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Normal and invertible composition operators

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Let N denote the set of natural numbers and let ϕ be a mapping from N into itself. Then the composition transformation C_ϕ on the weighted l^2 space with weights a^{2n}, where $n \in N$ and $0 < a < 1$ is defined by $C_\phi f = f \circ \phi$. If C_ϕ is a bounded operator, then it is called a composition operator. The adjoint of the composition operator C_ϕ is computed, and it is used to characterise normal, unitary, isometric, and co-isometric composition operators. Not every invertible ϕ induces an invertible composition operator, as is shown by examples. At the end of this note all invertible composition operators are characterised.

1. Preliminaries

Let N denote the set of non-zero positive integers and let λ be the measure on N defined by $\lambda(\{n\}) = \lambda_n = a^{2n}$ for every $n \in N$, where $0 < a < 1$. Let l_a^2 denote the space of all complex sequences such that

$$l_a^2 = \left\{g \mid g : N \to C \text{ and } \sum_{n=1}^{\infty} \lambda_n |g(n)|^2 < \infty\right\} .$$

Then l_a^2 is a Hilbert space under pointwise addition and scalar multiplication with the inner product defined by

$$\langle f, g\rangle = \sum_{n=1}^{\infty} \lambda_n f(n)\bar{g}(n) .$$

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If ϕ is a mapping from N into itself, we define a composition transformation C_ϕ on l_a^2 into the space of all complex valued functions on N by

$$C_\phi f = f \circ \phi \quad \text{for all} \quad f \in l_a^2 .$$

If the range of C_ϕ is in l_a^2 and C_ϕ is bounded, then we call C_ϕ a composition operator induced by ϕ. By $B\left(l_a^2\right)$ we mean the Banach algebra of bounded linear operators on l_a^2.

In Section 2 of this paper we compute the adjoint of C_ϕ and, using this, we characterise normal, unitary, and isometric composition operators. In Section 3 of this paper invertible composition operators are characterised.

If ϕ is a mapping on N into itself such that $C_\phi \in B\left(l_a^2\right)$, then the measure $\lambda\phi^{-1}$ is absolutely continuous with respect to λ. We denote the Radon-Nikodym derivative of $\lambda\phi^{-1}$ with respect to λ by f_0. In the case of l^2 there is a ϕ such that ϕ is not the identity map, but $f_0 = 1$ (for example any bijection other than the identity). In the case of l_a^2, it is not so, as is proved in the following lemma.

LEMMA 1.1. *Let ϕ be a mapping from N into itself and f_0 be the Radon-Nikodym derivative of the measure $\lambda\phi^{-1}$ with respect to λ. Then $f_0 = 1$ if and only if ϕ is the identity.*

Proof. Let ϕ be the identity. Then $\phi(n) = n$ for all $n \in N$. Hence $f_0(n) = d\lambda\phi^{-1}(n)/d\lambda n = d\lambda n/d\lambda n = 1$ for all $n \in N$.

The converse is proved by induction. We first prove that $\phi(1) = 1$. Since $f_0(n) = d\lambda\phi^{-1}(n)/d\lambda n = 1$, we get $\lambda\phi^{-1}(n) = \lambda n$ for all $n \in N$. If $\phi(1) \neq 1$, let $\phi(1) = m$ where $m \neq 1$. Then $1 \in \phi^{-1}(m)$. Hence

$\lambda_1 \le \lambda\phi^{-1}(m) = \lambda_m$, which is impossible, for λ is a decreasing measure. Thus $\phi(1) = 1$.

Let us suppose that this result is true for $1, 2, \ldots, k$, that is $\phi(j) = j$ for $j = 1, 2, \ldots, k$; we prove $\phi(k+1) = k + 1$. If this is not so, then $\phi(k+1) = m$, where $m \ne k + 1$.

CASE I. If $m > k + 1$, then $k + 1 \in \phi^{-1}(m)$. Therefore $\lambda_k+1 \le \lambda\phi^{-1}(m) = \lambda_m$, which is a contradiction, since $\lambda_m < \lambda_{k+1}$.

CASE II. If $m < k + 1$, then $\phi(m) = m$. Thus $\{m, k+1\} \subset \phi^{-1}(m)$. Hence $\lambda_m + \lambda_{k+1} \le \lambda\phi^{-1}(m) = \lambda_m$, which is again a contradiction, since $\lambda_{k+1} \ne 0$. Therefore $\phi(k+1) = k + 1$, and hence the induction process is complete. Thus the proof of the lemma is finished.

2. Normal and unitary composition operators

For the characterisation of normal composition operators we need a familiarity with the nature of the adjoint of such operators. The computation of the adjoint of a composition operator C_ϕ on the L^2 of a general measure space is very hard. But in the case of l_a^2 , the adjoint C_ϕ^* is computable. The following theorem computes the adjoint of C_ϕ .

THEOREM 2.1. *Let* $C_\phi \in B\left(l_a^2\right)$ *and* C_ϕ^* *be defined by*

$$\left(C_\phi^* g\right)(n) = \frac{1}{\lambda_n} \int_{\phi^{-1}(n)} g d\lambda$$

for all $g \in l_a^2$ *and* $n \in N$. *Then* $\langle C_\phi f, g\rangle = \langle f, C_\phi^* g\rangle$ *for all* $f, g \in l_a^2$.

Proof. Since

$$\begin{aligned}\langle C_\phi f, g\rangle &= \int_N (C_\phi f)(m)\overline{g}(m)d\lambda \\ &= \sum_{n=1}^{\infty} \int_{\phi^{-1}(n)} (f\circ\phi)(m)\overline{g}(m)d\lambda \\ &= \sum_{n=1}^{\infty} \int_{\phi^{-1}(n)} f(n)\overline{g}(m)d\lambda \\ &= \sum_{n=1}^{\infty} f(n) \int_{\phi^{-1}(n)} \overline{g}(m)d\lambda \\ &= \sum_{n=1}^{\infty} f(n)\lambda_n \overline{(C^*g)}(n) \\ &= \langle f, C_\phi^* g\rangle ,\end{aligned}$$

C_ϕ^* is the adjoint of C_ϕ .

On l^2 there are plenty of normal composition operators other than the identity operator, as every invertible composition operator in this is normal [4]. But strangely enough on l_a^2 there is no non-trivial normal composition operator. This is shown in the following theorem.

THEOREM 2.2. *Let* $C_\phi \in B\left(l_a^2\right)$. *Then* C_ϕ *is normal if and only if* ϕ *is the identity.*

In order to prove this theorem we need the following lemma.

LEMMA 2.3. *Let* $\phi : N \to N$ *be a one-to-one and onto mapping. Then* $\phi(m) + \phi^{-1}(m) = 2m$ *for all* $m \in N$ *implies that* ϕ *is the identity.*

Proof. Let $\phi(1) = n$. Then $\phi(n) = 2n - 1$. Let $\phi^{-1}(1) = m$. Now since $\phi(1) + \phi^{-1}(1) = 2$, we have $n + m = 2$ which is possible only when $n = m = 1$. Thus $\phi(1) = 1$. Let us suppose that the result is true for $1, 2, \ldots, k$. We prove it for $k + 1$. Let $\phi(k+1) = n$; then $\phi(n) = 2n - (k+1)$. Let $\phi^{-1}(k+1) = m$. Then since $\phi(k+1) + \phi^{-1}(k+1) = 2(k+1)$, $n + m = 2(k+1)$. Since n and m are not less than or equal to k , we conclude $\phi(k+1) = k + 1$ and

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$\phi^{-1}(k+1) = k+1$. Hence, by induction, $\phi(n) = n$ for all $n \in N$, which implies that ϕ is the identity.

Proof of Theorem 2.2. The sufficiency is obvious. To prove the necessary part, let C_ϕ be normal and e^m be the sequence defined by $e^m(p) = \delta_{mp}$ (the Kronecker delta). Then $\left\|C_\phi^* e^m\right\|^2 = \left\|C_\phi e^m\right\|^2$ for all $m \in N$. Now $\left\|C_\phi^* e^m\right\|^2 = \sum_{n=1}^{\infty} \lambda_n \left|\left(C_\phi^* e^m\right)(n)\right|^2$. But since $\left(C_\phi^* e^m\right)(p) = 1/\lambda_p \int_{\phi^{-1}(p)} e^m(p)d\lambda$ and $m \in \phi^{-1}(p)$ for only one value of p , we get

$$\left(C_\phi^* e^m\right)(p) = 1/\lambda_p \int_{\{m\}} e^m(m)d\lambda = \lambda_m/\lambda_p .$$

Therefore $\left\|C_\phi^* e^m\right\|^2 = \lambda_p \lambda_m^2/\lambda_p^2 = \lambda_m^2/\lambda_p$, where $m \in \phi^{-1}(p)$. Also

$$\left\|C_\phi e^m\right\|^2 = \|C_\phi X_{\{m\}}\|^2 = \sum_{n=1}^{\infty} \lambda_n |C_\phi X_{\{m\}}(n)|^2 = \sum_{n=1}^{\infty} \lambda_n |X_{\phi^{-1}(m)}(n)|^2 ,$$

where X_E stands for the characteristic function of the set E . But if ϕ is not onto, $\phi^{-1}(m)$ is empty for some $m \in N$, and hence $X_{\phi^{-1}(m)}(n) = 0$ for every $n \in N$, which implies that $\left\|C_\phi e^m\right\|^2 = 0$. But $\left\|C_\phi^* e^m\right\|^2 > 0$ for all $m \in N$, so that

$$\left\|C_\phi^* e^m\right\| \neq \left\|C_\phi e^m\right\| ,$$

which is a contradiction to the normality of C_ϕ . Hence C_ϕ is normal implies that ϕ is onto. By Corollary 2.3 of Theorem 2.1 of [6], C_ϕ is one-to-one. Since C_ϕ is normal it has dense range. Thus by Corollary 2.6 of Theorem 2.4 of [6], ϕ is one-to-one. Now since ϕ is one-to-one, a simple computation shows that

$$\left\|C_\phi e^m\right\|^2 = \lambda\phi^{-1}(m) .$$

By normality of C_ϕ and left invertibility of ϕ we have

$$\lambda\phi^{-1}(m) = \lambda_m^2/\lambda\phi(m) .$$

This after further simplication reduces to

$$\phi(m) + \phi^{-1}(m) = 2m .$$

Hence by the above lemma, ϕ is the identity.

COROLLARY 1. *Let* $C_\phi \in B\left(l_a^2\right)$. *Then* C_ϕ *is an isometry if and only if* ϕ *is the identity.*

Proof. The sufficiency is obvious. To prove the necessary part, suppose C_ϕ is an isometry. Then we have, by [2],

$$M_{f_0} = C_\phi^* C_\phi = I .$$

From this we conclude that

$$f_0(n) = 1 \text{ for every } n \in N ,$$

and hence by Lemma 1.1, ϕ is the identity.

COROLLARY 2. *Let* $C_\phi \in B\left(l_a^2\right)$. *Then* C_ϕ *is unitary if and only if* ϕ *is the identity.*

THEOREM 2.4. *Let* $C_\phi \in B\left(l_a^2\right)$. *Then* C_ϕ *is a co-isometry if and only if* ϕ *is the identity.*

Proof. The sufficiency is again obvious. To prove the necessary part, let C_ϕ be a co-isometry. Then

$$\left\|C_\phi^* e^m\right\| = \|e^m\| \text{ for all } m \in N .$$

But $\left\|C_\phi^* e^m\right\|^2 = \lambda_m^2/\lambda_p$, where $m \in \phi^{-1}(p)$ and $\|e^m\|^2 = \lambda_m$. Therefore we have $\lambda_m = \lambda_p$, which implies that $m = \phi(m)$ for all $m \in N$.

This shows that ϕ is the identity.

3. Invertible composition operators

The invertibility of ϕ is a necessary and sufficient condition for the invertibility of C_ϕ on l^2 [4, Theorem 2.2]. But this is not true in the case of l_a^2, as is shown in the next example.

EXAMPLE. Let ϕ be a mapping from N into itself defined as

$$\phi(n) = \begin{cases} n/3 & \text{when } n = p_n \text{ where } p_n = 3(p_{n-1}+1) \text{ with } p_0 = 0 , \\ n+1 & \text{otherwise.} \end{cases}$$

Then ϕ is invertible. But since

$$\frac{\|C_\phi X_{\{p_{n/3}\}}\|^2}{\|X_{\{p_{n/3}\}}\|^2} = \frac{\|X_{\phi^{-1}\{p_{n/3}\}}\|^2}{\|X_{\{p_{n/3}\}}\|^2}$$

$$= \frac{\|X_{\{p_n\}}\|^2}{\|X_{\{p_{n/3}\}}\|^2}$$

$$= \frac{a^{2p_n}}{a^{(2/3)p_n}} = a^{(4/3)p_n} ,$$

which goes to zero as n goes to infinity, we have that C_ϕ is not bounded away from zero, and consequently C_ϕ is not invertible.

It is clear from the above example that characterization of invertibility of C_ϕ in terms of invertibility of ϕ (and *vice versa*) is not possible in this case. But the invertibility of ϕ together with an extra condition characterises the invertibility of C_ϕ, as is shown in the following theorem.

THEOREM 3.1. *Let $C_\phi \in B\left(l_a^2\right)$. Then C_ϕ is invertible if and only if ϕ is invertible and there exists an integer $k \geq 0$ such that*

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$$\phi^{-1}(n) \le k + n \quad \textit{for all} \quad n \in N \ .$$

In order to prove the theorem we need the following lemma.

LEMMA 3.2. *Let* $\phi : N \to N$ *be a mapping. Then* $C_\phi \in B\left(l_a^2\right)$ *if and only if there exists an integer* $M > 0$ *such that* $\lambda\left(\phi^{-1}(n)\right) \le M\lambda(\{n\})$ *for all* $n \in N$.

Proof. Proof of this lemma follows from Theorem 1 of [3].

Proof of Theorem 3.1. Let C_ϕ be invertible. If ϕ is not one-to-one, then $\phi(n) = \phi(m)$ for at least two distinct m and n in N , and hence $g_n = g_m$ for all g in the range of C_ϕ . This shows that C_ϕ is not onto, which is a contradiction. If ϕ is not onto, then there exists a positive integer m such that $m \notin \phi(N)$. Hence $C_\phi X_{\{m\}} = X_{\phi^{-1}\{m\}} = 0$ which shows that C_ϕ is not one-to-one. This is again a contradiction. Further let there exist no integer $k \ge 0$ such that $\phi^{-1}(n) \le k + n$ for every $n \in N$. This implies that for each integer $p \ge 0$ there exists an integer n_p such that

$$\phi^{-1}\left(n_p\right) > p + n_p \quad [p = 1, 2, 3, \ldots] \ .$$

Consider the sequence $\langle X_{\{n_p\}} \rangle$. Then

$$\begin{aligned}
\frac{\left\|C_\phi X_{\{n_p\}}\right\|^2}{\left\|X_{\{n_p\}}\right\|^2} &= \frac{\left\|X_{\phi^{-1}(n_p)}\right\|^2}{\left\|X_{\{n_p\}}\right\|^2} \\
&= \frac{a^{2\phi^{-1}(n_p)}}{a^{2n_p}} \\
&< \frac{a^{2(p+n_p)}}{a^{2n_p}} \\
&= a^{2p} \to 0 \quad \text{as} \quad p \to \infty \ .
\end{aligned}$$

This implies that C_ϕ is not bounded away from zero and hence it is not invertible. This is a contradiction. Hence there exists an integer $k \geq 0$ such that $\phi^{-1}(n) \leq k + n$.

Conversely, suppose ϕ is invertible and there exists an integer $k \geq 0$ such that $\phi^{-1}(n) \leq k + n$ for every $n \in N$. Then there exists a function ψ such that

$$(\phi \circ \psi)(n) = (\psi \circ \phi)(n) = n ,$$

and $\phi(n) \geq n - k$ for every $n \in N$. From this it follows that

$$\begin{aligned}\lambda\psi^{-1}(n) = \lambda(\phi(n)) &= a^{2\phi(n)} \\ &\leq a^{-2k}\lambda(\{n\}) .\end{aligned}$$

Hence by Lemma 1.1 we conclude that C_ϕ is bounded. Since

$$C_\psi C_\phi = C_{\phi\circ\psi} = I = C_{\psi\circ\phi} = C_\phi C_\psi ,$$

C_ϕ is invertible.

This completes the proof of Theorem 3.1.

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