A generalisation of a theorem of al-Kuhi

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1. *Introduction*

Several problems that were composed and/or solved by the tenth century Islamic mathematician Abu Sahl al-Kuhi reached us via the writings of Abd al-Jalil al-Sijzi, another tenth century Islamic mathematician who, according to [1], was presumably a student of al-Kuhi's. Twelve of these (sets of) problems and theorems are discussed in [1] and are referred to as *The Fragments*. The theorem of al-Kuhi alluded to in the title is Fragment #9, which is presented below, together with Fragment #5 and their proofs, in Section 2. Section 3 is devoted to the generalisation referred to in the title. Section 4 describes a relation to angle trisection and Sections 5 and 7 a relation too a configuration of Serenus. Section 6 contains a speculation on what motivated Fragment #9.

Al-Kuhi's theorem (Fragment #9) and his proof are presented in Theorem 1. Our generalisation is presented, with two proofs, in Theorem 2. Since al-Kuhi lived more that ten centuries ago, and since our generalisation is fairly substantial, one would expect that our two proofs of the generalisation use much more advanced tools than al-Kuhi's proof. This is true of our first proof which uses calculus. However, when we compare al-Kuhi's proof of his theorem and the second proof of our generalisation, we see that the exact opposite happens here: our proof does not use anything beyond Euclid's *Elements*, while al-Kuhi's proof depends heavily on the advanced properties of hyperbolas that appeared in Apollonius's *Conic Sections*, decades after the death of Euclid.

In all that follows, Ω will be a given circular arc. The minor arc joining two points X and Y on Ω will be denoted by $\widehat{XY}$, and its length by $|\widehat{XY}|$. The area of a triangle ABC will be denoted by $[ABC]$.

2. *Al-Kuhi's Fragments* #5 *and* #9 *and their proofs*

Al-Kuhi's Fragment #9 refers to the configuration in Figure 1, where Ω is a semicircle with diameter BC and where X is an arbitrary point on Ω with orthogonal projection X' on BC. It asks for a proof that the area $[XBX']$ of triangle XBX' is maximal when X is a third of the way from C to B, i.e. when the length $|\widehat{CX}|$ of the arc $\widehat{CX}$ is one third of that of Ω. For ease of reference, we shall state this again below as Theorem 1, and we shall give al-Kuhi's own proof.

Fragment #9 is closely related, and possibly motivated by, Fragment #5, which together with its solution [1, pp. 634-635] treats the construction of the point X as shown in Figure 1 such that triangle XBX' has a prescribed area v. Naturally, the least upper bound for feasible values of v is the maximal area of XBX', and hence Fragment #9.

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For the solution of Fragment #5, we form the quadrant T enclosed between the perpendicular lines BC and BJ, as in Figure 2. Denoting by X' the orthogonal projection of any point X in T on BC, we use the fact that the locus of points X in T for which the rectangle with diagonal XB (or equivalently $[XBX']$) is fixed is a hyperbola. This is a special case of [3, Proposition 34, p. 59 (Conics II 12)]. Now we take any point Z in T for which $[ZBZ']$ is equal to the required value v of $[XBX']$, and we pass a hyperbola H through Z. Obviously, any point X at which H intersects Ω is a solution to the problem.

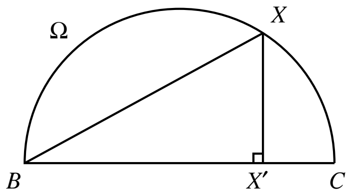

FIGURE 1: (Al-Kuhi's Fragment #9) The area of XBX' is maximal when $|\widehat{XC}| = |\Omega|/3$

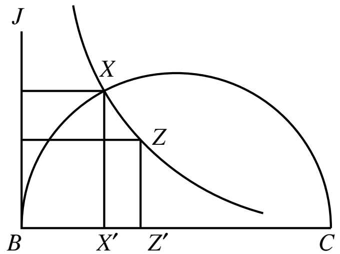

FIGURE 2: (Al-Kuhi's Fragment #5) Construction of X with given $[XBX']$

We now state Fragment #9, and we give al-Kuhi's own proof.

Theorem 1: (Al-Kuhi's Fragment #9) Referring to Figure 1, let Ω be a semicircle with diameter BC. For any X on Ω, let X' denote the orthogonal projection of X on BC. Then, among all points X on Ω, the area $[XBX']$ of XBX attains its maximum at the point P for which

$$|\widehat{CP}| = \frac{|\Omega|}{3}. \tag{1}$$

Proof: Referring to Figure 3, it is obvious that the point X for which $[XBX']$ is maximal is the point where the hyperbola H that is tangent to Ω touches Ω. For if such an H is moved up, it would not have any point in common with Ω, and if it is moved down, it would intersect Ω at points Y with $[YBY'] < [XBX']$. Let the tangent at X to both H and Ω meet the lines BC and BJ at U and W, respectively. Then it follows from [3, Proposition 30, p. 56 (Conics II 3)] that X is the midpoint of UW. Therefore X' is the midpoint of BU, because $XX' // WB$. It also follows from [3, Proposition 64, p. 106 (Conics III 37–40)] that the points B, X', C and U form a harmonic set, i.e.

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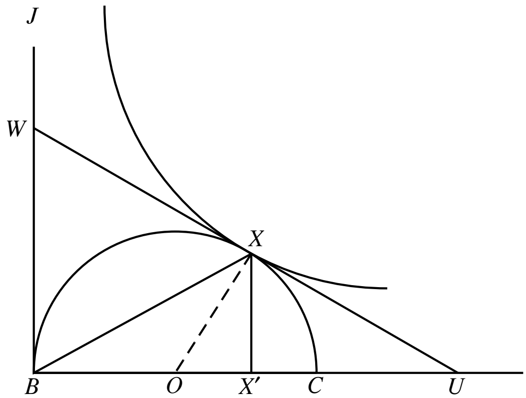
$(BX')(CU) = (X'C)(BU)$. Letting $BX' = z$ and $X'C = t$, this reduces to $(z)(z - t) = (t)(2z)$, i.e. $z = 3t$.

The proof in [1] stops here, i.e. on line 3 of page 620. One can complete it by letting O be the centre of Ω and $OX' = u$, and use $z = 3t$ to obtain

$$R + u = 3(R - u),$$

and hence $R = 2u$ and thus $t = u$. Hence X' is the midpoint of OC, and thus $XO = XC$, and XOC is equilateral. Therefore $\angle XCB = 60°$. Hence $\angle XBC = 30°$. Hence $\angle XCB = 2\angle XBC$, and therefore $|\widehat{BX}| = 2|\widehat{CX}|$, and hence (1), as desired.



$BX' = z$, $X'C = t$, $OX' = u$.

FIGURE 3: Illustrating al-Kuhi's proof of Fragment #9

The solutions above are essentially the ones given on pages 634–635 (for Fragment #5) and on pages 619–620 (for Fragment #9) of [1]. Notice that these proofs depend on properties of hyperbolas and on notions such as harmonic sets, and were thus inaccessible to Euclid and his contemporaries. Hence these solutions are not exactly Euclidean, but rather Apollonian. For more on conics, one may consult [2].

Note: The condition (1) given by al-Kuhi in Theorem 1 is obviously equivalent to the condition

$$\angle PBC = 30°. \tag{2}$$

Since (2) is simpler and possibly more elegant than (1), one wonders why al-Kuhi chose to use (1) and not (2). In view of our generalisation in Theorem 2 and the condition (3) therein, one observes that al-Kuhi's choice of (1) is more suitable for that generalisation than (2). Since one may safely assume that al-Kuhi was not aware of the condition (3) in our generalisation, his choice of (1) and not (2) may sound even more puzzling.

Finally, an apparent relation of (1) and (3) to angle trisection was pointed out by a referee and is discussed in Section 4.

3. *A generalisation of al-Kuhi's Fragment* #9

In this section, we generalise al-Kuhi's theorem in two ways. Instead of the semicircle, we take a minor arc Ω joining points B and C of some circle, as shown in Figure 4, and instead of finding the point X for which the area of XBX' is maximal, we study the subarcs of Ω for which the area of XBX' decreases and increases. Interestingly, the area of XBX' is maximal again when X is a third of the way from C to B. The theorem below describes these results.

Theorem 2: (A generalisation of al-Kuhi's theorem) Referring to Figure 4, let Ω be a circular arc with endpoints B and C and midpoint M, and assume that Ω is a minor arc, i.e. $\angle BMC \geqslant 90°$. Let P and Q be the points that trisect Ω, i.e. the points defined by

$$\left|\widehat{CP}\right| = \left|\widehat{PQ}\right| = \left|\widehat{QB}\right| = \frac{|\Omega|}{3}. \tag{3}$$

For any X on Ω, let X' denote the orthogonal projection of X on BC.

Then as X moves on Ω from C to P, $[XBX']$ increases, and as X moves from P to B, $[XBX']$ decreases. In particular, as X moves on Ω, $[XBX']$ attains its maximum when $X = P$, and $[XCX']$ attains its maximum when $X = Q$.

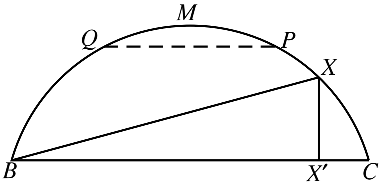


FIGURE 4:

Trisectors P, Q of Ω defined by (3)

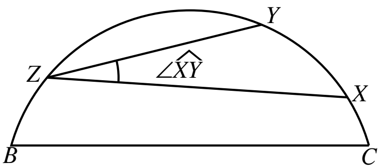


FIGURE 5:

The angle $\angle\widehat{XY}$

Proofs: We give two proofs. In both proofs, we let R denote the radius of Ω. We also use the obvious fact that as X moves from M to B, both XX' and BX' decrease, and hence $[XBX']$ decreases. So we restrict X to move from M to C.

If Z is any point on Ω that does not lie on $\widehat{XY}$, then the angle $\angle XZY$ will be called the angle of $\widehat{XY}$ and will be denoted by $\angle\widehat{XY}$; see Figure 5. It is clear that $\left|\widehat{XY}\right|$ is proportional to $\angle\widehat{XY}$, namely

$$\left|\widehat{XY}\right| = 2R\angle\widehat{XY}.$$

First proof (using calculus): As shown in Figure 6, we let

$$\angle MCB = \angle MBC = \alpha, \angle MBX = x.$$

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Then as X moves from M to C, x increases from 0 to α. Also $0 \leqslant \alpha \leqslant 45°$, since $\angle BMC \geqslant 90°$.

By the sine rule, we have

$$XB = 2R\sin\angle BCX = 2R\sin\left(\alpha + \angle MCX\right) = 2R\sin\left(\alpha + \angle MBX\right)$$
$$= 2R\sin(\alpha + x).$$

Therefore

$$\begin{aligned}[BXX'] &= \frac{(BX')(XX')}{2} = \frac{1}{2}\left((XB)\cos(\alpha - x)\right)\left((XB)\sin(\alpha - x)\right)\\ &= 2R^2\sin^2(\alpha + x)\cos(\alpha - x)\sin(\alpha - x)\\ &= 2R^2\left(\sin(\alpha + x)\cos(\alpha - x)\right)\left(\sin(\alpha + x)\sin(\alpha - x)\right)\\ &= \frac{R^2}{2}\left(\sin 2\alpha + \sin 2x\right)\left(\cos 2x - \cos 2\alpha\right)\\ &= \frac{R^2}{4}\left(\sin 4x - \sin 4\alpha + 2\sin 2(\alpha - x)\right)\end{aligned}$$

$$\frac{d}{dx}\left([BXX']\right) = R^2\left(\cos 4x - \cos 2(\alpha - x)\right).$$

Hence

$$\begin{aligned}[BXX'] \text{ increases with } x &\Leftrightarrow \cos 4x > \cos 2(\alpha - x)\\ &\Leftrightarrow 4x < 2(\alpha - x), \text{because } 0 < 4x, 2\alpha - 2x < 180°\\ &\Leftrightarrow \alpha - x > \frac{2\alpha}{3}.\end{aligned}$$

Therefore as X moves from C to P, x decreases from α to $\alpha/3$, and $[BXX']$ increases. As X moves from P to M, x decreases from $\alpha/3$ to 0, and $[BXX']$ decreases. The last statement follows from this and from symmetry.

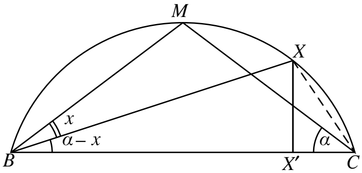

FIGURE 6:
Illustrating the calculus proof

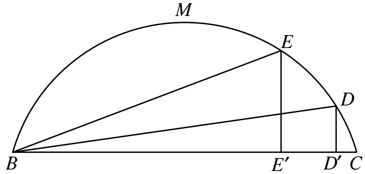

FIGURE 7:
Comparing areas of triangles DBD', EBE'

Second proof (using elementary geometry): Referring to Figure 6, and letting P be as in Figure 4, we are to prove that as a point X moves from C to P, $[XBX']$ increases, and as it moves from P to B, $[XBX']$ decreases. Thus, as shown in Figure 7, we take two points D and E on $\overset{\frown}{CM}$ and we compare the areas $[DBD']$ and $[EBE']$ of triangles DBD' and EBE'. We assume that D lies between C and E. As shown in Figure 8, we draw from E a line parallel to BC, and we let F and K be the points where this line meets the extension of $D'D$ and Ω, respectively. We join FB and we let G be the point where it crosses EE'. We also join FE' and GD'.

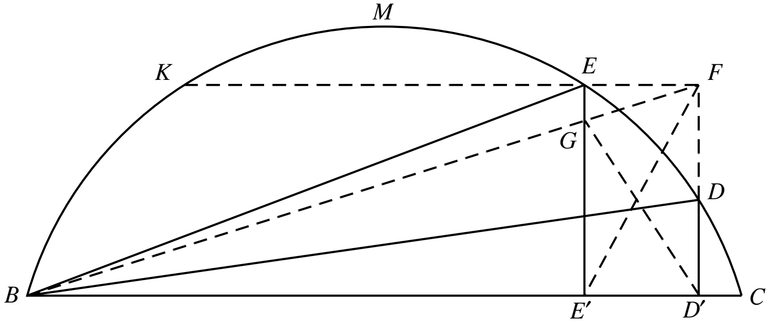

FIGURE 8: Illustrating the proof that $[EBE'] = [GBD']$

Then we have

$$\begin{aligned} [EBE'] &= [FBE'] \text{ because } EF \,//\, BC \\ &= [FGE'] + [GBE'] \\ &= [D'GE'] + [GBE'] \text{ because } FD' \,//\, GE' \\ &= [GBD']. \end{aligned}$$

Joining ED and KB, as shown in Figure 9, we see that

$$\begin{aligned} [EBE'] > [DBD'] &\Leftrightarrow [GBD'] > [DBD'] \\ &\Leftrightarrow GE' > DD' \text{ because triangles } GBD' \text{ and } DBD' \\ &\qquad\qquad \text{have the same base } BD' \\ &\Leftrightarrow GE' > LE' \\ &\Leftrightarrow \angle GFE < \angle LFE \\ &\Leftrightarrow \angle BFE < \angle DEF \\ &\Leftrightarrow \angle FBC < \angle DEF \text{ because } EF \,//\, BC \\ &\Leftrightarrow \angle FBC < \angle KBD \text{ because } EKBD \text{ is cyclic.} \end{aligned}$$

Therefore

$$[EBE'] > [DBD'] \Leftrightarrow \angle FBC < \angle KBD. \tag{4}$$

Using the fact that $\angle DBC < \angle FBC < \angle EBC$, it follows from (4) that

$$\angle EBC < \angle KBD \Rightarrow [EBE'] > [DBD'], \angle DBC > \angle KBD \Rightarrow [EBE'] < [DBD'].$$

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In terms of lengths of arcs, this says that

$$\left|\widehat{EC}\right| < \left|\widehat{KD}\right| \Rightarrow [EBE'] > [DBD'], \tag{5}$$

$$\left|\widehat{DC}\right| > \left|\widehat{KD}\right| \Rightarrow [EBE'] < [DBD']. \tag{6}$$

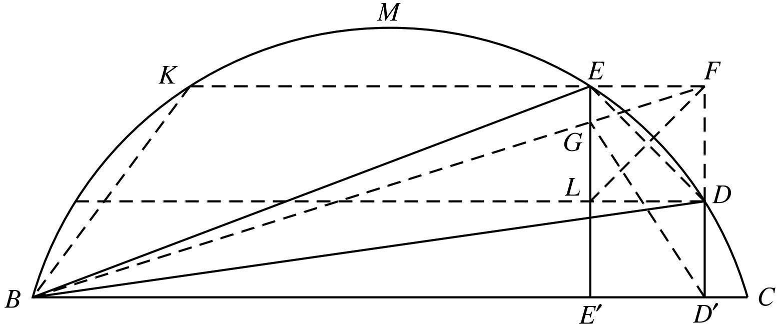

FIGURE 9: Illustrating the proof of Theorem 2

If D and E are both on $\widehat{CP}$, as in Figure 10, then

$$\left|\widehat{EC}\right| < \left|\widehat{PC}\right| = \frac{|\Omega|}{3}, \left|\widehat{DK}\right| > \left|\widehat{PQ}\right| = \frac{|\Omega|}{3},$$

and hence $\left|\widehat{EC}\right| < \left|\widehat{DK}\right|$. By (5), it follows that $[EBE'] > [DBD']$.

If D and E are both on $\widehat{PQ}$, as in Figure 11, then

$$\left|\widehat{DC}\right| > \left|\widehat{PC}\right| = \frac{|\Omega|}{3}, \left|\widehat{DK}\right| < \left|\widehat{PQ}\right| = \frac{|\Omega|}{3},$$

and hence $\left|\widehat{DC}\right| > \left|\widehat{DK}\right|$. By (6), it follows that $[EBE'] < [DBD']$.

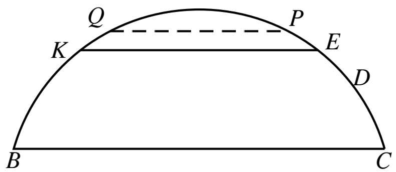

FIGURE 10

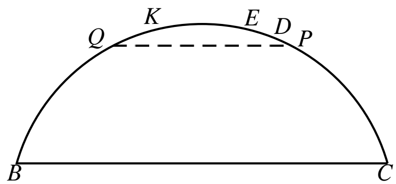

FIGURE 11

Thus as a point X moves from C to P along Ω, the area $[XBX']$ increases, and as X moves from P to M, the area $[XBX']$ decreases. As mentioned earlier, as X moves from M to B, each of the sides XX' and $X'B$ decreases and hence the area $[XBX']$ decreases. The last statement follows from this and from symmetry.

4. *Relation, albeit only apparent, to angle trisection*

Let θ be an acute angle that is to be trisected. We may assume that θ is placed in the plane so that its vertex O is the centre of a circle A and that its

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arms are the radii OB and OC. Referring to Figure 13, it is clear that the points P and Q that trisect the arc $\widehat{BC}$ trisect $\angle BOC$. Again letting X', for any X on $\widehat{BC}$, be the orthogonal projection of X on the chord BC, it follows from Theorem 2, that P is the point on $\widehat{BC}$ for which $[PBP']$ is maximal among all areas $[XBX']$, where X is on $\widehat{BC}$. This characterises the angle trisector P by being the point that maximises the area in a family of triangles. Needless to say, locating such a maximiser P is not in any way easier than trisecting $\angle BOC$.

Characterising the point that trisects a given angle in terms of another property (such as maximising a certain quantity) is not new. For example, Exercise 31 (p. 124) of [4] attributes to Pappus a characterisation of the angle trisector P. Referring to Figure 12, Pappus characterises P as the point for which

$$PC = 2(P'M'), \tag{7}$$

where M is the midpoint of arc $\widehat{BC}$. It may be interesting to survey the literature on angle trisection for similar characterisations.

5. *Relation to a configuration of Serenus*

The generalisation of al-Kuhi's theorem given in Theorem 2 is illustrated in the configuration shown in Figure 4. This is quite closely related to a configuration of Serenus that illustrates Proposition 53 (pp. 142–145) of his book [5]. Referring to Figure 12 showing a circular arc $\widehat{BC}$ (not necessarily minor) with midpoint M, Serenus' proposition states that as a point X moves on this arc from B to M, $XB + XC$ (or equivalently, the perimeter $\text{per}(XBC)$ of XBC) increases. Apparently unaware of this proposition, R. Honsberger included a weaker variant as Problem 9 (pp. 16–17) in his book [6] and gave a proof. He included it later with two more proofs (on pp. 21–24) in his book [7]. Three more proofs are given in [8] and one more proof can be found in the last paragraph of [9]. Needless to say, Serenus' result obviously holds if the perimeter is replaced by the area. It is this obvious area version that will be used in the section below.

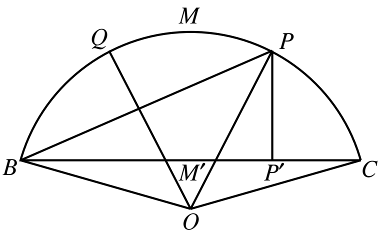

FIGURE 12

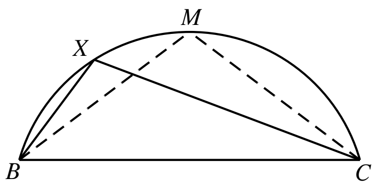

FIGURE 13: Illustrating Proposition 53 of Serenus saying that as X moves from B to M, $XB + XC$ increases

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6. *Using al-Kuhi's Fragment #9 to prove that the equilateral triangles are the largest among all triangles that can be inscribed in a given circle*

Let Ω be a given circle. We shall prove that among the triangles that are inscribed in Ω, the ones that have maximal area are the equilateral ones (which clearly are all congruent).

We observe first that if UVW is any triangle inscribed in Ω, then the isosceles triangle UNW, where N is the midpoint of the arc $\widehat{UVW}$ has a larger or equal area; see triangles BCX and BCM in Figure 13. Thus we may restrict ourselves to the family of isosceles triangles that are inscribed in Ω.

If we fix a diameter, say BC, in Ω, then we can restrict our attention to those isosceles triangles ABA^* having B as an apex and BC as an axis of symmetry; see Figure 14. This is because every isosceles triangle inscribed in Ω can be rotated about the centre of Ω to occupy a position like ABA^*. It is clear that maximising $[ABA^*]$ is equivalent to maximising $[ABA']$, where A' is the point of intersection of AA^* with BC. Triangles ABA' are precisely the triangles considered in al-Kuhi's Fragment #9; see Figure 1. Thus finding the inscribed triangles of maximal area is reduced to al-Kuhi's Fragment #9, which states that $[ABA']$ is maximal when $\angle A'AB = 60°$. It follows that $[A^*AB]$ is maximal when A^*AB is equilateral, as desired.

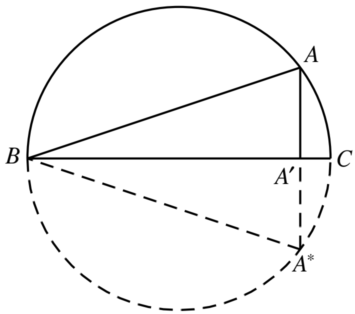

FIGURE 14: All isosceles triangles can be moved to become symmetric about BC

The proof above raises the question whether al-Kuhi's Fragment #9 was motivated by the problem of maximising the area of the triangle inscribed in a given circle.

7. *The perimeter version of Section 5*

Using Serenus' theorem, as stated in Section 5, one can argue as in Section 6 to prove that among the triangles that are inscribed in a given circle those with maximal perimeter are isosceles. To complete the proof that among all triangles inscribed in a given circle the equilateral triangles are the ones with maximal perimeter, one needs to prove that the sum $BA + AA'$, in Figure 1, is maximal when $\angle A'AB = 60°$. This is not very difficult to prove.

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