let $V=\pi^{-1}(\tilde{U})$ and $\mu:\tilde{V}\to V$ be a normalization of V. Being an analytic subset of the complete hyperbolic complex space $\tilde{U}\times T$, V is also complete hyperbolic. Then, from a result of Kwack [4], $\tilde{V}$ is complete hyperbolic. Moreover, since $\tilde{U}$ is irreducible, so are V and $\tilde{V}$. We now define meromorphic mappings $s:\tilde{V}\to\tilde{U}$ and $t:\tilde{U}\to\tilde{V}$ by $s=\psi\circ\mu$ and $t=\nu\circ\omega$, where $\psi: V\to\tilde{U}$ is the restriction of $\pi:\Gamma_f\to S$ to V, $\nu: V\to\tilde{V}$ is the inverse meromorphic mapping of the proper modification $\mu:\tilde{V}\to V$ and $\omega:\tilde{U}\to V$ is the restriction of $g: S\to\Gamma_f$ to $\tilde{U}$, respectively. Then we can show that $s:\tilde{V}\to\tilde{U}$ and $t:\tilde{U}\to\tilde{V}$ are bimeromorphic mappings and the inverse to each other. From our construction, it is clear that $t:\tilde{U}\to\tilde{V}$ is not holomorphic. q.e.d.

Proof of Theorem 2. Let $\tilde{U}$, $\tilde{V}$, $s:\tilde{V}\to\tilde{U}$ and $t:\tilde{U}\to\tilde{V}$ be complex spaces and bimeromorphic mappings as in Lemma 2. Putting $X=\tilde{U}\times\tilde{V}$, we define a bimeromorphic automorphism ϕ of X by $\phi(u,v)=(s(v),t(u))$ for $(u,v)\in X$. Then X is a normal irreducible complete hyperbolic complex space. Moreover ϕ cannot be a biholomorphic automorphism of X. In fact, if it were so, both $s:\tilde{V}\to\tilde{U}$ and $t:\tilde{U}\to\tilde{V}$ are necessarily biholomorphic mappings. This contradicts the fact $t:\tilde{U}\to\tilde{V}$ is not holomorphic. Therefore we have shown that $\mathrm{Aut}(X)\neq\mathrm{Bim}(X)$. q.e.d.

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Akita University

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