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# BOUNDARY INTERPOLATION FOR CONTINUOUS HOLOMORPHIC FUNCTIONS

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Let $B_n$ denote the unit ball in $C^n$ with boundary $S$. We will be concerned with spaces of holomorphic functions on $B_n$ and will use much of the notation and terminology found in W. Rudin's book [**16**]. Thus, if $f$ is holomorphic in $B_n$ and has homogeneous polynomial expansion

$$f(z) = \sum_{k=0}^{\infty} f_k(z)$$

the *radial derivative* of $f$ is given by

$$Rf(z) = \sum_{k=0}^{\infty} kf_k(z) = \sum_{j=1}^{n} z_j \frac{\partial f}{\partial z_j}(z).$$

Following and Beatrous and Burbea [**6**], for $\beta \in \Re$, we define

$$D^{\beta} f(z) = \sum_{k=0}^{\infty} (1+k)^{\beta} f_k(z).$$

If $\beta > 0$ then $D^{\beta} f$ may be interpreted as a fractional derivative of $f$, and, of course, $D = 1 + R$.

Let $dV$ denote the volume Lebesgue measure on $C^n$ and, following [**6**], for $p, q > 0$ let $A^p_q = A^p_q(B_n)$ be the space

$$A^p_q = \{f : f \text{ holomorphic on } B_n \text{ and } \|f\|_{p,q} < \infty\}$$

where

$$\|f\|_{p,q} = \left[\int |f|^p dV_q\right]^{1/p},$$

with

$$dV_q(z) = \frac{\Gamma(n+q)}{\Gamma(q)} (1 - |z|^2)^{q-1} dV(z).$$

As $q \longrightarrow 0^+$, the probability measures $dV_q$ converge in the weak* sense to the normalized surface measure $d\sigma$ on $S$. Therefore let $A^p_0 = H^p$, where $H^p$ is the

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usual Hardy class of functions holomorphic on $B_n$; see [16]. For $\beta \in \Re$ and $q \geqq 0$ define $A^p_{q,\beta} = D^{-\beta}A^p_q$, i.e.,

$$A^p_{q,\beta} = \{f : D^\beta f \in A^p_q\}$$

and set

$$\|f\|_{p,q,\beta} = \|D^\beta f\|_{p,q}.$$

For $\beta > 0$ and $q = 0$ we will also use the notation $H^p_\beta = A^p_{0,\beta}$; $H^p_\beta$ can be thought of as a Sobolev space of holomorphic functions [6].

Our main results concern the Sobolev spaces $H^p_\beta$ and the Besov spaces $B^p_\beta = A^p_{p,1+\beta}$. It will be important to know that

$$B^P_\beta = A^p_{p(\alpha-\beta),\alpha} \quad \text{for } \alpha > \beta, \tag{1}$$

with equivalent norms. For $n = 1$ this may be found in [9]. For $n > 1$ the result follows by slice integration; see [6].

If one allows $\alpha = \beta$ then (1) must be replaced by the well known continuous inclusions

$$B^p_\beta \subseteq H^p_\beta, \quad 0 < p \leqq 2 \tag{2}$$

and

$$H^p_\beta \subseteq B^p_\beta, \quad 2 \leqq p \leqq \infty. \tag{3}$$

In particular, $H^2_\beta = B^2_\beta$ with equivalent norms; see [6] and [18].

To motivate the problem studied here, consider the case where $n = 1$ so $B_1 = U$, the unit disk in the complex plane. Let $C(K)$ be the Banach space of functions continuous on a compact Hausdorff space $K$ equipped with the supremum norm $\|\cdot\|_K$. If $\mathcal{A}$ is a Banach space of holomorphic functions on $U$ one may consider the space

$$\mathcal{B} = \mathcal{A} \cap A(\bar{U}) = \mathcal{A} \cap C(T)$$

where $A(\bar{U})$ is the classical disk algebra of holomorphic functions continuous on the closed disk $\bar{U}$ whose boundary is the circle $T$.

We can make $\mathcal{B}$ a Banach space by using either one of the following norms:

$$\|f\|_{\mathcal{B}} = \max(\|f\|_{\mathcal{A}}, \|f\|_T) \quad \text{(the "max norm")}$$

or

$$\|f\|_{\mathcal{B}} = \|f\|_{\mathcal{A}} + \|f\|_T \quad \text{(the "sum norm").}$$

If $K$ is a compact subset of $T$ let $R_K$ denote the restriction operator $R_K\colon \mathcal{B} \longrightarrow C(K)$ which sends a function $f$ to its restriction on $K$.

*Definition* 1. $K$ is called a *boundary interpolation set* (B.I. set) for $\mathcal{B}$ if $R_K$ maps $\mathcal{B}$ onto $C(K)$.

*Definition* 2. $K$ is called a *strong boundary interpolation set* (S.B.I. set) for $\mathcal{B}$ if $R_K$ maps the unit ball of $\mathcal{B}$ onto the unit ball of $C(K)$.

The model for boundary interpolation theorems is the following result proved independently by Rudin and Carleson which considers the case where $\mathcal{A} = A(\bar{U})$; see [**16**], Chapter 10.

THEOREM A. *Let* $\mathcal{A} = A(\bar{U})$ *and give* $\mathcal{B}$ *the max norm. Then the following conditions on a compact subset* $K \subseteq T$ *are equivalent:*
(i) $|K| = 0$.
(ii) $K$ *is a* B.I. *set for* $\mathcal{B}$.
(iii) $K$ *is a* S.B.I. *set for* $\mathcal{B}$.

Here, $|K|$ denotes the one dimensional Lebesgue measure of $K$.

The next theorem we state on boundary interpolation motivates our own results. It should be considered in the following context.

If $f$ is holomorphic on $D$, let us say that the *exceptional set* of $f$ is the set

$$E(f) = \{\zeta : \zeta \in T \text{ and } f \text{ does not have a finite non-tangential limit at } \zeta\}.$$

If $1 < p < \infty$ and $1 - \beta p \geqq 0$, it is well known that the compact sets $K$ arising as exceptional sets for $H^p_\beta$ and $B^p_\beta$ functions are those sets for which $C^1_{p,\beta}(K) = 0$. Here, $C^1_{p,\beta}$ denotes a particular (one dimensional) Bessel capacity defined by means of potentials on $T$. See [**5**] and [**17**] for a discussion of this fact as well as [**2**] for a discussion of the equivalence of the zero sets of corresponding Bessel and Besov capacities.

THEOREM B. (Peller and Khruschev [**15**] and Sjödin [**17**]) *Let* $\mathcal{B} = B^p_{1/p} \cap C(T)$ *have the sum norm. If $K$ is a compact subset of $T$ then the following conditions are equivalent:*
(i) $C^1_{p,1/p}(K) = 0$.
(ii) $K$ *is a* S.B. I. *set for* $\mathcal{B}$.
(iii) $K$ *is a* B.I. *set for* $\mathcal{B}$.

Actually, in [**15**], Peller and Khruschev showed that (i) implies (ii). The implication (iii) implies (i) had been established earlier by Sjödin and so Theorem B follows.

We turn now to our main results. The reader should have no trouble in extending Definitions 1 and 2 to the context of $C^n$.

First, let $\zeta \in S$ and $\delta > 0$. Define the Koranyi ball

$$B(\zeta, \delta) = \{\eta : \eta \in S \text{ and } |1 - \langle \eta, \zeta \rangle| < \delta\}.$$

For $0 < m \leqq n$ and $K$ a compact subset of $S$ let

$$H_m(K) = \inf \sum \delta_k^m \tag{4}$$

where the infimum is taken over all covers $\{B(\zeta_k, \delta_k)\}$ of $K$. If $n > 1$, since the Koranyi ball is "non-isotropic", $H_m$ is called (non-isotropic) *Hausdorff capacity of dimension m*. See [**1**] for a discussion of (isotropic) Hausdorff capacity.

In [**4**] and [**8**] the compact subsets $K$ of $S$ arising as exceptional sets for $H^p_\beta(B_n)$ (and actually $B^p_\beta(B_n)$) functions were characterized as the ones for which $H_m(K) = 0$ where $m = n - \beta p$, provided $0 < p \leqq 1$ and $n - \beta p > 0$. These results were new even for $n = 1$. See [**4**] and [**8**] for the precise definition of "exceptional set", if $n > 1$.

We can now state our main result.

THEOREM 1. *Let* $0 < n - \beta < 1$ *and give* $B^1_\beta(B_n) \cap C(S)$ *and* $H^1_\beta(B_n) \cap C(S)$ *the sum norms. If K is a compact subset of S then the following conditions are equivalent:*

(i) $H_{n-\beta}(K) = 0$.
(ii) $K$ *is a* S.B.I. *set for* $B^1_\beta \cap C(S)$.
(iii) $K$ *is a* S.B.I. *set for* $H^1_\beta \cap C(S)$.

If $n > 1$ then easy examples show that Theorem 1 is false if $n - \beta > 1$.

It should be pointed out that we do not know whether the notions of S.B.I. set and B.I. set coincide for the spaces considered in Theorem 1, even in the case $n = 1$. However, we do have the following result which points in that direction.

THEOREM 2. *Let* $0 < n - \beta < 1$. *Suppose* $K \subseteq S$ *is a* B.I. *set for* $H^1_\beta \cap C(S)$. *Then* $H_\omega(K) = 0$ *for any increasing concave downward defining function* $\omega$ *on* $[0, \infty)$ *satisfying the condition* $\omega(t) = o(t^{n-\beta})$. *In particular,* $H_{n-\beta+\epsilon}(K) = 0$ *for all* $\epsilon > 0$.

Here $H_\omega(K)$ is defined by replacing $\delta_k^m$ by $\omega(\delta_k)$ in (4).

We are also able to provide new information concerning boundary interpolation for the case $n = 1$, $1 < p < \infty$, and $1 - \beta p \geqq 0$. Sjödin, [**17**], has shown that a necessary condition for $K$ to be a B.I. set for $B^p_\beta \cap C(T)$ or $H^p_\beta \cap C(T)$ is that $C^1_{p,\beta}(K) = 0$. (See also [**19**].) As has already been remarked, Peller and Khruschev established that the condition $C^1_{p,1/p}(K) = 0$ was sufficient for $K$ to be a S.B.I. set for $B^p_{1/p} \cap C(T)$. Koosis, [**12**], simplified Peller's and Khruschev's proof for the special case $B^2_{1/2}$ and left open the question of whether or not $C^1_{2,\beta}(K) = 0$ implies that $K$ is a B.I. set for $B^2_\beta \cap C(T)$ when $0 < \beta < 1/2$. We answer this in the affirmative. More generally, we have the following fact.

THEOREM 3. *Let* $1 - \beta p \geqq 0, 1 < p < \infty$ *and suppose* $\mathcal{B} = B^p_\beta \cap C(T)$ *is given the sum norm. If K is a compact subset of T and* $C^1_{p,\beta}(K) = 0$ *then K is a* S.B.I. *set for* $\mathcal{B}$.

When combined with Sjödin's results, Theorem 3 shows that the notions of S.B.I. and B.I. coincide for $B^p_\beta \cap C(T)$, $1 < p < \infty$ and $1 - \beta p \geqq 0$.

If $0 < p \leqq 2$ the inclusion $B^p_\beta \subseteq H^p_\beta$ and the proof of Theorem 3 will show that one may replace $B^p_\beta$ by $H^p_\beta$ in Theorem 3. However, we do not know if this is still true when $p > 2$.

For $n > 1$ the problem of characterizing exceptional sets for $H^p_\beta, 1 < p < \infty$, is unresolved; see [**4**] for some interesting examples. While our methods yield some results, it seem premature to consider boundary interpolation in that context.

The proof of Theorems 1 and 2 require the following lemma based on a construction similar to the one in [**8**]. In the sequel, the letter $C$ will denote a constant whose value may differ at each new occurrence.

LEMMA 1. *Let $0 < n - \beta < 1$ and suppose $K$ is a compact subset of $S$ for which $H_{n-\beta}(K) = 0$. Then there exists a sequence $\{f_k\}$ of holomorphic functions on $B_n$ satisfying the following conditions:*

(i) $\operatorname{Re} f_k$ *is positive and continuous in the extended sense on $\bar{B}_n$, the closed ball.*

(ii) $\operatorname{Re} f_k \equiv \infty$ *on $K$.*

(iii) $f_k$ *is continuous on $\bar{B}_n \backslash K$.*

(iv) $\lim_{k\to\infty} f_k(z) = 0$ *for $z \in \bar{B}_n \backslash K$.*

(v) $\exp(-f_k) \in B^1_\beta \cap C(S)$.

(vi) $\lim_{k\to\infty} \|1 - \exp(-f_k)\|_{B^1_\beta} = 0$.

*Proof.* Since $H_{n-\beta}(K) = 0$, for each $j = 1, 2, \ldots$ we may find a cover of $K, \{(\zeta_{l,j}, \delta_{l,j})\}_{l=1}^{N_j}$ such that

$$\sum_{l=1}^{N_j} (\delta_{l,j})^{n-\beta} \leqq 2^{-j}. \tag{5}$$

Choose $\epsilon$ so $0 < \epsilon < 1 - (n - \beta)$ and set $z_{l,j} = (1 - \delta_{l,j})\zeta_{l,j}$, $l = 1, \ldots, N_j$. Define

$$f_k(z) = \sum_{j \geqq k} \sum_{l=1}^{N_j} \frac{(\delta_{l,j})^{1-\epsilon}}{(1 - \langle z, z_{l,j} \rangle)^{1-\epsilon}}.$$

Since for $|z| \leqq 1$ and $|w| < 1$

$$\operatorname{Re} \frac{1}{(1 - \langle z, w \rangle)^{1-\epsilon}} \geqq \frac{\cos\left(\dfrac{\pi(1-\epsilon)}{2}\right)}{|1 - \langle z, w \rangle|^{1-\epsilon}}$$

it is clear that $f_k$ has positive real part on $\bar{B}_n$. If $z$ is any point in $B(\zeta_{l,j}, \delta_{l,j})$ then

$$\frac{1}{|1 - \langle z, z_{l,j} \rangle|} \geqq \frac{1}{2\delta_{l,j}}$$

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and since every point $z \in K$ lies in infinitely many balls $B(\zeta_{l,j}, \delta_{l,j})$ it follows easily that (i) and (ii) hold. If $z \in \bar{B}_n \backslash K$ then each term $(1 - \langle z, z_{l,j} \rangle)^{\epsilon - 1}$ is continuous and uniformly bounded on a neighborhood of $z$, for $j$ large. Therefore (iii) follows from (5) and the Weierstrass $M$-test since $n - \beta < 1 - \epsilon$. Condition (iv) is verified in the same fashion.

To verify that (v) and (vi) hold we need two additional lemmas.

LEMMA 2. *Let $g$ be a non-negative function on a positive measure space $(\Omega, \mu)$. For $k_1, \ldots, k_l > 0$ and $m_1, \ldots, m_l \geqq 1$, define*

$$m = \sum_{j=1}^{l} k_j m_j \quad \text{and} \quad L = \sum_{j=1}^{l} m_j - 1.$$

*Then*

$$\prod_{j=1}^{l} \left[ \int_\Omega g^{k_j+1} d\mu \right]^{m_j} \leqq \left[ \int_\Omega g^{m+1} d\mu \right] \left[ \int_\Omega g d\mu \right]^{L}.$$

*Proof.* Observe that $m_j k_j \leqq m$ and so $m/k_j \geqq m_j \geqq 1$. It follows from Hölder's inequality that

$$\int_\Omega g^{k_j+1} d\mu \leqq \left[ \int_\Omega g^{m+1} d\mu \right]^{k_j/m} \left[ \int_\Omega g d\mu \right]^{1-k_j/m},$$

and thus

$$\prod_{j=1}^{l} \left[ \int_\Omega g^{k_j+1} d\mu \right]^{m_j} \leqq \left[ \int_\Omega g^{m+1} d\mu \right]^{\frac{1}{m} \sum k_j m_j} \left[ \int_\Omega g d\mu \right]^{\sum m_j - \frac{1}{m} \sum k_j m_j},$$

which is the desired result.

LEMMA 3. *Let $0 < \epsilon < 1 - (n - \beta) < 1$. Suppose $|z_j| < 1, j = 1, \ldots N$ and set*

$$F(z) = \sum_{j=1}^{N} \frac{(1 - |z_j|)^{1-\epsilon}}{(1 - \langle z, z_j \rangle)^{1-\epsilon}}.$$

*Then there exists a constant $C$ depending only on $\beta$ and $\epsilon$ such*

$$\|1 - e^{-F}\|_{B^1_\beta} \leqq C \left[ \|1 - e^{-F}\|_{1,n-\beta} + \sum (1 - |z_j|)^{n-\beta} \right].$$

*Proof.* Since $B^1_\beta = A^1_{1,1+\beta} = A^1_{n-\beta,n}$ with equivalent norms, it follows that

$$\|1 - e^{-F}\|_{B^1_\beta} \leqq C \|D^n(1 - e^{-F})\|_{1,n-\beta} = C \|1 - D^n e^{-F}\|_{1,n-\beta}.$$

If $h$ is holomorphic on $B_n$, we may calculate $D^n h(z)$ by letting $h_z(\lambda)$ be the slice function

$$h_z(\lambda) = h(\lambda z), \quad \lambda \in U$$

and then

$$D^n h(z) = \left(\left(\frac{d}{d\lambda}\cdot\lambda\right)^n h_z\right)(1)$$

where $\left(\frac{d}{d\lambda}\cdot\lambda\right)^n$ is the operator $\frac{d}{d\lambda}\cdot\lambda$ iterated $n$ times. Using $e^{-F}$ in place of $h$ yields the equality

$$(D^n e^{-F})(z) = e^{-F(z)} + \sum_{k=1}^{n} c(k,n)\frac{d^k}{d\lambda^k}e^{-F_z}(1)$$

where $c(k,n)$ are positive integers depending only on $k$ and $n$. Therefore

$$|D^n(1-e^{-F})(z)| \leqq C\left[|1-e^{-F(z)}| + \sum_{k=1}^{n}\left|\left(\frac{d^k}{d\lambda^k}e^{-F_z}\right)(1)\right|\right].$$

We show how to obtain the estimate

$$\left\|\left(\frac{d^n}{d\lambda^n}e^{-F_z(\lambda)}\right)(1)\right\|_{1,n-\beta} \leqq C\sum(1-|z_j|)^{n-\beta}; \tag{6}$$

the lower order terms in the sum can be handled in a similar way. For each $|z|<1$, $\frac{d^n}{d\lambda^n}e^{-F_z(\lambda)}$ is dominated by a constant times a sum of terms of the form

$$|e^{-F(\lambda z)}|\prod_{j=1}^{l}\left|\frac{d^{k_j}}{d\lambda^{k_j}}F_z(\lambda)\right|^{m_j} \tag{7}$$

where

$$\sum_{j=1}^{l} k_j m_j = n, \quad m_j > 0 \quad \text{and} \quad 0 < k_j \leqq n.$$

Estimate that

$$\left|\frac{d^k}{d\lambda^k}F_z(\lambda)\right| \leqq c\sum\frac{(1-|z_j|)^{1-\epsilon}}{|1-\lambda\langle z,z_j\rangle|^{1-\epsilon+k}}$$
$$= C\int_{B_n} g^{k+1}(\zeta)d\mu(\zeta)$$

where

$$g(\zeta)=\frac{1}{|1-\lambda\langle z,\zeta\rangle|}$$

and $\mu$ is the sum of point-masses

$$d\mu(\zeta)=\sum\frac{(1-|z_j|)^{1-\epsilon}}{|1-\lambda\langle z,z_j\rangle|^{-\epsilon}}\delta_{z_j}(\zeta).$$

If no $k_j=n$, we may apply Lemma 2 and get that (7) is dominated by

$$(8)\qquad |e^{-F(\lambda z)}|\left[\int_{B_n}g(\zeta)^{n+1}d\mu\right]\left[\int_{B_n}g(\zeta)d\mu\right]^L.$$

Since

$$\begin{aligned}\operatorname{Re}F(\lambda z)&\geqq C\sum\frac{(1-|z_j|)^{1-\epsilon}}{|1-\lambda\langle z,z_j\rangle|^{1-\epsilon}}\\&=C\int_{B_n}g(\zeta)d\mu\end{aligned}$$

it follows that (8) is less than a constant times

$$(9)\qquad \int_{B_n}g(\zeta)^{n+1}d\mu=\sum\frac{(1-|z_j|)^{1-\epsilon}}{|1-\lambda\langle z,z_j\rangle|^{1-\epsilon+n}}.$$

For the single term where $l=1$ and $k_1=n$ the fact that (7) is less than a multiple of (9) is immediate. To obtain (6) use the integral estimate ([**16**], 17–18)

$$\int_{B_n}\frac{(1-|z|)^{n-\beta-1}}{|1-\langle z,z_j\rangle|^{n+1-\epsilon}}dV\leqq\frac{C}{(1-|z_j|)^{(1-\epsilon)-(n-\beta)}}$$

and the triangle inequality.

The same reasoning will give a stronger pointwise estimate on the lower order terms and the proof is therefore complete.

We can now complete the proof of Lemma 1. Property (v) follows as a consequence of (5) and Lemma 3 since $1-|z_{l,j}|=\delta_{l,j}$. Property (vi) follows similarly in conjunction with Lemma 3, (5), properties (i) and (iv), and the dominated convergence theorem.

To prove Theorem 3 we will need an analogue of Lemma 1 for the spaces $B^p_\beta(B_1)$.

LEMMA 4. *Let $1<p<\infty$, $1-\beta p\geqq 0$ and suppose $K$ is a compact subset of $T$ for which $C^1_{p,\beta}(K)=0$. Then there exists a sequence of holomorphic functions*

*$\{f_k\}$ on $U = B_1$ such that conditions* (i)–(vi) *of Lemma* 1 *hold, with $U$ in place of $B_n, T$ in place of $S$, and $B^p_\beta = B^p_\beta(U)$ in place of $B^1_\beta$.*

*Proof.* For $p = 2$ and $\beta = 1/2$ the functions $\{f_k\}$ were constructed explicitly in [**7**] and the same procedure works, if $p = 2$, for $0 < \beta < \frac{1}{2}$. The method is essentially the same for the general case of non-linear capacities, but the details are more technical. We will try to sketch the basic idea.

Since $C^1_{p,\beta}(K) = 0$, $K \subseteq \bigcap_{j=1}^\infty K_j$, where $K_j \subseteq K_{j-1}$, and each $K_j$ is a finite union of closed arcs whose one dimensional Besov capacity can be chosen to be arbitrarily small; see [**5**]. If we use the fact that the two dimensional Bessel capacity $C^2_{p,\beta+1/p}$, when restricted to the one dimensional set $T$, is equivalent to the above mentioned Besov capacity, [**13**, p. 114], it follows that we may find an equillibrium potential for $K_j$ of the form

$$h_j(z) = \int_U \frac{g_i(\zeta)}{|z-\zeta|^{2-(\beta+1/p)}}\,dA$$

where $dA$ is area measure on $C^1$ and

$$[g_j(\zeta)]^{p-1} = \int \frac{d\mu_j(\eta)}{|\eta-\zeta|^{2-(\beta+1/p)}}, \tag{10}$$

where $\mu_j$ is supported on $K_j$ and

$$\int_U |g_j|^p dA = \|\mu_j\|^p = C^2_{p,\beta+1/p}(K_j); \tag{11}$$

see [**14**]. Note that we may make $\|\mu_j\|$ arbitrarily small.

Since $h_j$ is the equillibrium potential for $K_j$,

$$h_j(e^{i\theta}) \geqq 1 \quad \text{for all } e^{i\theta} \in K_j$$

except for a set of capacity zero. We now modify $h_j$ to get a holomorphic $B^p_\beta$ function. Since $\beta > 0$ we may choose $0 < \gamma < 1/p'$, where $p' = p/(p-1)$, so $2 - (\beta + 1/p + \gamma) < 1$. For $|z| \leqq 1$ let

$$F_j(z) = \int_D \frac{g_j(\zeta)(1-|\zeta|)^{-\gamma}}{(1-z\bar{\zeta})^{2-(\beta+1/p+\gamma)}}\,dA. \tag{12}$$

Then Re $F_j(z) \geqq 0$. We will show later that $F_j \in B^p_\beta$ and this means that $F_j$ is the Poisson integral of its boundary function. Since

$$\operatorname{Re} F_j(e^{i\theta}) \geqq Ch_j(e^{i\theta})$$

it follows that

$$\operatorname{Re} F_j(e^{i\theta}) \geqq C \quad \text{for } e^{i\theta} \in K_j,$$

where $C > 0$ is independent of $j$. Now set

$$f_k(z) = \sum_{j \geqq k} F_j(z).$$

It follows that (i) and (ii) hold. To establish (iii) and (iv), let $z \in \bar{U} \backslash K_j$ and suppose $N$ is a small closed neighborhood containing $z$ which does not intersect $K_l$ for $l \geqq j$.

Estimate that

$$|F_l(z)| \leqq I_1 + I_2$$

where $I_1$ is the integral over $N$ of the absolute value of the integrand in (12) and $I_2$ is the integral over $U \backslash N$. To estimate $I_1$, use (10) to get the inequality

$$|g_l(\zeta) \leqq C \|\mu_l\|^{p'-1}, \quad \text{if } \zeta \in N,$$

where $C$ depends only on the distance of $N$ to $K_j$. Since

$$\int_U \frac{(1-|\zeta|)^{-\gamma}}{|1-\bar{\zeta}z|^{2-(\beta+1/p+\gamma)}}\, dA \leqq C$$

where $C$ does not depend on $z \in U$ it follows that

$$I_1 \leqq C \|\mu_l\|^{p'-1}.$$

To estimate $I_2$, notice that

$$I_2 \leqq C \int_{U \backslash N} |g_l(\zeta)|(1-|\zeta|)^{-\gamma} dA$$

where, again, $C$ depends only on the distance from $N$ to $K_j$. Thus

$$\begin{aligned} I_2 &\leqq C \left( \int_U |g_l|^p dA \right)^{1/p} \left( \int_U (1-|\zeta|)^{-\gamma p'} dA \right)^{1/p'} \\ &\leqq C \|\mu_l\| \end{aligned}$$

since $\gamma p' < 1$. We now have the estimate

$$|F_l(z)| \leqq C[\|\mu_l\|^{p'-1} + \|\mu_l\|]$$

valid for $z \in \bar{U} \backslash K_j$ and $l \geqq j$. By choosing $C^2_{p,\beta+1/p}(K_j)$ sufficiently small and, to insure continuity, replacing $F_l(z)$ by a dilation $F_l(r_l z)$ where $r_l$ is sufficiently close to 1, we will obtain (i)–(iv).

To verify (v) and (vi) it is necessary to estimate $\|F_j\|_{B^p_\beta}$. We will be very sketchy here. Since $B^p_\beta = A^p_{p,1+\beta} = A^p_{1,1/p+\beta}$ this amounts to estimating the $A^p_1$ norm of

$$H_j(z) = \int_U \frac{g_i(\zeta)(1-|\zeta|)^{-\gamma}}{(1-\bar{\zeta}z)^{2-\gamma}}\, dA.$$

Since $\gamma < 1/p'$ it follows from a theorem of Forelli and Rudin, [**16**] Chapter 7, that

$$\begin{aligned}\|H_j\|^p_{p,1} &\leqq C\int_U |g_i|^p dA \\ &= C\|\mu_j\|^p.\end{aligned}$$

Verifying (v) and (vi) is now routine:

$$\begin{aligned}\|1-e^{-f_k}\|_{B^p_\beta} &\leqq C\|D^1(1-e^{-f_k})\|_{p,p(1-\beta)} \\ &= C\|1-e^{-f_k}+e^{-f_k}f'_k\|_{p,p(1-\beta)} \\ &\leqq C\left[\|1-e^{-f_k}\|_{p,p(1-\beta)} + \|f'_k\|_{p,p(1-\beta)}\right] \\ &\leqq C\left[\|1-e^{-f_k}\|_{p,p(1-\beta)} + \|f_k\|_{B^p_\beta}\right] \\ &\leqq C\left[\|1-e^{-f_k}\|_{p,p(1-\beta)} + \sum_{j\geqq k}\|\mu_j\|\right].\end{aligned}$$

If $\|\mu_j\|$ is chosen sufficiently small then both terms will go to zero as $k \longrightarrow \infty$. This completes the proof.

*Remark.* We have been unable to obtain a version of Lemma 4 for the Sobolev spaces $H^p_\beta(B_1)$ when $p > 2$. There is no problem in using potential theory to find $\{f_k\}$ satisfying (i)–(iv). The difficulty arises when one tries to prove that $\exp(-f_k) \in H^p_\beta$.

We now prove Theorem 1, by showing that (i) implies (ii), (ii) implies (iii), and (iii) implies (i).

The proof that (i) implies (ii) is based on the duality argument used by Peller and Khruschev in [**15**] as outlined by Koosis in [**12**]. Suppose $K$ is compact in $S$ and $H_{n-\beta}(K) = 0$. Let $U_S$ be the unit ball in $B^1_\beta \cap C(S)$ and $U_K$ be the unit ball in $C(K)$. It is enough to show that the closure in $C(K)$ of $R_K(U_S)$ contains $U_K$.

Suppose this is false. Then the Hahn-Banach theorem gives a $\nu \in M(K)$, the set of complex Borel measures supported on $K$, such that $\|\nu\| = 1$ and

$$(13) \qquad \left|\int f d\nu\right| \leqq \lambda < 1$$

for all $f \in U_S$. Reasoning as in [**15**], we identify $B^1_\beta \cap C(S)$ with the diagonal $\Delta$ in the product space $B^1_\beta \times C(S)$. Thus

$$\Delta = \{(f,f) : f \in B^1_\beta \cap C(S)\}$$

and, since $B^1_\beta \cap C(S)$ has the sum norm, the map which sends $f$ to $(f,f)$ is an isometric isomorphism of $B^1_\beta \cap C(S)$ onto $\Delta$. It follows that the dual space of $B^1_\beta \cap C(S)$ is isometrically isomorphic to the quotient

$$(B^1_\beta)^* \times M(S)/\Delta^\perp.$$

Here, $\Delta^\perp$ is the annihilator of $\Delta$ in $(B^1_\beta)^* \times M(S)$, and, if $(\Lambda, \mu) \in (B^1_\beta)^* \times M(S)$ then

$$(\Lambda, \mu)(f, g) = \Lambda(f) + \int g d\mu$$

for $(f, g) \in B^1_\beta \times C(S)$. Inequality (13) asserts that the quotient norm of the coset $[(0, \nu)]$ is less than 1. It follows that there is a pair $(\Lambda, \mu) \in \Delta^\perp$ such that

$$\max(\|\Lambda\|_{(B^1_\beta)^*}, \|\nu - \mu\|) \leqq \lambda. \tag{14}$$

Let $E$ be a compact subset of $K$ and apply Lemma 1 with $E$ in place of $K$. Since $(\Lambda, \mu) \in \Delta^\perp$ and $1 - \exp(-f_k)$ belongs to $B^1_\beta \cap C(S)$ it follows that

$$\Lambda(1 - e^{-f_k}) = -\int (1 - e^{-f_k}) d\mu$$

and therefore

$$|\Lambda(1 - e^{-f_k})| = \left| \int (1 - e^{-f_k}) d\mu \right|$$

which yields the inequality

$$\left| \int (1 - e^{-f_k}) d\mu \right| \leqq \|\Lambda\| \, \|1 - e^{-f_k}\|_{B^1_\beta}. \tag{15}$$

Properties (i)–(iv) of Lemma 1 assert that $1 - e^{-f_k}$ converges boundedly to the characteristic function of $E$, while property (vi) says that, as $k \longrightarrow \infty$, the right hand side of (15) goes to zero. The dominated convergence theorem implies therefore that

$$|\mu(E)| = 0.$$

Since $E$ was an arbitrary compact subset of $K$, it follows that $\mu$ is singular with respect to $\nu \in M(K)$. Thus

$$\|\nu - \mu\| \geqq \|\nu\| = 1$$

which contradicts (14). Thus (i) implies (ii).

*Remark.* The preceding argument together with Lemma 4 provides the proof of Theorem 3.

To see that (ii) implies (iii), let $K$ be a S.B.I. set for $B^1_\beta \cap C(S)$, $F \in C(K)$ and $\|F\|_K = 1$. If $0 < t < 1$ it follows that there exists $f$ in the unit ball of $B^1_\beta \cap C(S)$ such that $R_K f = tF$. Thus $\|f\|_S \geqq t$ and since $\|f\|_S + \|f\|_{B^1_\beta} < 1$ it follows that $\|f\|_{B^1_\beta} < 1 - t$. From inclusion (2) there exists a constant $C$ independent of $t$ such that $\|f\|_{H^1_\beta} < C(1-t)$. Thus

$$\|f\|_S + \|f\|_{H^1_\beta} < 1 + C(1-t)$$

and if

$$g = \frac{1}{1 + C(1-t)} f$$

it follows that $g$ is the unit ball of $H^1_\beta \cap C(S)$ and

$$R_K g = \frac{tF}{1 + C(1-t)}.$$

As $t$ varies from 0 to 1, so does $t/(1 + C(1-t))$ and it follows that $K$ is a S.B.I. for $H^1_\beta \cap C(S)$.

To prove that (iii) implies (i) we need some results which can be found, essentially, in [**4**] and [**8**]. First, for $0 < m \leqq n$ and $\mu$ a positive measure on $S$ define

$$|||\mu|||_m = \sup \mu(B(\zeta, \delta))\delta^{-m}$$

where the supremum is taken over all $\zeta \in S$ and $\delta > 0$.

THEOREM C. (Ahern, [**4**]). *Let $0 < p \leqq 1$ and $m = n - \beta p > 0$. Then there is a constant $C$ such that*

$$\int |f|^p d\mu \leqq C|||\mu|||_m \|f\|^p_{p,0,\beta}$$

*for all $f \in H^p_\beta \cap C(S)$.*

THEOREM D. *Let $K$ be a compact subset of $S$. Then $H_m(K) > 0$ if and only if $K$ supports a positive measure $\mu$ such that $|||\mu|||_m < \infty$.*

For $n = 1$ Theorem D is due to Frostman, [**10**]. A proof for $n > 1$ is contained in [**8**].

Suppose now that (iii) of Theorem 1 holds. By Theorem D it is enough to show that if $\mu$ is a positive measure in $M(K)$ satisfying $|||\mu|||_{n-\beta} < \infty$ then $\mu \equiv 0$. Let $\mu$ be such a measure. Denote by $\phi$ the function in $C(K)$ which is identically 1 on $K$. By (iii) there is a sequence of functions $\{f_k\}$ in the unit ball of $H^1_\beta \cap C(S)$ (with the sum norm) such that

$$R_K f_k = \frac{k}{1+k}\phi.$$

Since

$$\|f_k\|_S \geqq \frac{k}{1+k}$$

it follows that

$$\|f_k\|_{1,0,\beta} < \frac{1}{1+k}.$$

Now $\mu$ is a positive measure with support on $K$, so

$$\begin{aligned} \mu(K) &= \int \phi d\mu \\ &= \frac{1+k}{k}\int f_k d\mu \\ &= \frac{1+k}{k}\int |f_k| d\mu \\ &\leqq C\frac{1+k}{k}|||\mu|||_{n-\beta}\|f_k\|_{1,0,\beta} \\ &\leqq \frac{C|||\mu|||_{n-\beta}}{k}. \end{aligned}$$

Therefore $\mu(K) = 0$. This completes the proof of Theorem 1.

We now prove Theorem 2. Suppose $R_K$ maps $H^1_\beta \cap C(S)$ onto $C(K)$. Standard duality theory says then that the adjoint map

$$R_K^* : M(K) \longmapsto [H^1_\beta \cap C(S)]^*$$

is bounded below. Thus, there exists a constant $C$ such that

$$\|\mu\| \leqq C \sup\left|\int F d\mu\right| \tag{16}$$

whenever $\mu \in M(K)$ and the supremum is taken over all functions $F$ in the unit ball of $H^1_\beta \cap C(S)$. If $\omega$ is increasing and concave downward on $[0, \infty)$

then a simple modification of the proof of Theorem D given in [**8**] shows that $H_\omega(K) > 0$ if and only if there exists a positive measure $\nu$ supported on $K$ such that

$$(17) \qquad \nu(B(\zeta,\delta)) = O(\omega(\delta))$$

for all $\zeta \in S$ and $\delta > 0$. Theorem 2 will therefore be proved if we show that no inequality of the form (16) can hold for all measures $d\mu = \psi d\nu$, where $\nu$ satisfies (17) and $\psi$ is a unimodular function.

To this end, find unimodular functions $\psi_k$ such that $d\mu_k = \psi_k d\nu$ converges weak* to zero in $M(K)$. (See [**12**] for a way to do this when $n = 1$; the general case presents no difficulty because the key point is that, by (17), $\nu$ can have no point-masses since $\omega(t) = O(t^{n-\beta})$.) Now let $F \in H^1_\beta$. It follows that $f = D^\beta F \in H^1$, and thus $F = D^{-\beta}f$ with $f \in H^1$, and, of course, $\|F\|_{1,0,\beta} = \|f\|_{1,0}$. By definition (see also [**4**] and [**6**]), for every $z \in B$,

$$F(z) = D^{-\beta}f(z) = \frac{1}{\Gamma(\beta)}\int_0^1 [\log 1/t]^{\beta-1} f(tz)dt.$$

We make the additional assumptions that $f$, and hence also $F$ are continuous on the closed ball and that the representation above holds for $z \in \partial S$; this will not invalidate our argument. It follows that

$$(18) \qquad \left|\Gamma(\beta)\int_S F d\mu_k\right| = \left|\int_S\int_0^1 \left[\log\frac{1}{t}\right]^{\beta-1} f(tz)dt\psi_k(z)d\nu(z)\right|.$$

We will show that the right hand side of (18) goes to zero as $k \longrightarrow \infty$ at a rate depending only on $\|f\|_{1,0}$ and this will suffice to contradict (16).

Break the right hand side of (18) into two integrals, $I = I_1 + I_2$ where, in $I_1, 0 \leqq t \leqq r_0$ and in $I_2, r_0 \leqq t \leqq 1$, for some $r_0, 0 < r_0 < 1$. Consider $I_1$. For fixed $r_0 < 1$ it is easy to see that, as $f$ ranges over the unit ball in $H^1$, the functions

$$f_{r_0}(z) = \int_0^{r_0} [\log 1/t]^{\beta-1} f(tz)dt$$

lie in a compact subset of $C(S)$. Since $\psi_k d\nu$ converges weak* to zero it is also easy to see that, given $\epsilon > 0$ there is an $N = N(r_0,\epsilon)$ such that

$$|I_1| = \left|\int_S f_{r_0}(z)\psi_k(z)d\nu(z)\right| \leqq \epsilon\|f\|_{1,0}$$

for all $f \in H^1$, provided $k \geqq N$.

We must therefore consider $I_2$. Estimate that

$$\begin{aligned}|I_2| &\leqq C\int_S\int_{r_0}^1 (1-t)^{\beta-1}|f(tz)|dtd\nu(z)\\ &= C\int_{A_{r_0}} |f|d\lambda\end{aligned}$$

where $A_{r_0} = \{tz : z \in S \text{ and } r_0 < t < 1\}$ and $\lambda$ is the measure on $B_n$ defined by

$$d\lambda(t,z) = (1-t)^{\beta-1} dt d\nu(z).$$

The two conditions

$$\nu(B(\zeta,\delta)) = O(\omega(\delta))$$

and

$$\omega(\delta) = o(\delta^{n-\beta})$$

easily imply that if $S(\zeta,\delta)$ is the "Carleson region"

$$S(\zeta,\delta) = \{r\eta : \eta \in B(\zeta,\delta), 1-\delta < r < 1\}$$

then

$$\lambda(S(\zeta,\delta)) = o(\delta^n) \quad \text{as } \delta \longrightarrow 0.$$

In other words, $\lambda$ is a vanishing Carleson measure. From this it follows without difficulty that

$$\lim_{r_0 \to 1} \left[ \sup \int_{A_{r_0}} |f| d\lambda \right] = 0$$

where the supremum is taken over all functions $f$ in the unit ball of $H^1$; this is well known for $n = 1$ and we refer to [**11**, p. 33 and 63]. A proof for the general case can be easily devised using the atomic decomposition of the modulus of an $H^1$ function described in [**3**].

If we now fix $\epsilon > 0$ and choose $r_0$ so close to 1 that

$$\sup \left[ \int_{A_{r_0}} |f| d\lambda \right] < \epsilon$$

then with $k \geqq N = N(r_0, \epsilon)$ it follows that the right hand side of (18) is less than $(2\epsilon)\|f\|_{1,0}$. This completes the argument.

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