Canad. Math. Bull. Vol. **27** (4), 1984

FINITE UNIONS OF QUASI-INDEPENDENT SETS

BY

DAVID GROW AND WILLIAM C. WHICHER

ABSTRACT. The analogue of Horn's theorem characterizing finite unions of linearly independent sets in a vector space is shown to fail in the group of integers.

A finite subset $E=\{n_j\}$ of a discrete abelian group Γ is called quasi-independent if each finite sum $\sum c_j n_j$ ($c_j \in \{-1, 0, 1\}$, $n_j \in E$) is nonzero unless $c_j = 0$ for all j. A recent theorem of Pisier [2, theorem 2] has reduced an open problem concerning the arithmetic characterization of Sidon subsets of Γ to the following purely combinatorial problem.

PROBLEM 1. Suppose that a set $E \subset \Gamma$ has the property that there exists $k > 0$ such that every finite set $F \subset E$ contains a quasi-independent subset F' satisfying $|F| \leq k\,|F'|$. (Here $|X|$ denotes the number of elements of X.) Can E be written as the union of a finite number of quasi-independent subsets?

It is natural to wonder whether a set E satisfying the property in Problem 1 can be written as the union of k quasi-independent subsets, especially in light of the following theorem by Horn [1].

THEOREM. *Suppose that a subset E of a vector space has the property that there exists a positive integer k such that every finite set $F \subset E$ contains a linearly independent subset F' satisfying $|F| \leq k\,|F'|$. Then E can be written as the union of k linearly independent subsets.*

The example in the next proposition shows that the analogue of Horn's theorem fails in the case $\Gamma = Z$, the additive group of integers.

PROPOSITION. *Consider $E = E_0 \cup E_1 \cup E_2$ where $E_k = \{3^j + kj : 1 \leq j \leq 5\}$. Then E has the property that every set $F \subset E$ contains a quasi-independent set F' satisfying $|F| \leq 2\,|F'|$, and yet E cannot be written as the union of two quasi-independent subsets.*

Sketch of the proof. To prove that every $F \subset E$ contains a quasi-independent subset F' satisfying $|F| \leq 2\,|F'|$, it suffices to establish it when $|F|$ is odd and $|F'| = (|F|+1)/2$. It is a trivial exercise to prove that if $F \subset Z^+$ and

Received by the editors November 3, 1983 and, in revised form, January 19, 1984.
AMS (1980) subject classification: 43A46.

https://doi.org/10.4153/CMB-1984-078-0 Published online by Cambridge University Press

$|F| \leq 5$, then F contains a quasi-independent subset F' satisfying $|F| \leq 2|F'|$. Therefore only the cases $|F| = 7$, 9, 11, 13, and 15 $(= |E|)$ need to be considered. There are $C(15, 7) + C(15, 9) + C(15, 11) + C(15, 13) + C(15, 15) =$ 12, 911 such sets $F \subset E$. (Here $C(n, k) = n!/[(n-k)!\ k!]$.) Using a Hewlett-Packard 3000 computer and a FORTRAN program designed by the second author, each one of these 12, 911 subsets F were found to contain at least one quasi-independent subset F' satisfying $|F'| = (|F| + 1)/2$.

The proof that E cannot be written as the union of two quasi-independent subsets consists of two parts. First, the computer was used to establish that none of the $C(15, 9) = 5005$ subsets $F \subset E$ satisfying $|F| = 9$ are quasi-independent. It follows that E has no quasi-independent subsets F satisfying $|F| \geq 9$, and hence the only possible partitions $E = F \cup G$ where both F and G are quasi-independent must be of the form $|F| = 8$ and $|G| = 7$. Next, a computer check showed that there are precisely 5 quasi-independent subsets $F \subset E$ with $|F| = 8$:

$$F_1 = \{4, 11, 27, 81, 89, 243, 248, 253\},$$
$$F_2 = \{4, 11, 30, 81, 89, 243, 248, 253\},$$
$$F_3 = \{11, 27, 81, 85, 89, 243, 248, 253\},$$
$$F_4 = \{11, 30, 81, 85, 89, 243, 248, 253\},$$
$$F_5 = \{11, 33, 81, 85, 89, 243, 248, 253\}.$$

It is easy to verify that for each F_i, the complement of F_i in E is not quasi-independent. This completes the proof.

Copies of the FORTRAN programs used in the proof of the Proposition may be obtained from the authors upon request.

It is interesting to note that in the case $\Gamma = Z$, Problem 1 can be reduced to the consideration of finite sets E.

PROBLEM 2. Suppose that E is any finite subset of Z with the property that there exists a positive integer k such that every set $F \subset E$ contains a quasi-independent subset F' satisfying $|F| \leq k\,|F'|$. Does there exist a positive integer $n = n(k)$, independent of E, such that E can be written as the union of n quasi-independent subsets?

Proof of the equivalence of Problems 1 and 2 for $\Gamma = Z$. If Problem 2 is settled in the affirmative, then Problem 1 would be answered affirmatively as well. This results from the following lemma by R. Rado [3, Lemma 1] and a short argument in [1] which will be reproduced here.

LEMMA. *Let E be any set and let T be a function which associates with each $x \in E$ a finite set $T(x)$. Suppose that for every finite subset F of E there exists a* *function f_F with domain F such that $f_F(x)\in T(x)$ for all $x\in F$. Then there exists a function f^* with domain E such that for any finite subset G of E there is a finite set F with $G\subset F\subset E$ such that $f^*(x)=f_F(x)$ for all $x\in G$.*

Suppose that $E\subset Z$ has the property in Problem 1 and that the answer to Problem 2 is affirmative. In the lemma, take $T(m)=\{1,\ldots,n\}$ for all $m\in E$. Any finite subset F of E can be written as the union of n disjoint quasi-independent sets $H(F, i)$, $1\le i\le n$. Define $f_F(m)=i$ for $m\in H(F,i)$. Let f^* be a function guaranteed by the lemma and let H_i be the set of all m for which $f^*(m)=i$. Then $E=H_1\cup\cdots\cup H_n$ and the H_i are disjoint. Furthermore, let G be any finite subset of H_i. There is a finite set F with $G\subset F\subset E$ and $f_F(m)=f^*(m)=i$ for $m\in G$. Therefore $G\subset H(F,i)$, which is quasi-independent. Thus each H_i is quasi-independent and Problem 1 is settled in the affirmative.

On the other hand, suppose that the answer to Problem 2 is negative. Then there exists a positive integer k and a sequence of finite sets $E_m=\{n_{m,j}\}\subset Z$, $m=1, 2, 3,\ldots$, each with the property that every set $F\subset E_m$ contains a quasi-independent set F' satisfying $|F|\le k\,|F'|$, and yet E_m cannot be written as the union of m quasi-independent subsets. Set $p_1=1$ and construct inductively an increasing sequence of integers $\{p_m\}$ which satisfies

$$p_{m+1}>p_m\sum_{j=1}^{|E_m|}|n_{m,j}|+\cdots+p_1\sum_{j=1}^{|E_1|}|n_{1,j}|$$

for $m=1,2,3,\ldots$ Define $E=\bigcup p_mE_m$ where $pX=\{px : x\in X\}$. The construction of $\{p_m\}$ implies that if we have any quasi-linear combination from E representing 0, say

$$p_m\left(\sum_{j=1}^{|E_m|}c_{m,j}n_{m,j}\right)+\cdots+p_1\left(\sum_{j=1}^{|E_1|}c_{1,j}n_{1,j}\right)=0,\quad (c_{i,j}\in\{-1,0,1\}),$$

then

$$\sum_{j=1}^{|E_i|}c_{i,j}n_{i,j}=0\quad\text{for}\quad i=1,\ldots,m.$$

It follows that every finite set $F\subset E$ contains a quasi-independent subset F' satisfying $|F|\le k\,|F'|$. Now E_m cannot be written as the union of m quasi-independent subsets and thus neither can p_mE_m. Consequently, $E=\bigcup p_mE_m$ cannot be written as the union of any finite number of quasi-independent subsets. This answers Problem 1 in the negative.

The authors would like to express their gratitude to Professor Gordon Woodward for several insightful suggestions.

References

1. A. Horn, *A characterization of unions of linearly independent sets*, J. London Math. Soc. **30** (1955), 494–496.

2. G. Pisier, *Arithmetic characterizations of Sidon sets*, Bull. Amer. Math. Soc. **8** (1983), 87–89.

3. R. Rado, *Axiomatic treatment of rank in infinite sets*, Canadian J. of Math. **1** (1949), 337–343.

Parks College of Saint Louis University
Cahokia, Illinois 62206