Glasgow Math. J. **55** (2013) 399–409.
doi:10.1017/S001708951200064X.

UNIQUENESS FOR SINGULAR SEMILINEAR ELLIPTIC BOUNDARY VALUE PROBLEMS

D. D. HAI AND R. C. SMITH

Department of Mathematics, Mississippi State University
Mississippi State, MS 39762, USA
e-mail: dang@math.msstate.edu, smith@math.msstate.edu

(Received 14 September 2011; revised 27 January 2012; accepted 22 May 2012; first published online 25 February 2013)

Abstract. We prove uniqueness of positive solutions for the boundary value problems

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega, \lambda$ is a positive parameter and $f : (0, \infty) \to (0, \infty)$ is sublinear at ∞ and is allowed to be singular at 0.

2010 *Mathematics Subject Classification.* 35J75, 35J92.

1. Introduction.

Consider the boundary value problem

$$\begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega, f : (0, \infty) \to (0, \infty)$ is possibly singular at 0 and λ is a positive parameter.

We are interested in the uniqueness of positive solutions to (1.1) when f is sublinear at ∞ and is singular at 0. Let us briefly recall the literature on uniqueness of positive solutions for (1.1) when f is nonsingular. Schuchman in [**7**] showed that (1.1) has a unique positive solution for λ large when $f(0) > 0$ and there exists $\alpha > 1$ such that

$$0 \le f'(u) \le K(1 + u)^{-\alpha} \quad \text{for } u \ge 0.$$

The result in [**7**] was improved by Dancer [**2**], in which the uniqueness and asymptotic behaviour of positive solutions to (1.1) for λ large were established for C^1 functions f satisfying $f(u) \to C > 0$, $uf'(u) \to 0$ as $u \to \infty, f > 0$ on $(0, \infty)$, and either $f(0) > 0$ or $f'(0) > 0$. The cases when there exist $\beta \in (0, 1)$ and $C > 0$ such that $u^{1-\beta} f'(u) \to \beta C$ as $u \to \infty$, or when there is an $a > 0$ such that $f > 0$ on $(0, a)$ and $f(a) = 0$ were also studied in [**2**]. Wiegner in [**9**] included cases where $f(u) \to 0$ as $u \to \infty$ or $f(u)$ does not behave like u^β at ∞ for some $\beta \in (0, 1)$ such as $(1 + u)^{-\gamma}$ for $\gamma > 0$ small and $\ln(2 + u)$, but required $f(0) > 0$. Related results when $f \in C^1(0, \infty)$ and f' is possibly singular at 0 (but not f) were obtained by Lin [**5**] and Hai and Smith [**4**]. In [**5**], uniqueness was established when $u^2 f'(u)$ is bounded near 0 and $f(u) \sim u^\beta$ at ∞ for some $\beta \in (0, 1)$, while in [**4**], nonlinearities such as $u^\beta \ln(2 + u)$ for some $\beta \in [0, 1)$ are allowed but required that $f(u)$ be nondecreasing for u large. In this paper, we shall establish uniqueness and asymptotic behaviour of positive solutions to (1.1) for λ large when f is sublinear at ∞ and is possibly singular at 0, which have not been considered in the literature to the best of our knowledge. Note that in the case when $f(u) \sim u^{\beta}$ at ∞ for some $\beta \in [0, 1)$, we do not require that f be nondecreasing for u large. Thus, our results provide an extension of the corresponding results in [**2**, **4**, **7**] to the singular case. In particular, our results when applied to the model case

$$\begin{cases} -\Delta u = \lambda\left(\frac{a}{u^{\gamma}} + u^{\beta}e^{\frac{1}{1+u}}\right) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where $a \geq 0$, $\gamma, \beta \in [0, 1)$, give the existence of a unique positive solution for λ large. Our approach depends on sharp upper and lower estimates on the solutions when λ is large.

2. Main results. We make the following assumptions:

(A1) $f : (0, \infty) \to (0, \infty)$ is differentiable and there exist a constant $A > 0$ and a continuous function $g : (0, \infty) \to (0, \infty)$ such that $g(u)$ is nondecreasing and $\frac{g(u)}{u}$ is decreasing for $u > A$,

$$\lim_{u\to\infty} \frac{f(u)}{g(u)} = 1, \quad \lim_{u\to\infty} \frac{g(u)}{u} = 0.$$

(A2) For each $c > 0$, there exist constants $A_c, B_c > 0$ such that

$$H^{-1}(cu) \leq A_c H^{-1}(u) \quad \text{for } u > B_c,$$

where $H(u) = \frac{u}{g(u)}$.

(A3) There exists a constant $\gamma \in (0, 1)$ such that

$$\limsup_{u\to 0^+} u^{\gamma+1}|f'(u)| < \infty.$$

(A4) $\liminf\limits_{u\to 0^+} \frac{f(u)}{u} > 0$.

(A5) $\liminf\limits_{u\to\infty} (f(u) - uf'(u)) > 0$.

(B1) $f : (0, \infty) \to (0, \infty)$ is differentiable and there exist constants $\beta \in [0, 1)$ and $C > 0$ such that

$$\lim_{u\to\infty} \frac{f(u)}{u^{\beta}} = C,$$

and

$$\limsup_{u\to\infty} \frac{u|f'(u)|}{f(u)} < 1.$$

THEOREM 2.1. *Let (A1)–(A5) hold and let f be nondecreasing for $u > A$, or let (B1),(A3) and (A4) hold. Then there exists a positive number λ_0 such that (1.1) has a unique positive solution for $\lambda > \lambda_0$.*

THEOREM 2.2. *Let (A1)–(A4) hold with $g(u) = Cu^{\beta}$ for some $\beta \in [0, 1)$, $C > 0$. Let u_{λ} be a solution of (1.1). Then*

$$\lambda^{\frac{1}{\beta-1}} u_{\lambda} \to C^{\frac{1}{1-\beta}} w_{\beta} \ \ in\ C^1(\bar{\Omega}) \quad as\ \lambda \to \infty,$$

where w_{β} is the unique positive solution of $-\Delta w_{\beta} = w_{\beta}^{\beta}$ in Ω, $w_{\beta} = 0$ on $\partial\Omega$.

Remark 2.1. Theorems 2.1 and 2.2 extend corresponding results in Theorem 1 and Remark 1 in [**2**] to include singular nonlinearities f. *Note that the assumption (B1) is weaker than the condition* $\lim_{u\to\infty} u^{1-\beta} f'(u) = \beta C$ *for some* $\beta \in [0, 1)$ *and* $C > 0$ *in [2].*

Remark 2.2. It follows from (A2) that

$$H^{-1}(u) = H^{-1}(c^{-1}(cu)) \le A_{1/c} H^{-1}(cu) \quad \textit{for } cu > B_{1/c},$$

i.e.

$$H^{-1}(cu) \ge A_{1/c}^{-1} H^{-1}(u) \quad \textit{for } u > c^{-1} B_{1/c}.$$

Remark 2.3. Note that condition (A2) is satisfied if for each $b > 1$*there exists a function* $h : (1, \infty) \to \mathbb{R}$ *such that*

$$\limsup_{u\to\infty} \frac{g(bu)}{g(u)} \le h(b),$$

and

$$\limsup_{b\to\infty} \frac{h(b)}{b} = 0.$$

Indeed, let $c > 1$ *and choose* $b > 1$ *so that* $\frac{h(b)+1}{b} < \frac{1}{c}$. *Since* $\lim_{x\to\infty} H^{-1}(x) = \infty$, *there exists a constant* $B_c > H(A)$ *be such that*

$$\frac{g(bH^{-1}(x))}{g(H^{-1}(x))} \le h(b) + 1 < \frac{b}{c} \quad \textit{for } x > B_c,$$

which implies

$$H(H^{-1}(cx)) = \frac{H^{-1}(cx)}{g(H^{-1}(cx))} = \frac{cH^{-1}(x)}{g(H^{-1}(x))} \le \frac{bH^{-1}(x)}{g(bH^{-1}(x))} = H(bH^{-1}(x))$$

for $x > B_c$. *Hence,* $H^{-1}(cx) \le bH^{-1}(x)$ for $x > B_c$.

Remark 2.4. It should be noted that the assumptions

(i) (A1)–(A5) and f *are nondecreasing for u large, and*

(ii) (B1), (A3) and (A4) are different.

Indeed, it follows from Remark 2.3 that the function $g(u) = Cu^\beta$ with $C > 0$ satisfies (A2). Hence, it is easily seen that (B1) implies (A1),(A2) and (A5). However, (B1) does not imply that f is nondecreasing for u large as the following example shows:

Let $\beta \in [0, 1)$ and $k \in (\beta, 1)$. Let $\zeta : [0, \infty) \to \mathbb{R}$ be a continuous function such that $0 \le \zeta \le k + \beta$, $\zeta(n) = k + \beta$ for all $n \in \mathbb{N}$, $\zeta = 0$ on $[0, 1/2]$, and $\int_0^\infty \zeta(t)dt < \infty$. For $u > 0$, define $f(u) = u^\beta \phi(u)$, where $\phi(u) = e^{-\int_1^u \frac{\zeta(t)}{t} dt}$. Then $f > 0$ on $(0, \infty)$,

$$u^{-\beta} f(u) \to C \quad \text{as } u \to \infty,$$

where $C = e^{-\int_1^\infty \frac{\zeta(t)}{t} dt}$, and

$$\frac{uf'(u)}{f(u)} = \beta + \frac{\phi'(u)}{\phi(u)} = \beta - \zeta(u) \equiv z(u) \quad \text{for } u > 0.$$

Since $-k \le z \le \beta$ and $z(n) = -k$ for all $n \in \mathbb{N}$, it follows that $f'(n) < 0$ or all $n \in \mathbb{N}$ and

$$\limsup_{u\to\infty} \frac{u|f'(u)|}{f(u)} \le k < 1.$$

Thus, f satisfies (B1) but f is not nondecreasing on (A, ∞) for any $A > 0$. Note that f also satisfies (A3) and (A4). On the other hand, a function such as $f(u) = u^{-\gamma} + u^{\delta}\ln(1+u)$, where $\gamma, \delta \in (0, 1)$, is nondecreasing for u large and satisfies (A1)–(A5), but does not satisfy (B1) since there do not exist $\beta \in [0, 1)$ and $C > 0$ such that $u^{-\beta} f(u) \to C$ as $u \to \infty$.

3. Preliminary results. Let λ_1 be the first eigenvalue of $-\Delta$ with Dirichlet boundary conditions and ϕ_1 be the corresponding normalized positive eigenfunction, i.e. $||\phi_1||_\infty = 1$.

LEMMA 3.1. *[5] Let (A4) hold. Then there exist positive numbers δ and β such that any positive solution u of (1.1) satisfies*

$$u \ge \delta\phi_1 \quad \text{in } \Omega \quad \text{for } \lambda > \lambda_1/\beta.$$

Proof. Lemma 3.1 was proved in [**5**] using Serrin's sweeping principle. Here we give a short, new proof. By (A4), there exist positive numbers $\delta, \beta > 0$ such that

$$f(u) > \beta u \quad \text{for } u \in (0, \delta).$$

Let u be a positive solution of (1.1) and $v = u - \delta\phi_1$. Define $D = \{x \in \Omega : u(x) < \delta\}$. Then $v \ge 0$ on ∂D and

$$-\Delta v - \lambda_1 v = -\Delta u - \lambda_1 u > (\lambda\beta - \lambda_1)u > 0 \ \text{ in } D$$

for $\lambda > \lambda_1/\beta$. By the maximum principle [**8**, Theorem 2], $v \ge 0$ in D. Clearly, $v \ge 0$ in $\Omega \backslash D$, and so $v \ge 0$ in Ω,
i.e. $u \ge \delta\phi_1$ in Ω.

□

LEMMA 3.2. *Let (A5) hold and $\alpha_0 \in (0, 1)$. Then there exist positive numbers K and C_{α_0} such that*

$$f(\alpha u) - \alpha f(u) \ge K(1-\alpha)$$

for $\alpha \in [\alpha_0, 1), u \ge C_{\alpha_0}$.

Proof. By (A5), there exist constants $C, K > 0$ such that

$$f(u) - uf'(u) \ge K \quad \text{for } u > C,$$

which implies

$$\left(\frac{f(u) - K}{u}\right)' = \frac{uf'(u) - f(u) + K}{u^2} \le 0$$

for $u > C$. Hence, if $u > C/\alpha_0$,

$$\frac{f(\alpha u) - K}{\alpha u} \geq \frac{f(u) - K}{u}$$

for $\alpha \in [\alpha_0, 1)$, and therefore

$$f(\alpha u) - \alpha f(u) \geq K(1 - \alpha) \quad \text{for } u > C/\alpha_0.$$

□

The next Lemma gives sharp lower and upper estimates for positive solutions of (1.1). Let ϕ be the solution of $-\Delta\phi = 1$ in Ω, $\phi = 0$ on $\partial\Omega$.

LEMMA 3.3. *Let (A1)–(A4) hold. Then there exist positive constants C_1, C_2 and $\tilde{\lambda}$ such that any positive solution of (1.1) satisfies*

$$C_1 H^{-1}(\lambda)\phi \leq u \leq C_2 H^{-1}(\lambda)\phi \quad \text{in } \Omega \tag{3.1}$$

for $\lambda > \tilde{\lambda}$.

Proof. Let u be a positive solution of (1.1). By (A1), for each $c > 0$, there exist constants $K_c, \tilde{K}_c > 0$ such that

$$\tilde{K}_c g(z) \geq f(z) \geq K_c g(z) \quad \text{for } z \geq c. \tag{3.2}$$

In particular, there exists a constant $M_c > 0$ such that

$$f(z) \geq M_c \quad \text{for } z \geq c.$$

Let D be an open subset of Ω with $\bar{D} \subset \Omega$. By Lemma 3.1, $u \geq \delta\phi \geq \delta_0 > 0$ in D for λ large. Hence,

$$-\Delta u = \lambda f(u) \geq \begin{cases} \lambda M_{\delta_0} & \text{in } D, \\ 0 & \text{in } \Omega|D, \end{cases}$$

and the weak comparison principle [**6**, Lemma A2] implies $u \geq \lambda M_{\delta_0}\tilde{\phi}$ in Ω, where $\tilde{\phi}$ is the solution of

$$-\Delta\tilde{\phi} = \begin{cases} 1 & \text{in } D, \\ 0 & \text{in } \Omega\backslash D, \end{cases} \quad \tilde{\phi} = 0 \text{ on } \partial\Omega.$$

Let c_λ be the largest number such that $u \geq c_\lambda\tilde{\phi}$ in Ω, and $k_0 = \inf_D \tilde{\phi}$. Then

$$u \geq c_\lambda k_0 > \lambda M_{\delta_0} k_0 > A \quad \text{in } D$$

for $\lambda > (M_{\delta_0} k_0)^{-1} A$, which we shall assume. Hence, it follows from (3.2) and the fact that g is nondecreasing on (A, ∞) that

$$-\Delta u = \lambda f(u) \geq \begin{cases} \lambda K_A g(c_\lambda k_0) & \text{in } D, \\ 0 & \text{in } \Omega|D, \end{cases}$$

which implies $u \geq \lambda K_A g(c_\lambda k_0)\tilde{\phi}$ in Ω and therefore

$$c_\lambda \geq \lambda K_A g(c_\lambda k_0).$$

https://doi.org/10.1017/S001708951200064X Published online by Cambridge University Press

Consequently,

$$H(c_\lambda k_0) = \frac{c_\lambda k_0}{g(c_\lambda k_0)} \geq \lambda K_A k_0.$$

This, together with Remark 2.2, implies the existence of a positive constant k_1 such that

$$c_\lambda k_0 \geq H^{-1}(\lambda K_A k_0) \geq k_1 H^{-1}(\lambda)$$

for λ large. Let $\tilde{k} > 0$ be such that $\tilde{\phi} \geq \tilde{k}\phi$ in Ω. Then

$$u \geq c_\lambda \tilde{k}\phi \geq C_1 H^{-1}(\lambda)\phi \quad \text{in } \Omega \tag{3.3}$$

for λ large, where $C_1 = \tilde{k}(k_1/k_0)$.

Next, let ψ be the solution of

$$\begin{cases} -\Delta\psi = \frac{1}{\phi^\gamma} & \text{in } \Omega, \\ \quad \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

and let $k_2 > 0$ be such that $\psi \leq k_2\phi$ in Ω. (see e.g. [**1**, Theorem 2.25], [**3**, Lemma 3.1]. From (3.3), (A3) and (3.2) with $c = A$, we deduce the existence of positive constants M and $\tilde{K} \equiv \tilde{K}_A$ such that

$$-\Delta u \leq \lambda\left(\frac{M}{u^\gamma} + \tilde{K}g(u)\right) \leq \lambda\left(\frac{M}{(C_1 H^{-1}(\lambda)\phi)^\gamma} + \tilde{K}g(||u||_\infty)\right) \quad \text{in } \Omega.$$

By the comparison principle,

$$u \leq \lambda\left(\frac{M}{(C_1 H^{-1}(\lambda))^\gamma}\psi + \tilde{K}g(||u||_\infty)\phi\right)$$

$$\leq \lambda\left(\frac{Mk_2}{(C_1 H^{-1}(\lambda))^\gamma} + \tilde{K}g(||u||_\infty)\right)\phi \leq 2\lambda\tilde{K}g(||u||_\infty)\phi \quad \text{in } \Omega \tag{3.5}$$

if λ is large enough. Consequently,

$$H(||u||_\infty) = \frac{||u||_\infty}{g(||u||_\infty)} \leq 2\lambda\tilde{K}||\phi||_\infty,$$

which implies

$$||u||_\infty \leq H^{-1}(2\lambda\tilde{K}||\phi||_\infty). \tag{3.6}$$

From (3.5), (3.6) and (A2), we get

$$u \leq 2\lambda\tilde{K}g(H^{-1}(2\lambda\tilde{K}||\phi||_\infty))\phi = \frac{H^{-1}(2\lambda\tilde{K}||\phi||_\infty)}{||\phi||_\infty}\phi \leq C_2 H^{-1}(\lambda)\phi \ \text{ in } \Omega$$

for λ large. This completes the proof of Lemma 3.3. □

https://doi.org/10.1017/S001708951200064X Published online by Cambridge University Press

4. Proofs of main results. We are now ready to give the proof of main results.

Proof of Theorem 2.1. By Theorem 2.1 in **[3]**, (1.1) has a positive solution $u \in C^{1,\alpha}(\bar{\Omega})$ for λ large when f is sublinear at ∞ and there exist constants $a > 0$, $\beta \in (0, 1)$ such that $\limsup_{u\to 0^+} u^\beta |f(u)| < \infty$ and $f(u) \geq \frac{a}{u^\beta}$ for u large. Thus, we only need to establish the uniqueness part. Let u_1, u_2 be positive solutions of (1.1). Since (A1)–(A4) hold, it follows from Lemma 3.3 that for λ large enough, $\alpha_0 u_2 \leq u_1 \leq \alpha_0^{-1} u_2$ in Ω, where $\alpha_0 = C_1/C_2$. Let α be the largest number such that $\alpha u_2 \leq u_1 \leq \alpha^{-1} u_2$ in Ω and suppose $\alpha < 1$. Then

$$|u_1 - u_2| \leq \frac{1-\alpha}{\alpha} u_2 \leq \frac{1-\alpha}{\alpha_0} u_2 \quad \text{in } \Omega. \tag{4.1}$$

Suppose (A1)–(A5) hold and f is nondecreasing for $u > A$. Then it follows from Lemma 3.2 that

$$f(u_1) - \alpha f(u_2) \geq f(\alpha u_2) - \alpha f(u_2) \geq K(1-\alpha) \tag{4.2}$$

for $u_2 > \max\{A/\alpha_0, C_{\alpha_0}\} \equiv B$.

By (A4), there exists a constant $K_1 > 0$ such that

$$z|f'(z)| \leq \frac{K_1 \alpha_0^{2+\gamma}}{z^\gamma} \quad \text{for } z \in (0, B/\alpha_0], \tag{4.3}$$

and

$$f(z) \leq \frac{K_1}{z^\gamma} \quad \text{for } z \in (0, B]. \tag{4.4}$$

Using (4.1), (4.3) and the Mean Value Theorem, we obtain for $u_2 \leq B$,

$$\begin{aligned} |f(u_1) - f(u_2)| &= |u_1 - u_2||f'(\zeta)| \leq \frac{1-\alpha}{\alpha_0} u_2 |f'(\zeta)| \\ &= \frac{1-\alpha}{\alpha_0} \zeta |f'(\zeta)| \left(\frac{u_2}{\zeta}\right) \leq \frac{K_1(1-\alpha)}{u_2^\gamma}, \end{aligned} \tag{4.5}$$

for some ζ between u_1 and u_2. Here we have used the fact that $\zeta \leq u_2/\alpha_0 \leq B/\alpha_0$ and $\zeta \geq \alpha_0 u_2$. This, together with (4.4), implies

$$|f(u_1) - \alpha f(u_2)| \leq |f(u_1) - f(u_2)| + (1-\alpha) f(u_2) \leq \frac{2(1-\alpha)K_1}{u_2^\gamma} \tag{4.6}$$

for $u_2 \leq B$. Since

$$K \leq \frac{K_2}{u_2^\gamma} \quad \text{for } u_2 \leq B,$$

where $K_2 = KB^\gamma$, we deduce from (4.2), (4.6) and Lemma 3.3 that

$$\begin{aligned} -\Delta(u_1 - \alpha u_2) &= \lambda(f(u_1) - \alpha f(u_2)) \geq \lambda\left(K - \frac{K_3}{u_2^\gamma}\right)(1-\alpha) \\ &\geq \lambda\left(K - \frac{K_3}{(C_1 H^{-1}(\lambda)\phi)^\gamma}\right)(1-\alpha) \ \text{ in } \Omega, \end{aligned}$$

https://doi.org/10.1017/S001708951200064X Published online by Cambridge University Press

where $K_3 = 2K_1 + K_2$. By the comparison principle,

$$u_1 - \alpha u_2 \geq \lambda(1-\alpha)\left(K\phi - \frac{K_3}{(C_1 H^{-1}(\lambda))^\gamma}\psi\right)$$
$$\geq \lambda(1-\alpha)\left(K - \frac{K_3 k_2}{(C_1 H^{-1}(\lambda))^\gamma}\right)\phi \geq \frac{\lambda(1-\alpha)K}{2}\phi \text{ in } \Omega$$

if λ is large enough, where ψ is defined in (3.4). Consequently, there exists a number $\tilde{\alpha} > \alpha$ such that $\tilde{\alpha} u_2 \leq u_1 \leq \tilde{\alpha}^{-1} u_2$ in Ω, a contradiction with the maximality of α. Thus, $\alpha = 1$, and $u_1 = u_2$ in Ω.

Suppose next that (B1), (A3) and (A4) hold. In view of Remark 2.4, we see that (A1)–(A4) hold and therefore Lemma 3.3 applies. We need only to show the existence of a positive constant $K > 0$ such that

$$f(u_1) - \alpha f(u_2) \geq K(1-\alpha) \text{ for } u_2 \text{ large.} \tag{4.7}$$

The rest of the proof then follows the same way as above. Let $\alpha_1 \in (0, 1)$ be such that

$$\limsup_{\zeta\to\infty} \frac{\zeta|f'(\zeta)|}{f(\zeta)} < \alpha_1^{2+\beta}. \tag{4.8}$$

By the Mean Value Theorem,

$$|f(u_1) - f(u_2)| = |(u_1 - u_2)f'(\zeta)| \leq \frac{1-\alpha}{\alpha} u_2 |f'(\zeta)|,$$

where ζ is between u_1 and u_2. Note that $\alpha\zeta \leq u_2 \leq (1/\alpha)\zeta$. Suppose $\alpha > \alpha_1$. Then

$$f(u_1) - \alpha f(u_2) = f(u_1) - f(u_2) + (1-\alpha)f(u_2) \geq (1-\alpha)[f(u_2) - \alpha^{-1}u_2|f'(\zeta)|]$$
$$\geq (1-\alpha)f(\zeta)\left[\frac{f(u_2)}{f(\zeta)} - \frac{\zeta|f'(\zeta)|}{\alpha_1^2 f(\zeta)}\right]. \tag{4.9}$$

Since

$$\frac{f(u_2)}{f(\zeta)} = \frac{f(u_2)}{u_2^\beta}\left(\frac{u_2}{\zeta}\right)^\beta\left(\frac{\zeta^\beta}{f(\zeta)}\right) \geq \alpha_1^\beta\left(\frac{f(u_2)}{u_2^\beta}\right)\left(\frac{\zeta^\beta}{f(\zeta)}\right)$$

for $\alpha > \alpha_1$, it follows from (4.8) that

$$\liminf_{u_2\to\infty}\left(\frac{f(u_2)}{f(\zeta)} - \frac{\zeta|f'(\zeta)|}{\alpha_1^2 f(\zeta)}\right) \geq \alpha_1^\beta - \alpha_1^{-2}\limsup_{\zeta\to\infty}\frac{\zeta|f'(\zeta)|}{f(\zeta)} > 0,$$

which, together with (4.9), implies (4.7) when $\alpha > \alpha_1$.

Next, suppose $\alpha \leq \alpha_1$. Then we have

$$f(u_1) - \alpha f(u_2) = u_1^\beta\left(\frac{f(u_1)}{u_1^\beta} - \alpha\frac{f(u_2)}{u_2^\beta}\left(\frac{u_2}{u_1}\right)^\beta\right)$$
$$\geq u_1^\beta\left(\frac{f(u_1)}{u_1^\beta} - \frac{\alpha_1^{1-\beta}f(u_2)}{u_2^\beta}\right).$$

https://doi.org/10.1017/S001708951200064X Published online by Cambridge University Press

Since $u_1^{-\beta}f(u_1)-\alpha_1^{1-\beta}u_2^{-\beta}f(u_2)\to C(1-\alpha_1^{1-\beta})>0$ as $u_2\to\infty$, (4.7) follows. This completes the proof of Theorem 2.1.

Proof of Theorem 2.2. Let $v_\lambda=(C\lambda)^{\frac{1}{1-\beta}}w_\beta$ and note that v_λ is the solution of

$$\begin{cases}-\Delta v_\lambda=\lambda C v_\lambda^\beta & \text{in } \Omega,\\ v_\lambda=0 & \text{on } \partial\Omega.\end{cases}$$

Note that Lemma 3.3 holds for u_λ and v_λ when λ is large enough, which we shall assume. As in the proof of Theorem 2.1, let $\alpha_0=C_1/C_2$ and α be the largest number such that $\alpha v_\lambda\le u_\lambda\le\alpha^{-1}v_\lambda$ in Ω. Let $0<\varepsilon<1-\alpha_0$ and suppose that $\alpha\le 1-\varepsilon$. Let $0<\varepsilon_0<1-(1-\varepsilon)^{1-\beta}$ and choose $A_0>0$ so that

$$\frac{Cz^\beta}{1-\varepsilon_0}\ge f(z)\ge(1-\varepsilon_0)Cz^\beta \quad \text{for } z>A_0.$$

Hence, for $v_\lambda>A_0/\alpha_0\equiv A_1$,

$$f(u_\lambda)-\alpha C v_\lambda^\beta\ge\left[(1-\varepsilon_0)Cu_\lambda^\beta-\alpha Cv_\lambda^\beta\right]\ge\left[(1-\varepsilon_0)C(\alpha v_\lambda)^\beta-\alpha Cv_\lambda^\beta\right]$$
$$=C\alpha^\beta[1-\varepsilon_0-\alpha^{1-\beta}]v_\lambda^\beta\ge\delta_0, \tag{4.10}$$

where $\delta_0=C\alpha_0^\beta\left[1-\varepsilon-(1-\varepsilon)^{1-\beta}\right]A_1^\beta>0$, and

$$f(u_\lambda)-\alpha^{-1}Cv_\lambda^\beta\le\frac{Cu_\lambda^\beta}{1-\varepsilon_0}-\frac{Cv_\lambda^\beta}{\alpha}\le\frac{Cv_\lambda^\beta}{(1-\varepsilon_0)\alpha^\beta}-\frac{Cv_\lambda^\beta}{\alpha}$$
$$=\frac{C}{\alpha^\beta}\left(\frac{1}{1-\varepsilon_0}-\frac{1}{\alpha^{1-\beta}}\right)v_\lambda^\beta\le\frac{C}{\alpha^\beta}\left(\frac{1}{1-\varepsilon_0}-\frac{1}{(1-\varepsilon)^{1-\beta}}\right)v_\lambda^\beta\le\delta_1, \tag{4.11}$$

where $\delta_1=C((1-\varepsilon_0)^{-1}-(1-\varepsilon)^{\beta-1})A_1^\beta<0$.

On the other hand, for $v_\lambda\le A_1$, there exists a constant $\tilde K>0$ such that

$$\left|f(u_\lambda)-\alpha Cv_\lambda^\beta\right|\le\frac{\tilde K}{v_\lambda^\gamma}. \tag{4.12}$$

Combining (4.7)–(4.9), we obtain

$$f(u_\lambda)-\alpha Cv_\lambda^\beta\ge\delta_0-\frac{\tilde K+\delta_0A_1^\gamma}{v_\lambda^\gamma}\quad\text{in } \Omega,$$

and

$$f(u_\lambda)-\alpha^{-1}Cv_\lambda^\beta\le\delta_1+\frac{\tilde K-\delta_1A_1^\gamma}{v_\lambda^\gamma}\quad\text{in } \Omega.$$

Hence, by Lemma 3.3,

$$-\Delta(u_\lambda-\alpha v_\lambda)=\lambda\big(f(u_\lambda)-\alpha Cv_\lambda^\beta\big)\ge\lambda\left(\delta_0-\frac{\tilde K+\delta_0A_1^\gamma}{(C_1H^{-1}(\lambda)\phi)^\gamma}\right)\quad\text{in } \Omega,$$

https://doi.org/10.1017/S001708951200064X Published online by Cambridge University Press

and

$$-\Delta(u_\lambda - \alpha^{-1}v_\lambda) = \lambda\big(f(u_\lambda) - \alpha^{-1}Cv_\lambda^\beta\big) \le \lambda\left(\delta_1 + \frac{\tilde{K} - \delta_1 A_1^\gamma}{v_\lambda^\gamma}\right) \quad \text{in } \Omega.$$

Hence, by the comparison principle,

$$u_\lambda - \alpha v_\lambda \ge \lambda\left(\delta_0\phi - \frac{\tilde{K} + \delta_0 A_1^\gamma}{(C_1 H^{-1}(\lambda))^\gamma}\psi\right) \ge \lambda(\delta_0/2)\phi \quad \text{in } \Omega,$$

and

$$u_\lambda - \alpha^{-1}v_\lambda \le \lambda\left(\delta_1\phi + \frac{\tilde{K} - \delta_1 A_1^\gamma}{(C_1 H^{-1}(\lambda))^\gamma}\psi\right) \le \lambda\,(\delta_1/2)\,\phi \quad \text{in } \Omega$$

if λ is large enough, where ψ is defined in (3.4), which is a contradiction. Therefore, $\alpha > 1 - \varepsilon$ for λ large, i.e. $(1-\varepsilon)v_\lambda \le u_\lambda \le (1-\varepsilon)^{-1}v_\lambda$ in Ω for λ large. Consequently,

$$-\varepsilon C^{\frac{1}{1-\beta}} w_\beta \le \lambda^{\frac{1}{\beta-1}} u_\lambda - C^{\frac{1}{1-\beta}} w_\beta \le C^{\frac{1}{1-\beta}} w_\beta \varepsilon(1-\varepsilon)^{-1} \quad \text{in } \Omega$$

for λ large. In particular, $\lambda^{\frac{1}{\beta-1}} u_\lambda \to C^{\frac{1}{1-\beta}} w_\beta$ in $C(\bar{\Omega})$ as $\lambda \to \infty$. To show the $C^1(\bar{\Omega})$ convergence, let $\tilde{u}_\lambda = \lambda^{\frac{1}{\beta-1}} u_\lambda - C^{\frac{1}{1-\beta}} w_\beta$. Then

$$-\Delta \tilde{u}_\lambda = \lambda^{\frac{\beta}{\beta-1}} f(u_\lambda) - C^{\frac{1}{1-\beta}} w_\beta^\beta \;\equiv h_\lambda \quad \text{in } \Omega.$$

By writing

$$h_\lambda = (\lambda^{\frac{1}{\beta-1}} u_\lambda)^\beta \frac{f(u_\lambda)}{u_{\lambda^\beta}} - C^{\frac{1}{1-\beta}} w_\beta^\beta,$$

we see that there exist constants $A_2, K_0 > 0$ such that $|h_\lambda| < K_0$ for λ large and $u_\lambda > 1$.

On the other hand, it follows from (A3) and Lemma 3.3 that there exists a constant $K_1 > 0$ such that

$$|h_\lambda| \le \frac{1}{\phi^\gamma} + K_1$$

for λ large and $u_\lambda \le 1$. Thus, there exists a constant $K_2 > 0$ such that $|h_\lambda| \le \frac{K_2}{\phi^\gamma}$ in Ω for λ large. By [**3**, Lemma 3.1], there exist constants $\nu \in (0,1)$ and $\bar{K} > 0$ such that $\tilde{u}_\lambda \in C^{1,\nu}(\bar{\Omega})$ and $|\tilde{u}_\lambda|_{1,\nu} < \bar{K}$. Since $C^{1,\nu}(\bar{\Omega})$ is compactly imbedded in $C^1(\bar{\Omega})$ and $\tilde{u}_\lambda \to 0$ in $C(\bar{\Omega})$ as $\lambda \to \infty$, it follows that $\tilde{u}_\lambda \to 0$ in $C^1(\bar{\Omega})$ as $\lambda \to \infty$. This completes the proof of Theorem 2.2.

REFERENCES

1. M. G. Crandall, P. H. Rabinowitz and L. Tarta, On a Dirichlet problem with a singular nonlinearitiy, *Comm. Partial Differ. Equ.* **2** (1977), 193–222.

2. E. N. Dancer, On the number of positive solutions of semilinear elliptic systems, *Proc. Lond. Math. Soc.* **53** (1986), 429–452.

https://doi.org/10.1017/S001708951200064X Published online by Cambridge University Press

3. D. D. Hai, On a class of singular p-Laplacian boundary value problems, *J. Math. Anal. Appl.* **383** (2011), 619–626.

4. D. D. Hai and R. C. Smith, On uniqueness for a class of nonlinear boundary value problems, *Proc. Roy. Soc. Edinburgh A* **136A** (2006), 779–784.

5. S. S. Lin, On the number of positive solutions for nonlinear elliptic equations when a parameter is large, *Nonlinear Anal.* **16** (1991), 283–297.

6. S. Sakaguchi, Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems, *Ann. Scuola Norm. Sup. Pisa Cl. Sci.* **14** (1987), 403–421.

7. V. Schuchman, About uniqueness for nonlinear boundary value problems, *Math. Ann*. **267** (1984), 537–542.

8. W. Walter, A theorem on elliptic differential inequalities with an application to gradient bounds, *Math. Z.* **200** (1989), 293–299.

9. M. Wiegner, A uniqueness theorem for some nonlinear boundary value problems with a large parameter, *Math. Ann*. **270** (1985), 401–402.