ON SPECTRAL SYNTHESIS FOR ONE POINT

B. Natzitz

In [2, page 3831], Varopoulos proves that for any $\varepsilon > 0$ there exists a function $f = \sum_{n=-\infty}^{\infty} a_n e^{inx}$, $\sum_{n=-\infty}^{\infty} |a_n| < \varepsilon$ and $f(x) = 1 - e^{ix}$ on a neighbourhood of 0. Indeed, if $0 < \varepsilon < \pi/2$, then $f(x)$ defined to be equal to $1 - e^{ix}$ when $-\varepsilon \leq x \leq \varepsilon$, linear on $[\varepsilon, 2\pi - \varepsilon]$ and of period 2π, is an example of such a function.

The above result can be used to give a direct proof of the following result without reference to the L_2 theory [1, Theorem 2.6.4].

THEOREM. Let $\{G, +\}$ be a compact abelian group with dual $\{\Gamma, \cdot\}$ and let $A(G) = \{f \in C(G) : f = \sum_{x \in \Gamma} a_x$ $\|f\|_{A(G)} = \sum |a_x| < \infty\}$. Let $f \in A(G)$, $f(0) = 0$ and let $\varepsilon > 0$. Then there exists $g \in A(G)$ such that $g = f$ on a neighbourhood U of 0 in G and $\|g\|_{A(G)} < \varepsilon$.

Proof. I. Let $f = \sum_{m \leq n} a_m \chi_n$ be a trigonometric polynomial. If $n = 2$, we get essentially the case discussed above. Indeed, it suffices to observe it for $f = (1 - \chi)$.

Choose a neighbourhood U of zero in G so that $|\chi(\alpha) - 1| < |e^{i\varepsilon} - 1|$ for $\alpha \in U$. If $\sum_{m=-\infty}^{\infty} a_n e^{inx} = 1 - e^{ix}$ for $-\varepsilon < x < \varepsilon$ and $\sum |a_n| < \varepsilon$ then $g = \sum_{n=-\infty}^{\infty} a_n \chi^n = (1 - \chi)$ in U and $\|g\|_{A(G)} < \varepsilon$.

To proceed from n to $n + 1$, one simply considers $f_1 = \sum_{m \leq n} a'_m x_m$ and $f_2 = a_{n+1} (\chi_{n+1} - \chi_n)$ where $a'_m = a'_m$ for $m = 1, \ldots, n-1$, $a'_n = a_n + a_{n+1}$ and $f = f_1 + f_2$.

II. The general case $f = \sum a_m \chi_m$, $\sum |a_m| < \infty$, $\sum a_m = 0$ is proved by first choosing n so large so that $|\sum_{m \leq n} a_m| < \varepsilon/2$ and $\sum_{m \leq n} |a_m| < \varepsilon/2$, and

https://doi.org/10.4153/CMB-1969-113-4 Published online by Cambridge University Press

then applying I to $\sum_{m \leq n} a_m \chi_m - \sum_{m \leq n} a_m = f_1$; we get the required result since $f = f_1 + \sum_{m \leq n} a_m + \sum_{m \leq n} a_m \chi_m$.

REFERENCES

1. W. Rudin, Fourier analysis on groups. (Interscience, 1962).

2. N. Th. Varopoulos, Sur les ensembles parfaits et les séries trigonométriques. C.R. Acad. Sc. Paris 260 (1965) 4668-4670, 5165-5168, 5997-6000.