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CONGRUENCES ON SEMIGROUPS GENERATED BY INJECTIVE NILPOTENT TRANSFORMATIONS

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To Gordon Preston with respect and gratitude on his 80th birthday

In 1987, Sullivan characterised the elements of the semigroup $NI(X)$ generated by the nilpotents in $I(X)$, the symmetric inverse semigroup on an infinite set X; and, in the same year, Gomes and Howie did the same for finite X. In 1999, Marques-Smith and Sullivan determined all the ideals of $NI(X)$ for arbitrary X. In this paper, we use that work to describe all the congruences on $NI(X)$.

1. Introduction

Throughout this paper, X is a non-empty set. In addition, $P(X)$ denotes the semigroup under composition of all *partial* transformations of X (that is, all transformations α whose *domain*, $\operatorname{dom}\alpha$, and *range*, $\operatorname{ran}\alpha$, are subsets of X). Note that $P(X)$ contains a zero (namely, the empty mapping $\emptyset$): we say $\alpha \in P(X)$ is *nilpotent* with *index* r if $\alpha^r = \emptyset$ and $\alpha^{r-1} \neq \emptyset$, and we let $NP(X)$ denote the semigroup generated by all nilpotents in $P(X)$. In like manner, if $I(X)$ denotes the symmetric inverse semigroup on X, we write $NI(X)$ for the semigroup generated by all nilpotents in $I(X)$.

In [6] the authors described the ideals of $NP(X)$ and $NI(X)$ as a prelude to determining all congruences on these semigroups. In fact, in [**6**, Section 4], they found all the congruences on every principal factor of $NI(X)$ for infinite X. Here, we use the notation and results of [6], as well as ideas from [**1**, Section 10.8], to describe all congruences on $NI(X)$.

2. Preliminary results

All notation and terminology will be from [1] and [6] unless specified otherwise. In particular, if $\alpha \in P(X)$, we let $r(\alpha)$ denote the *rank* of α (that is, $|X\alpha|$) and put

$$D(\alpha) = X\backslash X\alpha, \qquad d(\alpha) = |D(\alpha)|,$$
$$G(\alpha) = X\backslash \operatorname{dom}\alpha, \quad g(\alpha) = |G(\alpha)|.$$

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The cardinal numbers $d(\alpha)$ and $g(\alpha)$ are called the *defect* and the *gap* of α and were used by Sullivan to characterise the elements of $NI(X)$ for infinite X [**8**, Corollary 4]. Note that if $\alpha \in I(X)$ then $g(\alpha^{-1}) = d(\alpha)$ and $d(\alpha^{-1}) = g(\alpha)$. Hence, when X is infinite, the fact that $NI(X)$ is an inverse semigroup follows from the first part of the following result.

Theorem 1. *Suppose X is an infinite set with cardinal k and let $\alpha \in I(X)$. Then α is a product of nilpotents in $I(X)$ if and only if $d(\alpha) = g(\alpha) = k$. Moreover, when this occurs, $NI(X)$ is an inverse semigroup and each $\alpha \in NI(X)$ is a product of 3 or fewer nilpotents with index 2.*

To state the corresponding result for finite sets, we need some notation. If X is an arbitrary set with cardinal k and $1 \leqslant r \leqslant k$, we write

$$D_r = \{\alpha \in I(X) : r(\alpha) = r\}$$

(1) $$I_r = \{\alpha \in I(X) : r(\alpha) < r\}$$

and recall that each D_r is a $\mathcal{D}$-class of $I(X)$ and that the I_r constitute all the proper ideals of $I(X)$. Moreover, if $k = n < \aleph_0$ then each $\alpha \in D_{n-1}$ has a unique *completion* $\overline{\alpha} \in G(X)$, the symmetric group on X, defined by:

$$x\overline{\alpha} = \begin{cases} x\alpha, & \text{if } x \in \operatorname{dom}\alpha, \\ b, & \text{if } x = a, \end{cases}$$

where $X \setminus \operatorname{dom}\alpha = \{a\}$ and $X \setminus \operatorname{ran}\alpha = \{b\}$ ([**2**, p. 388]). We write

$$E_{n-1} = \{\alpha \in D_{n-1} : \overline{\alpha} \text{ is an even permutation}\}.$$

By [**2**, Lemma 2.1], if X is finite then $\alpha \in I(X)$ is nilpotent if and only if $A\alpha \neq A$ for each non-empty $A \subseteq \operatorname{dom}\alpha$. Clearly, if this condition holds for α, it also holds for α^{-1}. Hence, if X is finite and β is a product of nilpotents in $I(X)$ then β^{-1} is also, and thus again $NI(X)$ is an inverse semigroup. In [**2**, Theorem 3.18], the authors proved the following result.

Theorem 2. *If X is finite and $|X| = n \geqslant 3$, then $NI(X)$ is an inverse semigroup. In fact,*

(a) *if n is even then $NI(X) = I_n$, and*
(b) *if n is odd then $NI(X) = I_{n-1} \cup E_{n-1}$.*

Moreover, in each case, each non-zero $\alpha \in NI(X)$ is a product of $n-1$ or fewer nilpotents, each with index n (and rank $n-1$).

In what follows, we extend the convention introduced in [**1**, Vol. 2, p. 241]: namely, if $\alpha \in P(X)$ is non-zero then we write

$$\alpha = \begin{pmatrix} A_i \\ x_i \end{pmatrix}$$

and take as understood that the subscript i belongs to some (unmentioned) index set I, that the abbreviation $\{x_i\}$ denotes $\{x_i : i \in I\}$, and that $\operatorname{ran}\alpha = \{x_i\}$, $x_i\alpha^{-1} = A_i$ and $\operatorname{dom}\alpha = \bigcup\{A_i : i \in I\}$. In particular, if $\operatorname{dom}\alpha = \{a\}$ and $\operatorname{ran}\alpha = \{b\}$, we write α more simply as a_b. Also, we let id_A denote the identity on A.

For notational convenience, if ρ is a congruence on a transformation semigroup, we often write $\alpha \sim \beta$ to mean $(\alpha, \beta) \in \rho$. Also, sometimes we write $x\alpha = \emptyset$ to mean $x \notin \operatorname{dom}\alpha$.

The following result is comparable with [**1**, Lemma 10.64].

Lemma 1. *Suppose $|X| \geqslant 3$ and let ρ be a non-identity congruence on $NI(X)$. Then the ρ–class containing $\emptyset$ is an ideal of $NI(X)$ and it contains D_1.*

Proof: Suppose $(\alpha, \beta) \in \rho$ where $\alpha \neq \beta$. Then $x\alpha \neq x\beta$ for some $x \in X$ and, without loss of generality, we can assume $x\alpha = y \neq \emptyset$. Let $a, b \in X$ and $\lambda = a_x, \mu = y_b$. Then $\lambda, \mu \in NI(X)$, and $\lambda\alpha\mu = a_b$ and $\lambda\beta\mu = \emptyset$ (even if $x \in \operatorname{dom}\beta$). Hence $a_b \sim \emptyset$ and it follows that D_1 is contained in $\emptyset\rho$, the ρ–class containing $\emptyset$, which is clearly an ideal of $NI(X)$. ∎

The proper ideals of $NI(X)$ were described in [**6**, Theorems 6 and 14] as follows. However, note that if $|X| = k \geqslant \aleph_0$ and $\alpha \in I(X)$ satisfies $r(\alpha) < r \leqslant k$ then $d(\alpha) = g(\alpha) = k$ and so $\alpha \in NI(X)$ by Theorem 1. Hence, $I_r \subseteq NI_r$ and it follows that $NI_r = I_r$. In fact, a similar statement holds in almost all cases when X is finite. Despite this, we prefer to retain a distinctive notation for the ideals of $NI(X)$.

Theorem 3. *For any set X with (finite or infinite) cardinal $k \geqslant 3$, the proper ideals of $NI(X)$ are precisely the sets*

$$NI_r = \{\alpha \in NI(X) : r(\alpha) < r\}$$

where $1 \leqslant r \leqslant k$.

Consequently, if ρ is a non-identity and non-universal congruence on $NI(X)$ then $\emptyset\rho = NI_r$ for some r such that $2 \leqslant r \leqslant |X|$. We call r the *primary rank* of ρ and denote it by $\eta(\rho)$ (compare [**1**, Vol. 2, p. 231]). For what follows, we also need the characterisation of Green's $\mathcal{D}$–relation on $NI(X)$ given in [**6**, p. 309 and Theorem 17].

Theorem 4. *If X is any set with at least three elements, and if $\alpha, \beta \in NI(X)$, then $\beta = \lambda\alpha\mu$ for some $\lambda, \mu \in NI(X)$ if and only if $r(\beta) \leqslant r(\alpha)$. Hence, $\mathcal{D} = \mathcal{J}$ for $NI(X)$.*

If $1 \leqslant r \leqslant |X|$, we let DI_r denote the $\mathcal{D}$–class of $NI(X)$ which contains all elements with rank r. Also, as in [**1**, Vol. 2, p. 227], we let NI_r^* denote the Rees congruence on $NI(X)$ determined by the ideal NI_r. The following result is similar to [**1**, Theorem 10.65].

Lemma 2. *If ρ is a non-identity congruence on $NI(X)$ and $\eta = \eta(\rho)$ then*

$$NI_\eta^* \subseteq \rho \subseteq NI_\eta^* \cup \mathcal{D}.$$

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PROOF: We have $NI_\eta^* \subseteq \rho$ since $NI_\eta^* = \mathrm{id}_{NI(X)} \cup (NI_\eta \times NI_\eta)$ and $NI_\eta \times NI_\eta \subseteq \rho$ by the definition of $\eta(\rho)$. For the other inclusion, let $(\alpha, \beta) \in \rho$ and assume $r(\beta) < r(\alpha) = r$ (if $r(\alpha) = r(\beta)$ then $(\alpha, \beta) \in \mathcal{D}$ and the required inclusion holds). We aim to show that $r < \eta$, which clearly implies the desired result.

(a) r IS INFINITE. This means X is infinite and $NI(X)$ is described by Theorem 1. Also $|\operatorname{ran}\alpha \setminus \operatorname{ran}\beta| = r(\alpha)$ since $r = r(\alpha)$ is infinite and $r(\beta) < r(\alpha)$. Hence, if $|X| = k$ and γ is any bijection from $\operatorname{ran}\alpha \setminus \operatorname{ran}\beta$ onto $\operatorname{ran}\alpha$, then $g(\gamma) \geqslant d(\alpha) = k$ and $d(\gamma) = d(\alpha)$. Therefore $\gamma \in NI(X)$ and it follows that $\alpha\gamma \sim \emptyset$. Since $r(\alpha\gamma) = r$, this implies $r < \eta$, as required.

(b) r IS FINITE. In this case, X may be finite or infinite, but the following argument holds in both situations with appropriate justification. Let $|X| = n$ (finite or infinite) and write $r(\beta) = s < r = r(\alpha) < n$: note that if X is infinite, then $r < \aleph_0 \leqslant n$; and if X is finite, then $r < n$ since $\alpha \notin G(X)$. Now suppose $\operatorname{ran}\alpha \cap \operatorname{ran}\beta = \emptyset$. If this happens, then $\gamma = \mathrm{id}_{\operatorname{ran}\alpha}$ is an element of $NI(X)$ (for example, in the finite case, if n is odd and $r = n-1$ then $\overline{\gamma} = \mathrm{id}_X$, an even permutation of X, hence $\gamma \in E_{n-1}$; and in the infinite case, the gap and defect of γ equal $|X|$ since r is finite). Now $\alpha\gamma = \alpha$ and $\beta\gamma = \emptyset$, so $\alpha \sim \emptyset$ and hence $r < \eta$. Therefore, we may suppose

$$\operatorname{ran}\alpha \cap \operatorname{ran}\beta = C = \{c_1, \ldots, c_t\}$$

where $0 < t \leqslant s < r < n$. Let $\gamma_0 = \mathrm{id}_{\operatorname{ran}\alpha} \in NI(X)$ (as before) and note that $\alpha\gamma_0 = \alpha$ and $\operatorname{ran}(\beta\gamma_0) = C$. For each $i = 1, \ldots, t$, let γ_i be the idempotent in $I(X)$ with domain $\operatorname{ran}\alpha \setminus \{c_i\}$. Note that, since $r(\gamma_i) = r - 1$ and this is at most $n-2$ if n is finite, each $\gamma_i \in NI(X)$ by Theorems 1 and 2 (that is, regardless of whether X is infinite or finite). Now

$$\operatorname{ran}(\alpha\gamma_0\gamma_1) = \operatorname{ran}\alpha \setminus \{c_1\}, \qquad \operatorname{ran}(\beta\gamma_0\gamma_1) = C \setminus \{c_1\},$$
$$\operatorname{ran}(\alpha\gamma_0\gamma_1\gamma_2) = \operatorname{ran}\alpha \setminus \{c_1, c_2\}, \qquad \operatorname{ran}(\beta\gamma_0\gamma_1\gamma_2) = C \setminus \{c_1, c_2\},$$

and so on. Write $\alpha_i = \alpha\gamma_0 \cdots \gamma_i$ and $\beta_i = \beta\gamma_0 \cdots \gamma_i$ for each $i = 0, \ldots, t$. Clearly, $\beta_t = \emptyset$ but $\alpha_t \neq \emptyset$ (since $s < r$). That is, $r(\alpha_t) \geqslant 1$ and, since $\alpha_t \sim \beta_t$, this implies $\eta \geqslant 2$ and $\alpha_t \in \emptyset\rho$. Since $r(\beta_{t-1}) = 1$, this implies $\beta_{t-1} \in \emptyset\rho$. But $r(\alpha_{t-1}) \geqslant 2$ and $\alpha_{t-1} \sim \beta_{t-1}$, so $\eta \geqslant 3$ and $\alpha_{t-1} \in \emptyset\rho$. In like manner, we deduce that $\beta_{t-2}, \alpha_{t-2}, \beta_{t-3}, \ldots, \alpha_0 = \alpha$ all belong to $\emptyset\rho$, and hence $r < \eta$. ∎

Next we recall Hall's Theorem [**3**, Proposition II.4.5]: namely, if S is a regular subsemigroup of a semigroup T then the $\mathcal{L}$ and $\mathcal{R}$ relations on S are the restrictions to S of the corresponding ones on T. Now, the $\mathcal{L}$ and $\mathcal{R}$ relations on $I(X)$ are well-known: namely, $\alpha \mathcal{L} \beta$ if and only if $\operatorname{ran}\alpha = \operatorname{ran}\beta$; and $\alpha \mathcal{R} \beta$ if and only if $\operatorname{dom}\alpha = \operatorname{dom}\beta$ [**3**, Exercise V.8.2]. And $NI(X)$ is a regular (in fact, inverse) subsemigroup of $I(X)$ by Theorems 1 and 2. Therefore we can prove a result for $NI(X)$ which is analogous to [**1**, Theorem 10.66].

Lemma 3. *Let ρ be a congruence on $NI(X)$ and suppose $\eta(\rho)$ is finite. If $(\alpha,\beta)\in\rho$ and $\eta(\rho)\leqslant r(\alpha)<\aleph_0$ then $(\alpha,\beta)\in\mathcal{H}$.*

Proof: Clearly we may assume ρ is not the identity congruence, so $\eta(\rho)>1$ and, by Lemma 2, $r(\alpha)=r(\beta)=r$, say. Suppose $\operatorname{ran}\alpha\neq\operatorname{ran}\beta$ and let $\gamma=\mathrm{id}_{\operatorname{ran}\alpha}$, which is an element of $NI(X)$, as discussed in case (b) for the proof of Lemma 2. Now $\alpha\gamma=\alpha$ and $r(\beta\gamma)\leqslant r-1$ (note that $\operatorname{ran}\beta\setminus\operatorname{ran}\alpha\neq\emptyset$ since α and β have the same finite rank but, by supposition, their ranges are not equal, so one cannot be contained in the other). Since $\alpha\gamma\sim\beta\gamma$, Lemma 2 implies $r<\eta(\rho)$, a contradiction. Therefore, $\operatorname{ran}\alpha=\operatorname{ran}\beta$ and hence $\alpha\;\mathcal{L}\;\beta$.

Suppose $\operatorname{dom}\alpha\neq\operatorname{dom}\beta$ and let $\delta=\mathrm{id}_{\operatorname{dom}\alpha}$. Then $\delta\in NI(X)$ (as for γ) and $r(\delta\beta)\leqslant r-1$ (also as before). Since $\alpha=\delta\alpha\sim\delta\beta$, this implies $r<\eta(\rho)$, a contradiction. Hence $\operatorname{dom}\alpha=\operatorname{dom}\beta$ and so $\alpha\;\mathcal{R}\;\beta$. ∎

Lemma 4. *Let ρ be a non-identity congruence on $NI(X)$ and suppose $\eta(\rho)$ is finite. If $(\alpha,\beta)\in\rho$ where $\alpha\neq\beta$ and $\eta(\rho)\leqslant r(\alpha)<\aleph_0$ then $r(\alpha)=\eta(\rho)$.*

Proof: By Lemma 3, $(\alpha,\beta)\in\mathcal{H}$. Hence $\operatorname{dom}\alpha=\operatorname{dom}\beta=\{a_1,\dots,a_r\}$, say, and $\operatorname{ran}\alpha=\operatorname{ran}\beta$. Thus we can write

$$\alpha=\begin{pmatrix}a_1&\dots&a_r\\b_1&\dots&b_r\end{pmatrix},\quad\beta=\begin{pmatrix}a_1&\dots&a_r\\b_{1\pi}&\dots&b_{r\pi}\end{pmatrix}$$

for some permutation π of $\{1,\dots,r\}$. Since $\alpha\neq\beta$, there exists i such that $i\neq i\pi$; and, since ρ is not the identity congruence, we know $\eta(\rho)\geqslant 2$ and thus $r\geqslant 2$. If γ is the identity on $\{a_1,\dots,a_{i-1},a_{i+1},\dots,a_r\}$, then $\gamma\in NI(X)$ (via the usual justification when X is finite or infinite) and so $\gamma\alpha\sim\gamma\beta$. But, since $i\pi^{-1}\neq i$, $\operatorname{ran}(\gamma\beta)$ contains b_i, whereas $\operatorname{ran}(\gamma\alpha)$ does not. Therefore $(\gamma\alpha,\gamma\beta)\notin\mathcal{H}$ and so, by Lemma 3, $r(\gamma\alpha)=r-1$ must be less than $\eta(\rho)$. Since $r(\alpha)=r\geqslant\eta(\rho)$ by supposition, it follows that $r=\eta(\rho)$. ∎

3. Finite primary rank

In [**6**, p. 316], the authors observed that, if X is finite and $r<|X|$, then NI_{r+1}/NI_r is completely 0–simple. For what follows, we require a more general result: compare [**1**, Vol. 2, Lemma 10.54 and p. 227, Exercise 3], and also [**7**, Lemma 2.4]. If r is any infinite cardinal then r' denotes the *successor* of r (that is, the least cardinal greater than r).

Lemma 5. *If X is any set with at least five elements and $4\leqslant r<|X|$ then $NI_{r'}/NI_r$ is 0–bisimple, and it contains a primitive idempotent if and only if r is finite. Consequently, if r is finite then NI_{r+1}/NI_r is completely 0–simple.*

Proof: Suppose $\alpha,\beta\in NI(X)$ and $r(\alpha)=r(\beta)=r$ (finite or infinite) and write

$$\alpha=\begin{pmatrix}a_p\\x_p\end{pmatrix},\quad\beta=\begin{pmatrix}b_p\\y_p\end{pmatrix},\quad\gamma=\begin{pmatrix}b_p\\x_p\end{pmatrix},\quad\lambda=\begin{pmatrix}a_p\\b_p\end{pmatrix}.$$

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Note that if X is infinite then $|P| = r < |X|$ implies $d(\gamma) = g(\gamma) = |X|$, hence $\gamma \in NI(X)$ and likewise $\lambda \in NI(X)$. Also, $\alpha = \lambda\gamma$ and $\gamma = \lambda^{-1}\alpha$, thus $\alpha\ \mathcal{L}\ \gamma$ and similarly $\gamma\ \mathcal{R}\ \beta$. In other words, if X is infinite then all elements of $NI(X)$ with rank r are $\mathcal{D}$-related, and so $NI_{r'}/NI_r$ is 0-bisimple.

Since $g(\gamma) = g(\beta) \neq 0$ and $r(\gamma) = r(\alpha) < |X|$, the same conclusion holds, by Theorem 4, when $|X| = n < \aleph_0$ and n is even, or n is odd and $r < n-1$. If n is odd and $r = n-1$, then $NI_{r+1}/NI_r = E_{n-1} \cup \{0\} = S$, say. Hence, the completions $\overline{\alpha}$ and $\overline{\beta}$ are even permutations of S. Also, $\operatorname{dom}\alpha$ and $\operatorname{dom}\beta$ differ in at most one element.

If $\operatorname{dom}\alpha = \operatorname{dom}\beta$, we can write (after a re-ordering of $\operatorname{dom}\beta$, if necessary)

$$\alpha = \begin{pmatrix} a_1 & \dots & a_{n-1} \\ x_1 & \dots & x_{n-1} \end{pmatrix}, \quad \beta = \begin{pmatrix} a_1 & \dots & a_{n-1} \\ z_1 & \dots & z_{n-1} \end{pmatrix}, \quad \mu = \begin{pmatrix} z_1 & \dots & z_{n-1} \\ x_1 & \dots & x_{n-1} \end{pmatrix}.$$

By [**2**, p. 388], $\overline{\alpha} = \overline{\beta}.\overline{\mu}$ (since $\alpha = \beta\mu$), hence $\overline{\mu}$ is an even permutation of X and thus $\mu \in E_{n-1}$. Clearly, $\beta = \alpha\mu^{-1}$. It follows that $\alpha\ \mathcal{L}\ \alpha\ \mathcal{R}\ \beta$ in S, so $\alpha\ \mathcal{D}\ \beta$ in S, as desired.

If $\operatorname{dom}\alpha \neq \operatorname{dom}\beta$, we suppose $a_1 \neq b_1$ and $a_i = b_i$ for $i = 2, \dots, n-1$ (after a possible re-ordering of $\operatorname{dom}\beta$, hence a possible re-labelling of $\operatorname{ran}\beta$, but without loss of generality). Thus, we now have:

$$\alpha = \begin{pmatrix} a_1 & a_2 & \dots & a_{n-1} \\ x_1 & x_2 & \dots & x_{n-1} \end{pmatrix}, \qquad \beta = \begin{pmatrix} b_1 & a_2 & \dots & a_{n-1} \\ z_1 & z_2 & \dots & z_{n-1} \end{pmatrix},$$

$$\gamma = \begin{pmatrix} b_1 & a_2 & a_3 & a_4 & \dots & a_{n-1} \\ x_1 & x_3 & x_2 & x_4 & \dots & x_{n-1} \end{pmatrix},$$

$$\lambda = \begin{pmatrix} b_1 & a_2 & a_3 & a_4 & \dots & a_{n-1} \\ a_1 & a_3 & a_2 & a_4 & \dots & a_{n-1} \end{pmatrix}, \qquad \mu = \begin{pmatrix} z_1 & z_2 & z_3 & z_4 & \dots & z_{n-1} \\ x_1 & x_3 & x_2 & x_4 & \dots & x_{n-1} \end{pmatrix}.$$

Note that, in this case, we have redefined γ and λ (but only after changing β, if necessary) and this is possible since $r \geqslant 4$. Also, observe that the completion of λ equals the even permutation $(a_1, b_1)(a_2, a_3)$ of X, hence $\lambda \in E_{n-1}$. Moreover, $\gamma = \lambda\alpha$, so $\gamma \in E_{n-1}$ and clearly $\alpha = \lambda^{-1}\gamma$. Hence, $\alpha\ \mathcal{L}\ \gamma$ in S. Next, we see that $\mu = \beta^{-1}\gamma \in E_{n-1}$ (since both β and γ belong to E_{n-1}). Since $\gamma = \beta\mu$ and $\beta = \gamma\mu^{-1}$, it follows that $\gamma\ \mathcal{R}\ \beta$ in S. Hence, $\alpha\ \mathcal{D}\ \beta$ in S, and we conclude that NI_n/NI_{n-1} is 0-bisimple when n is odd.

Suppose r is finite and let $\alpha = \alpha\beta = \beta\alpha$ for non-zero idempotents $\alpha, \beta \in NI(X)$, each with rank r. Then $\operatorname{ran}\alpha \subseteq \operatorname{ran}\beta$, and both these sets contain r elements, so $\operatorname{ran}\alpha = \operatorname{ran}\beta$. Hence, $\alpha = \mathrm{id}_{\operatorname{ran}\alpha} = \mathrm{id}_{\operatorname{ran}\beta} = \beta$; that is, every non-zero idempotent in NI_{r+1}/NI_r is primitive. Conversely, suppose β is a non-zero idempotent in $NI_{r'}/NI_r$ and assume $r \geqslant \aleph_0$. Then $\beta = \mathrm{id}_B$ where $|B| = r$. If $a \in B$ and $A = B \setminus \{a\}$ then $|A| = r$ and $\alpha = \mathrm{id}_A \in NI(X)$ (since the gap and defect of α equals $|X|$); also, we have $\alpha = \alpha\beta = \beta\alpha$. In other words, if $r \geqslant \aleph_0$ then no non-zero idempotent in $NI_{r'}/NI_r$ is primitive. ∎

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Next we prove a result which is similar to [**1**, Theorem 10.60]. However, although NI_{r+1}/NI_r is completely 0–simple when $r \geqslant 4$ is finite, and hence its congruences are known in that case, our proof differs from the one given in [**1**].

Lemma 6. *Suppose X is any set and r is any positive integer with $r+1 \leqslant |X|$. If σ is a non-universal congruence on NI_{r+1}/NI_r, then the relation σ^+ defined on $NI(X)$ by*

$$\sigma^+ = \mathrm{id}_{NI(X)} \cup \left[\sigma \cap (DI_r \times DI_r)\right] \cup (NI_r \times NI_r)$$

is a congruence on $NI(X)$.

Proof: Clearly σ^+ is an equivalence, so we aim to show it is left and right compatible with composition on $NI(X)$. To do this, we consider only the case when $(\alpha, \beta) \in \sigma$ and $r(\alpha) = r(\beta) = r$ (the other possibilities are easy to check). First suppose $|\operatorname{ran}\alpha \cap \operatorname{ran}\beta| = s < r$ and write $B = \operatorname{ran}\beta$. Then $\mathrm{id}_B \in DI_r$ (by the usual argument) and hence, in the semigroup NI_{r+1}/NI_r, $\alpha.\mathrm{id}_B = 0$ but $\beta.\mathrm{id}_B = \beta$. Since σ is a congruence on NI_{r+1}/NI_r, it follows that $(0, \beta) \in \sigma$ and hence σ is universal on NI_{r+1}/NI_r, a contradiction. Thus, $s = r$ and this implies $\operatorname{ran}\alpha = \operatorname{ran}\beta = Y$ say. Let $\mu \in NI(X)$, and note that the ranks of $\alpha\mu$ and $\beta\mu$ are equal and at most r. In fact, if $r(\alpha\mu) = r(\beta\mu) < r$, then $(\alpha\mu, \beta\mu) \in NI_r \times NI_r \subseteq \sigma^+$, as required. On the other hand, if $r(\alpha\mu) = r(\beta\mu) = r$ then $\operatorname{ran}\alpha \subseteq \operatorname{dom}\mu$. So, if $\mu' = \mu|Y$ then $\mu' \in DI_r$ (by the usual argument); also, $\alpha\mu' = \alpha\mu$ and $\beta\mu' = \beta\mu$. Therefore, $(\alpha\mu, \beta\mu) \in \sigma \cap (DI_r \times DI_r) \subseteq \sigma^+$. Hence σ^+ is right compatible.

Now let $\lambda \in NI(X)$ and suppose $r(\lambda\alpha) = r(\lambda\beta) = r$ for the same α, β as at the start. Let $|\operatorname{dom}\alpha \cap \operatorname{dom}\beta| = t$ and $C = \operatorname{dom}\beta$. Then an argument similar to the one above leads us to conclude that $t = r$ and hence that $\operatorname{dom}\alpha = \operatorname{dom}\beta = Z$ say. Moreover, $\operatorname{dom}\alpha \subseteq \operatorname{ran}\lambda$ since $r(\lambda\alpha) = r = r(\alpha)$ and α is injective. Therefore, if $\lambda' = \lambda \mid (Z\lambda^{-1})$ then $\lambda' \in DI_r$; and, since $\lambda'\alpha = \lambda\alpha$ and $\lambda'\beta = \lambda\beta$, we conclude that $(\lambda\alpha, \lambda\beta) \in \sigma^+$. ∎

Remark 1. Recall that every non-universal congruence ρ on a 0–simple semigroup is 0–*restricted*: that is, $0\rho = \{0\}$; and clearly, by Lemma 5, NI_{r+1}/NI_r is 0–simple for each (finite or infinite) $r \geqslant 4$. Consequently, in the above result, $\sigma_1^+ = \sigma_2^+$ implies $\sigma_1 = \sigma_2$. For, if $\sigma_1^+ = \sigma_2^+$ then, by their definition, $\sigma_1 \cap (DI_r \times DI_r) = \sigma_2 \cap (DI_r \times DI_r)$; and, since each σ_i is 0–restricted, this implies $\sigma_1 = \sigma_2$.

Using the results in section 2, we now determine all congruences ρ on $NI(X)$ for which $\eta(\rho)$ is finite (compare [**1**, Theorem 10.68] and [**7**, Lemma 2.6]).

Theorem 5. *Let ρ be a non-identity and non-universal congruence on $NI(X)$ and suppose $r = \eta(\rho)$ is finite. Then $\rho = \sigma^+$ where σ is a non-universal congruence on NI_{r+1}/NI_r.*

Proof: Suppose $(\alpha, \beta) \in \rho$. By the definition of $\eta(\rho)$, if one of α or β has rank less than r, then the other also has rank less than r, and thus $(\alpha, \beta) \in NI_r^*$. By Lemma 2, if the rank of α or β is at least r, then $r(\alpha) = r(\beta) = s$ say. We assert that if s is infinite

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then $\alpha = \beta$.

To see this, assume $s \geqslant \aleph_0$ and $x\alpha \neq x\beta$ for some $x \in \operatorname{dom}\alpha$ (without loss of generality). Write $x\alpha = a$ and choose $Y \subseteq \operatorname{dom}\alpha$ such that $x \in Y$, $|Y| = r$ and $a \notin Y\beta$ (this is possible since $s \geqslant \aleph_0$ and $r < \aleph_0$). Let $Z = Y\alpha$ and observe that $\alpha' = \operatorname{id}_Y . \alpha . \operatorname{id}_Z$ has rank r, whereas $\beta' = \operatorname{id}_Y . \beta . \operatorname{id}_Z$ has rank at most $r - 1$ (since $a \in Z \setminus Y\beta$). Moreover, by Theorem 1, id_Y and id_Z belong to $NI(X)$ since, by assumption, X is infinite but Y and Z are finite. Therefore, $(\alpha', \beta') \in \rho$. Since this contradicts the choice of $r = \eta(\rho)$, the assertion follows.

Consequently, if $s \geqslant \aleph_0$ then $(\alpha, \beta) \in \operatorname{id}_{NI(X)}$. On the other hand, if $r \leqslant s < \aleph_0$ and $\alpha \neq \beta$, then Lemma 4 implies $r = s$. That is, $(\alpha, \beta) \in \rho \cap (DI_r \times DI_r)$. We assert that

$$\sigma = \rho \cap (DI_r \times DI_r) \cup \{(0,0)\}$$

is a congruence on NI_{r+1}/NI_r. For, clearly it is an equivalence on NI_{r+1}/NI_r. Also, if $(\alpha, \beta) \in \rho \cap (DI_r \times DI_r)$ and $\mu \in DI_r$ then $(\alpha\mu, \beta\mu) \in \rho$, where the ranks of $\alpha\mu$ and $\beta\mu$ are at most r. However, by the choice of $r = \eta(\rho)$, either $r(\alpha\mu) = r(\beta\mu) = r$ or both $r(\alpha\mu)$ and $r(\beta\mu)$ is less than r: in the former case, $(\alpha\mu, \beta\mu) \in \rho \cap (DI_r \times DI_r)$ and, in the latter case, $\alpha\mu = \beta\mu = 0$ in the Rees factor semigroup NI_{r+1}/NI_r. That is, σ is right compatible on NI_{r+1}/NI_r, and similarly it is left compatible. Thus, we have shown that $\rho \subseteq \sigma^+$ as defined in Lemma 6, and clearly $\sigma^+ \subseteq \rho$, so equality follows. Moreover, σ is non-universal on NI_{r+1}/NI_r: otherwise, $\rho \cap (DI_r \times DI_r) = DI_r \times DI_r$ and hence

$$\rho = \operatorname{id}_{NI(X)} \cup (DI_r \times DI_r) \cup (NI_r \times NI_r)$$

which is not a congruence on $NI(X)$ (for example, if $|A| = |B| = r < \aleph_0$ and $A \neq B$ then $(\operatorname{id}_A, \operatorname{id}_B) \in \rho$, but $(\operatorname{id}_A . \operatorname{id}_A, \operatorname{id}_A . \operatorname{id}_B) \notin \rho$). ∎

Given the above result, we need more information about the congruences on NI_{r+1}/NI_r. In fact, by Lemma 5, NI_{r+1}/NI_r is a completely 0–simple semigroup for finite $r \geqslant 4$, and thus all of its congruences can be described (see [**1**, Section 10.7]). To avoid the complication which that entails, we prove the following result. But, first we recall the fact: if ρ is a congruence on an inverse semigroup and $(a, b) \in \rho$ then $(a^{-1}, b^{-1}) \in \rho$ (see [**3**, Proposition V.1.6]).

Lemma 7. *Suppose X is any set with at least six elements, and let r be a positive integer such that $r + 1 \leqslant |X|$. If σ is a non-universal congruence on NI_{r+1}/NI_r then, for each $Y \subseteq X$ with cardinal r, there exists $N \triangleleft G(Y)$ such that*

$$\sigma = \{(\lambda . \operatorname{id}_Y . \mu, \lambda . \gamma . \mu) : \lambda, \mu \in DI_r \text{ and } \gamma \in N\} \cup \{(0,0)\}.$$

Proof: Fix $Y \subseteq X$ with $|Y| = r$. If $\operatorname{id}_Y \sim \alpha$ and $\alpha\alpha^{-1} = \operatorname{id}_A$ then $\operatorname{id}_Y \sim \alpha^{-1}$, so $\operatorname{id}_Y \sim \operatorname{id}_A$ and hence $\operatorname{id}_Y \sim \operatorname{id}_{Y \cap A}$. Since σ is 0–restricted, we deduce that $|Y \cap A| = r$ and hence that $Y = A$ (since r is finite). In other words, $\operatorname{dom}\alpha = Y$ and similarly

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$\operatorname{ran}\alpha = Y$, and thus $\alpha \in G(Y)$. Put another way: the σ–class containing the idempotent id_Y is a subgroup N of $G(Y)$. We assert that $N \lhd G(Y)$. To see this, suppose $\alpha \in N$ and $\gamma \in G(Y)$. If X is infinite then $\gamma \in DI_r$ by Theorem 1 (since $r < \aleph_0$ by supposition), and hence $\gamma\alpha\gamma^{-1} \sim \gamma.\mathrm{id}_Y\gamma^{-1} = \mathrm{id}_Y$, so $\gamma\alpha\gamma^{-1} \in N$. On the other hand, if $|X| = n < \aleph_0$, then $r \leqslant n-1$ and, by Theorem 2, we deduce that $\gamma\alpha\gamma^{-1} \in N$ when n is even, and when n is odd and $r < n-1$. Hence, we assume n is odd and $r = n-1$. In this case, since each $\alpha \in N$ permutes Y, its extension $\overline{\alpha}$ to $X = Y \dot{\cup} \{z\}$ must fix z and be an even permutation of X. Consequently, α is an even permutation of Y and hence $\alpha \in \mathrm{Alt}(Y)$, the alternating group on Y. Clearly, each $\pi \in \mathrm{Alt}(Y)$ belongs to $E_{n-1} = DI_{n-1}$, so $\pi\alpha\pi^{-1} \sim \mathrm{id}_Y$ and thus $\pi\alpha\pi^{-1} \in N$. That is, N is a normal subgroup of $\mathrm{Alt}(Y)$, which is simple if $|Y| \geqslant 5$. Hence, for such Y, N equals $\{\mathrm{id}_Y\}$ or $\mathrm{Alt}(Y)$, and thus it is a normal subgroup of $G(Y)$.

Now suppose $\alpha \sim \beta$ and let $A = \operatorname{dom}\alpha$. Then $\alpha = \mathrm{id}_A.\alpha \sim \mathrm{id}_A.\beta$, so $A = \operatorname{dom}\beta$ (since σ is 0–restricted and β is injective) and similarly $\operatorname{ran}\alpha = \operatorname{ran}\beta$. Therefore, we can write

$$\alpha = \begin{pmatrix} a_1 & \dots & a_r \\ x_1 & \dots & x_r \end{pmatrix}, \quad \beta = \begin{pmatrix} a_1 & \dots & a_r \\ x_{1\pi} & \dots & x_{r\pi} \end{pmatrix}$$

for some permutation π of $\{1, \dots, r\}$. Let $Y = \{y_1, \dots, y_r\}$ and define

$$\lambda = \begin{pmatrix} a_1 & \dots & a_r \\ y_1 & \dots & y_r \end{pmatrix}, \quad \mu = \begin{pmatrix} y_1 & \dots & y_r \\ x_1 & \dots & x_r \end{pmatrix}, \quad \gamma = \begin{pmatrix} y_1 & \dots & y_r \\ y_{1\pi} & \dots & y_{r\pi} \end{pmatrix}.$$

If X is infinite then $\lambda, \mu \in DI_r = NI(X) \cap D_r$ by Theorem 1 (since $r < \aleph_0$). Suppose $|X| = n < \aleph_0$. If n is even then $r + 1 \leqslant n$ implies $r < n$, and so $\lambda, \mu \in DI_r$ by Theorem 2(a). Clearly, by Theorem 2(b), we reach the same conclusion if n is odd and $r < n-1$. Moreover, $\alpha = \lambda.\mathrm{id}_Y\,\mu$ and $\beta = \lambda.\gamma.\mu$, hence $\gamma = \lambda^{-1}\beta\mu^{-1} \in DI_r$ and so $\gamma \in N$: that is, the pair $(\alpha, \beta) \in \sigma$ has the desired form.

Now we assume n is odd and $r = n-1$. In this case, $\alpha, \beta \in E_{n-1}$ and we obtain

$$\text{(2)} \qquad \alpha\alpha^{-1} = \begin{pmatrix} a_1 & \dots & a_{n-1} \\ a_1 & \dots & a_{n-1} \end{pmatrix} \sim_\sigma \beta\alpha^{-1} = \begin{pmatrix} a_1 & \dots & a_{n-1} \\ a_{1\pi} & \dots & a_{(n-1)\pi} \end{pmatrix}$$

where π is the same permutation as before (but now $r = n-1$). Since $|X| = n$, the unordered sets $\{y_1, \dots, y_{n-1}\}$ and $\{a_1, \dots, a_{n-1}\}$ differ in at most one element. In fact, if

$$Y = \{y_1, \dots, y_{n-1}\} = \{a_1, \dots, a_{n-1}\} = A, \text{ say,}$$

then from (2) we deduce that $\mathrm{id}_Y \sim \beta\alpha^{-1} = \gamma'$ (say), where $\gamma' \in N$, $\alpha = \mathrm{id}_Y.\mathrm{id}_Y.\alpha$ and $\beta = \mathrm{id}_Y.\gamma'.\alpha$. Suppose instead that $Y \neq A$ and, after a possible re-ordering, but without loss of generality, assume that $y_i = a_i$ for each $i = 1, \dots, n-2$ and $y_{n-1} \neq a_{n-1}$. Define $\mu \in E_{n-1}$ and its completion in $G(X)$ as follows:

$$\mu = \begin{pmatrix} a_1 & \dots & a_{n-3} & a_{n-2} & y_{n-1} \\ a_1 & \dots & a_{n-3} & a_{n-1} & a_{n-2} \end{pmatrix}, \quad \overline{\mu} = \begin{pmatrix} a_1 & \dots & a_{n-3} & a_{n-2} & y_{n-1} & a_{n-1} \\ a_1 & \dots & a_{n-3} & a_{n-1} & a_{n-2} & y_{n-1} \end{pmatrix}.$$

Then, since $Y = \{a_1, \dots, a_{n-2}, y_{n-1}\}$ and $A = \{a_1, \dots, a_{n-2}, a_{n-1}\}$, from (2) we obtain

$$\mu. \operatorname{id}_A .\mu^{-1} = \operatorname{id}_Y \quad \sim \quad \mu.\beta\alpha^{-1}.\mu^{-1} = \gamma' \text{ (say)}.$$

This means $\gamma' \in N$, and we observe that $\alpha = \mu^{-1}. \operatorname{id}_Y .\mu\alpha$ and $\beta = \mu^{-1}.\gamma'.\mu\alpha$, where both μ^{-1} and $\mu\alpha$ belong to $E_{n-1} = DI_{n-1}$. Hence, in all cases, we have shown that each $(\alpha, \beta) \in \sigma$ has the desired form, and so

$$\sigma \subseteq \{(\lambda. \operatorname{id}_Y .\mu, \lambda.\gamma.\mu) : \lambda, \mu \in DI_r \text{ and } \gamma \in N\} \cup \{(0,0)\}.$$

Since the reverse containment is obvious, the result follows. ∎

REMARK 2. Suppose $N \lhd G(Y)$, where $Y \subseteq X$, $|Y| = r < \aleph_0$ and $r + 1 \leqslant X$. We assert that, if $(\alpha, \beta) \in \overline{\sigma}$, where

$$\overline{\sigma} = \{(\lambda. \operatorname{id}_Y .\mu, \lambda.\gamma.\mu) : \lambda, \mu \in DI_r \text{ and } \gamma \in N\}$$

then, in the Rees factor semigroup NI_{r+1}/NI_r, $\alpha = 0$ if and only if $\beta = 0$. That is, $\overline{\sigma}$ is never the universal relation on NI_{r+1}/NI_r. To see this, let $\lambda, \mu \in DI_r$ and $\gamma \in N$. Then, $\lambda\gamma\mu = 0$ in NI_{r+1}/NI_r if and only if $r(\lambda\gamma\mu) < r$ and, since the given mappings are injective, this is equivalent to saying: either $|\operatorname{ran}\lambda \cap \operatorname{dom}\gamma| < r$ or ($\lambda\gamma \neq 0$ and $|\operatorname{ran}(\lambda\gamma) \cap \operatorname{dom}\mu| < r$). Since $\operatorname{dom}\gamma = Y$, the first condition implies $r(\lambda. \operatorname{id}_Y .\mu) < r$ and so $\lambda. \operatorname{id}_Y .\mu = 0$ in NI_{r+1}/NI_r. Also, if $\lambda\gamma \neq 0$ then, since r is finite and $\lambda \in DI_r$, we deduce that $\operatorname{ran}\lambda = \operatorname{dom}\gamma$ and thus $\operatorname{ran}(\lambda\gamma) = Y$. Hence, the second condition implies $|Y \cap \operatorname{dom}\mu| < r$ and we again obtain $\lambda. \operatorname{id}_Y .\mu = 0$. Conversely, if $\lambda. \operatorname{id}_Y .\mu = 0$, then $|\operatorname{ran}\lambda \cap Y| < r$ or ($\operatorname{ran}\lambda = Y$ and $|Y \cap \operatorname{dom}\mu| < r$) and, in both cases, it follows that $\lambda\gamma\mu = 0$.

We now see, as a special case, that Theorem 5 describes the lattice of congruences on $NI(X)$ for finite X: compare the comment in [**1**, Vol. 2, p. 247] and in [**7**, p. 5]. However, the argument below does not require any knowledge of the congruences on arbitrary completely 0–simple semigroups.

COROLLARY 1. *For any finite set X with at least six elements, the lattice of congruences on $NI(X)$ forms a chain.*

PROOF: Let ρ_1 and ρ_2 be distinct congruences on $NI(X)$, neither of which equals the identity or the universal congruence on $NI(X)$, and write $r_i = \eta(\rho_i)$ for $i = 1, 2$. Then $\rho_i = \sigma_i^+$ for some (unique) congruence σ_i on NI_{r_i+1}/NI_{r_i}. If $r_1 < r_2$ then $NI_{r_1} \subsetneq NI_{r_2}$ and

$$\sigma_1 \cap (DI_{r_1} \times DI_{r_1}) \subsetneq NI_{r_2} \times NI_{r_2},$$

from which we deduce that $\rho_1 \subseteq \rho_2$. Suppose $r_1 = r_2 = r$, say. By Lemma 7, σ_1 is determined by some $N_1 \lhd G(Y)$ and σ_2 by some $N_2 \lhd G(Y)$ where $|Y| = r$ (note: the same Y can be used). Since the normal subgroups of $G(Y)$ form a chain, it follows from Lemma 7 that $\sigma_1 \subseteq \sigma_2$ or $\sigma_2 \subseteq \sigma_1$, and hence that $\rho_1 \subseteq \rho_2$ or $\rho_2 \subseteq \rho_1$. ∎

EXAMPLE. Suppose $|X| = 4$, an even integer. The normal subgroups of S_4 form a chain:

$$\{(1)\} \lhd K_4 \lhd A_4 \lhd S_4,$$

and hence there are four non-universal congruences $\sigma_{41}, \sigma_{42}, \sigma_{43}, \sigma_{44}$ on NI_5/NI_4. In turn, there are four congruences $\rho_{4i} = \sigma_{4i}^+$ on $NI(X)$. In fact, since $NI_5/NI_4 = DI_4 \cup \{0\}$ and $DI_4 = S_4$, each σ_{4i} is a congruence on a group with 0 adjoined and so the σ_{4i}–classes are simply the cosets of the corresponding normal subgroup of S_4 together with $\{0\}$ by itself. In particular, σ_{41} is the identity congruence on S_4^0 and so

$$\rho_{41} = \mathrm{id}_{NI(X)} \cup (NI_4 \times NI_4).$$

Similarly, there are exactly three non-universal congruences $\rho_{31} \subseteq \rho_{32} \subseteq \rho_{33}$ on $NI(X)$ corresponding to three congruences $\sigma_{31} \subseteq \sigma_{32} \subseteq \sigma_{33}$ on NI_4/NI_3 which are determined by the three normal subgroups of S_3. In particular,

$$\sigma_{33}^+ = \mathrm{id}_{NI(X)} \cup \left[\sigma_{33} \cap (DI_3 \times DI_3)\right] \cup (NI_3 \times NI_3),$$

which is properly contained in ρ_{41} as expected. In this way, we obtain the chain of non-universal congruences on $NI(X)$:

$$\mathrm{id}_{NI(X)} \subsetneq \rho_{21} \subsetneq \rho_{31} \subsetneq \rho_{32} \subsetneq \rho_{33} \subsetneq \rho_{41} \subsetneq \rho_{42} \subsetneq \rho_{43} \subsetneq \rho_{44}.$$

4. INFINITE PRIMARY RANK

Henceforth, X is an infinite set with cardinal k, and we write $Y = A \,\dot{\cup}\, B$ if $A \cap B = \emptyset$. Recall our comment before Theorem 3 and, in particular, the fact that if

$$I_k = \{\alpha \in I(X) : r(\alpha) < k\}$$

then $I_k \subseteq NI(X)$. Therefore, if ρ is a congruence on $NI(X)$ then

$$\rho = \left[\rho \cap (I_k \times I_k)\right] \cup \left[\rho \cap (DI_k \times DI_k)\right]. \tag{3}$$

Clearly, $\rho \cap (I_k \times I_k)$ is a congruence on the semigroup I_k. To say something about the other intersection in (3), we need some notation (see [**9**, Section 3]). First recall our convention: $x\alpha = \emptyset$ if and only if $x \notin \operatorname{dom}\alpha$. Now, for each $\alpha, \beta \in P(X)$ and $n \geqslant \aleph_0$, let

$$D(\alpha,\beta) = \{x \in X : x\alpha \neq x\beta\}, \quad \mathrm{dr}\,(\alpha,\beta) = \max\Big(\big|D(\alpha,\beta)\alpha\big|, \big|D(\alpha,\beta)\beta\big|\Big)$$

$$\Delta_n = \{(\alpha,\beta) \in P(X) \times P(X) : \mathrm{dr}\,(\alpha,\beta) < n\}$$

and note that, by [**7**, Theorem 3.1], each Δ_n is a congruence on $P(X)$. Hence, its *reduction*:

$$\delta_n = \left[\Delta_n \cap (Q_k \times Q_k)\right] \cup \{(0,0)\},$$

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to the Rees factor semigroup:

$$Q_k = NI_{k'}/NI_k = DI_k \cup \{0\}$$

is a congruence on Q_k (see [**6**, p. 313]). In fact, we have the following result [**6**, Theorem 18].

Theorem 6. *If $|X| = k \geqslant \aleph_0$ then every non-identity, non-universal congruence on Q_k equals δ_n for some n satisfying $\aleph_0 \leqslant n \leqslant k$.*

Clearly, if ρ is a congruence on $NI(X)$ then

$$\rho_k = \rho \cap (DI_k \times DI_k) \cup \{(0,0)\}$$

is an equivalence on Q_k. To show it is a congruence on Q_k, we need the following result [**9**, Lemma 3.4].

Lemma 8. *If $\alpha, \beta \in P(X)$ and $\mathrm{dr}\,(\alpha,\beta) = \xi \geqslant \aleph_0$ then there exists $Y \subseteq D(\alpha,\beta)$ such that $Y\alpha \cap Y\beta = \emptyset$ and $\max(|Y\alpha|, |Y\beta|) = \xi$.*

Lemma 9. *If ρ is a non-identity, non-universal congruence on $NI(X)$ then ρ_k is a congruence on Q_k.*

Proof: Suppose $(\alpha,\beta) \in \rho_k$ and $\mu \in Q_k$ is non-zero. If $r(\alpha\mu) < k$ and $r(\beta\mu) = k$ then the cardinal of $(\mathrm{ran}\,\beta \cap \mathrm{dom}\,\mu) \cap \mathrm{ran}\,\alpha$ is less than k, so

$$\big|(\mathrm{ran}\,\beta \cap \mathrm{dom}\,\mu) \setminus \mathrm{ran}\,\alpha\big| = k.$$

Therefore, $|\,\mathrm{ran}\,\beta \setminus \mathrm{ran}\,\alpha| = k$; and, if $(\mathrm{ran}\,\beta \setminus \mathrm{ran}\,\alpha)\beta^{-1} = \{x_i\}$, then $x_i\beta \neq x_i\alpha$ for each i (it is possible some $x_i \notin \mathrm{dom}\,\alpha$). In other words, $\mathrm{dr}\,(\alpha,\beta) = k$ and so, by Lemma 8, $Y\alpha \cap Y\beta = \emptyset$ for some $Y \subseteq D(\alpha,\beta)$ with $\max(|Y\alpha|, |Y\beta|) = k$. Without loss of generality, suppose $|Y\alpha| = k$ and choose disjoint sets $U, V \subseteq Y \cap \mathrm{dom}\,\alpha$ with cardinal k (possible since α is injective). Then $\mathrm{id}_U \in NI(X)$ (since $|X \setminus U| = k$), and so $\alpha \sim \beta$ implies $\mathrm{id}_U\,\alpha \sim \mathrm{id}_U\,\beta$. Let $U = \{u_i\}$, and suppose $\gamma \in I(X)$ has domain $\{u_i\alpha\}$ and $\gamma : u_i\alpha \mapsto u_i$ for each i. Then $g(\gamma) \geqslant d(\alpha) = k$ and $d(\gamma) = |X \setminus U| = k$, so $\gamma \in NI(X)$. Therefore, $\mathrm{id}_U = \mathrm{id}_U\,\alpha\gamma \sim \mathrm{id}_U\,\beta\gamma = \emptyset$ (the latter equality holds since $U \subseteq Y$ implies $U\beta \cap U\alpha = \emptyset$). In other words, an element of $NI(X)$ with rank k is ρ–equivalent to $\emptyset$, so $\eta(\rho) = k'$ and ρ is universal, a contradiction. In effect, this shows $r(\alpha\mu) < k$ if and only if $r(\beta\mu) < k$; that is, ρ_k is right compatible on Q_k.

Similarly, suppose $r(\lambda\alpha) < k$ and $r(\lambda\beta) = k$ for some non-zero $\lambda \in Q_k$. This implies $r(\alpha^{-1}\lambda^{-1}) < k$ and $r(\beta^{-1}\lambda^{-1}) = k$, where $\alpha^{-1} \sim \beta^{-1}$ and $\lambda^{-1} \in NI(X)$, contradicting what we have just shown. Therefore, $r(\lambda\alpha) < k$ if and only if $r(\lambda\beta) < k$, and so ρ_k is left compatible on Q_k. ∎

In view of (3), to describe all congruences on $NI(X)$, we need to know all congruences on I_k. To determine the latter, we recall Liber's Theorem regarding the congruences on $I(X)$ (compare [**7**, Lemma 3.10]). For convenience, we let Δ_1 denote the identity congruence on $I(X)$. Also, if ρ is a congruence on $I(X)$, we let $\eta(\rho)$ denote the least cardinal greater than $r(\alpha)$ for each α such that $(\alpha, \emptyset) \in \rho$ (compare the equivalent definition for $NI(X)$ before Theorem 4; and recall that the cardinals are naturally well-ordered: see [**5**, Theorem 7.2.6]).

LIBER'S THEOREM. *Suppose $|X| = k \geqslant \aleph_0$. If ρ is a congruence on $I(X)$ for which $\eta(\rho)$ is infinite then*

$$\rho = I^*_{\eta_1} \cup [\Delta_{\xi_1} \cap I^*_{\eta_2}] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap I^*_{\eta_r}] \cup \Delta_{\xi_r} \tag{4}$$

where $\eta_1 = \eta(\rho)$ and the cardinals ξ_i, η_i form a sequence:

$$\xi_r < \cdots < \xi_1 \leqslant \eta_1 < \cdots < \eta_r \leqslant k,$$

in which every term is infinite, except possibly ξ_r which equals 1 if it is finite.

LEMMA 10. *If σ is a congruence on I_k and $\sigma^\circ = \sigma \cup \mathrm{id}_{DI_k}$ then σ° is a congruence on $I(X)$.*

PROOF: Clearly, σ° is an equivalence on $I(X)$. To show it is right compatible on $I(X)$, suppose $(\alpha, \beta) \in \sigma$ and $\mu \in DI_k$. Then $r(\alpha\mu) < k$ and $r(\beta\mu) < k$. Let $\mu' \in I(X)$ be the restriction of μ to $(\operatorname{ran}\alpha \cup \operatorname{ran}\beta) \cap \operatorname{dom}\mu$. Then $\mu' \in I_k$, since $r(\alpha\mu) + r(\beta\mu) < k$; and, since $\alpha\mu = \alpha\mu'$ and $\beta\mu = \beta\mu'$, we conclude that $(\alpha\mu, \beta\mu) \in \sigma$.

Similarly, if $r(\lambda\alpha) < k$ and $r(\lambda\beta) < k$ for some $\lambda \in DI_k$, we let $\lambda' \in I(X)$ have domain $Z = (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)\lambda^{-1}$ and satisfy:

$$z\lambda' = z\lambda, \quad \text{for all } z \in (\operatorname{dom}\alpha \cup \operatorname{dom}\beta)\lambda^{-1}.$$

Then $|Z| < k$ (since λ is injective and $\alpha, \beta \in I_k$) and hence $\lambda' \in I_k$. Since $\lambda'\alpha = \lambda\alpha$ and $\lambda'\beta = \lambda\beta$, we conclude that $(\lambda\alpha, \lambda\beta) \in \sigma$ and hence σ is left compatible on $I(X)$. ∎

THEOREM 7. *Suppose $|X| = k \geqslant \aleph_0$. If σ is a congruence on I_k for which $\eta(\sigma)$ is infinite then*

$$\sigma = I^*_{\eta_1} \cup [\Delta_{\xi_1} \cap I^*_{\eta_2}] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap I^*_{\eta_r}] \tag{5}$$

where $\eta_1 = \eta(\sigma)$ and the cardinals ξ_i, η_i form a sequence:

$$\xi_{r-1} < \cdots < \xi_1 \leqslant \eta_1 < \cdots < \eta_r \leqslant k,$$

in which every term is infinite.

PROOF: Suppose σ is a congruence on I_k for which $\eta(\sigma) \geqslant \aleph_0$: that is, there exists $(\alpha, \emptyset) \in \sigma$ with $r(\alpha) \geqslant \aleph_0$. Then σ° is a congruence on $I(X)$ for which $\eta(\sigma^\circ) \geqslant \aleph_0$. Hence

$$\sigma \cup \mathrm{id}_{DI_k} = I^*_{\eta_1} \cup [\Delta_{\xi_1} \cap I^*_{\eta_2}] \cup \cdots \cup [\Delta_{\xi_{s-1}} \cap I^*_{\eta_s}] \cup \Delta_{\xi_s} \tag{6}$$

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where $\eta_1 = \eta(\sigma^\circ) = \eta(\sigma)$ and the cardinals ξ_i, η_i form a sequence:

$$\xi_s < \cdots < \xi_1 \leqslant \eta_1 < \cdots < \eta_s \leqslant k,$$

in which every term is infinite, except possibly ξ_s which equals 1 if it is finite. Clearly, $I(X)$ contains elements (in fact, idempotents) with rank k which differ in at least one place. Therefore, ξ_s must equal 1: otherwise, Δ_{ξ_s} in the right-hand side of (6) contains a pair of distinct elements of DI_k which does not appear on the left-hand side of (6). Consequently, (6) implies (5) where $r = s$. ∎

We need two more results before we can describe all congruences on $NI(X)$: these are comparable with [1, Lemmas 10.62(i) and 10.63(i)].

Lemma 11. *If the ranks of $\alpha, \beta \in NI(X)$ are not equal, and at least one of them is infinite, then* $\operatorname{dr}(\alpha, \beta) = \max(r(\alpha), r(\beta))$.

Proof: Suppose the condition holds and assume $r(\alpha) = r > s = r(\beta)$. Then, by supposition, r is infinite and $|X\alpha \cap X\beta| \leqslant s < r$, so $r(\alpha) = |X\alpha \setminus X\beta|$. If $X\alpha \setminus X\beta = \{x_i\alpha\}$, then $x_i \in D(\alpha, \beta)$ for each i, so

$$\operatorname{dr}(\alpha, \beta) \geqslant |I| = r(\alpha) = \max(r(\alpha), r(\beta)).$$

Since $\operatorname{dr}(\alpha, \beta) \leqslant r(\alpha)$ is always true, this gives the desired result. ∎

Lemma 12. *Suppose η_1, η_2 are infinite cardinals satisfying $\eta_1 \leqslant \eta_2$. If $\alpha, \beta \in NI(X)$ satisfy $r(\alpha) = r(\beta) = \eta_2$ and $\aleph_0 \leqslant \operatorname{dr}(\alpha, \beta) = \xi \leqslant \eta_1$, then there exists $\lambda \in NI(X)$ such that $r(\lambda\alpha) = r(\lambda\beta) = \eta_1$ and $\operatorname{dr}(\lambda\alpha, \lambda\beta) = \xi$.*

Proof: Let $D = D(\alpha, \beta)$ and, without loss of generality, suppose $|D\alpha| = \xi$ and $C = D\alpha \cup D\beta$. Then $\operatorname{ran}\alpha \setminus C = \operatorname{ran}\beta \setminus C = \{e_j\}$ say, and, for each j, there exists $r_j \in \operatorname{dom}\alpha \cap \operatorname{dom}\beta$ such that $r_j\alpha = e_j = r_j\beta$ (this is true by the definition of $D(\alpha, \beta)$ and our convention: $x\alpha = \emptyset$ if and only if $x \notin \operatorname{dom}\alpha$, at the start of this section). By Lemma 8, we can assume (again, without loss of generality) that there exists $Y = \{y_i\} \subseteq D \cap \operatorname{dom}\alpha$ such that $|Y\alpha| = \xi$ and $Y\alpha \cap Y\beta = \emptyset$. Since $g(\alpha) = k$, the identity transformation, λ say, on $Y \cup \{r_j\}$ belongs to $NI(X)$ and

$$\lambda\alpha = \begin{pmatrix} y_i & r_j \\ c_i & e_j \end{pmatrix}, \quad \lambda\beta = \begin{pmatrix} y_i & r_j \\ d_i & e_j \end{pmatrix},$$

where d_i may not exist for some i (that is, when $y_i \notin \operatorname{dom}\beta$). Since $|D\alpha| = \xi \geqslant |D\beta|$, we know $|C| = \xi = |Y\alpha| = |I|$. Hence $r(\lambda\alpha) = r(\alpha) = \xi + |J| = \eta_2 \geqslant \eta_1$ (by supposition). If $|J| = \eta_2$, choose $P \subseteq J$ with cardinal η_1, and let λ' be the identity on $\{y_i\} \cup \{r_p\}$. Then $\lambda' \in NI(X)$ since $g(\lambda') \geqslant g(\alpha) = k$, and we have

$$\lambda'\lambda\alpha = \begin{pmatrix} y_i & r_p \\ c_i & e_p \end{pmatrix}, \quad \lambda'\lambda\beta = \begin{pmatrix} y_i & r_p \\ d_i & e_p \end{pmatrix}.$$

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Since $\{c_i\} \cap \{d_i\} = \emptyset$, these are elements of $NI(X)$ with rank η_1 and difference rank ξ, as required. On the other hand, if $\xi = \eta_2$ (hence $\eta_1 = \eta_2$) then $\lambda\alpha$ and $\lambda\beta$ are elements of $NI(X)$ with rank η_1 and difference rank ξ. $\square$

THEOREM 8. *Suppose $|X| = k \geqslant \aleph_0$. If ρ is a non-universal congruence on $NI(X)$ for which $\eta(\rho)$ is infinite then*

(7) $$\rho = I^*_{\eta_1} \cup [\Delta_{\xi_1} \cap I^*_{\eta_2}] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap I^*_{\eta_r}] \cup [\Delta_n \cap (DI_k \times DI_k)]$$

where $\eta_1 = \eta(\rho)$ and the cardinals ξ_i, η_i form a sequence:

$$n \leqslant \xi_{r-1} < \cdots < \xi_1 \leqslant \eta_1 < \cdots < \eta_r \leqslant k,$$

in which ξ_{r-1} is infinite, either $n = 1$ or n is infinite, and if $n \geqslant \aleph_0$ then $\eta_r = k$.

Conversely, if ρ is a relation on $NI(X)$ defined as in (7) for a sequence of cardinals with the above properties, then ρ is a non-universal congruence on $NI(X)$.

PROOF: If $\sigma = \rho \cap (I_k \times I_k)$, then $\eta(\sigma) \geqslant \aleph_0$ (since $\sigma \subseteq \rho$ and $\eta(\rho) \geqslant \aleph_0$). Hence, we know there are cardinals

$$\aleph_0 \leqslant \xi_{r-1} < \cdots < \xi_1 \leqslant \eta_1 < \cdots < \eta_r \leqslant k$$

such that

(8) $$\rho \cap (I_k \times I_k) = I^*_{\eta_1} \cup [\Delta_{\xi_1} \cap I^*_{\eta_2}] \cup \cdots \cup [\Delta_{\xi_{r-2}} \cap I^*_{\eta_{r-1}}] \cup [\Delta_{\xi_{r-1}} \cap I^*_{\eta_r}]$$

and we also know

(9) $$\rho \cap (DI_k \times DI_k) = \Delta_n \cap (DI_k \times DI_k)$$

where $n = 1$ or $\aleph_0 \leqslant n \leqslant k$. Taking the union of (8) and (9), we find

(10) $$\rho = I^*_{\eta_1} \cup [\Delta_{\xi_1} \cap I^*_{\eta_2}] \cup \cdots \cup [\Delta_{\xi_{r-1}} \cap I^*_{\eta_r}] \cup \left[\Delta_n \cap (DI_k \times DI_k)\right].$$

If $n > \eta_1$, then ρ contains a pair of elements with rank k which differ at η_1 places and, from this, we can find a pair $(\alpha, \emptyset) \in \rho$ where $r(\alpha) = \eta_1$, contradicting the choice of η_1. Hence, $n \leqslant \eta_1$ and, if $n \neq 1$, then $\aleph_0 \leqslant n \leqslant k$. If $n > \xi_{r-1}$ then (9) implies that ρ contains each pair of elements with rank k which differ at $\xi_{r-1} < \eta_{r-1} < k$ places. Thus, by Lemma 12, there exists a pair of elements in ρ with rank $\eta_{r-1} < \eta_r$ which differ at ξ_{r-1} places, contradicting the expression for $\rho \cap (I_k \times I_k)$ in (8). Hence, $n \leqslant \xi_{r-1}$.

Now, if $n \geqslant \aleph_0$ then (9) implies that ρ contains all pairs of elements with rank k which differ at less than n places. In particular, if $X = A \dot\cup B \dot\cup C$, where $|A| = |B| = k$ and $|C| < n$, then $(\mathrm{id}_{A\cup C}, \mathrm{id}_A) \in \rho$. Consequently, if $\eta_r < k$ and $Y \subseteq A$ has cardinal η_r, then $\mathrm{id}_{Y\cup C} \in NI(X)$ and

$$(\mathrm{id}_{Y\cup C}, \mathrm{id}_Y) = (\mathrm{id}_{Y\cup C} \cdot \mathrm{id}_{A\cup C}, \mathrm{id}_{Y\cup C} \cdot \mathrm{id}_A) \in \rho.$$

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That is, ρ contains a pair of distinct elements with rank $\eta_r < k$ which differ at less than $n \leqslant \xi_{r-1}$ places. Since this again contradicts the expression for $\rho \cap (I_k \times I_k)$ in (8), we conclude that if $n \geqslant \aleph_0$ then $\eta_r = k$.

Conversely, suppose ρ is defined as in (7) and its associated cardinals have the stated properties. We now follow the first part of the proof of [**1**, Vol. 2, Theorem 10.72]. Clearly, ρ is reflexive and symmetric. To show it is transitive, first note that $i < j$ implies $\xi_j < \xi_i$ and so $\Delta_{\xi_j} \subsetneq \Delta_{\xi_i}$, and likewise $\eta_i < \eta_j$ implies $I^*_{\eta_i} \subsetneq I^*_{\eta_j}$.

For convenience, we write $\xi_0 = k'$, so that $I^*_{\eta_1} = \Delta_{\xi_0} \cap I^*_{\eta_1}$. Now suppose $(\alpha, \beta) \in \Delta_{\xi_i} \cap I^*_{\eta_{i+1}}$ and $(\beta, \gamma) \in \Delta_{\xi_j} \cap I^*_{\eta_{j+1}}$, where $i < j$. Assume $r(\alpha) \neq r(\beta)$. If both these cardinals are finite then $(\alpha, \beta) \in I^*_{\eta_1}$ (since $\eta_1 \geqslant \aleph_0$); and, if at least one of them is infinite, then Lemma 11 implies

$$\max\bigl(r(\alpha), r(\beta)\bigr) = \operatorname{dr}(\alpha, \beta) < \xi_i \leqslant \eta_1$$

and so $(\alpha, \beta) \in I^*_{\eta_1}$. Similarly, if $r(\beta) \neq r(\gamma)$ then $(\beta, \gamma) \in I^*_{\eta_1}$, and clearly the same is true if $r(\beta) = r(\gamma)$ since we already know $r(\beta) < \eta_1$. Therefore, in all cases, $r(\alpha) \neq r(\beta)$ implies $(\alpha, \gamma) \in I^*_{\eta_1}$. Hence, we may assume that $r(\alpha) = r(\beta) = r(\gamma)$. But, since $r(\alpha) < \eta_{i+1}$, we then deduce that $(\alpha, \gamma) \in \Delta_{\xi_i} \cap I^*_{\eta_{i+1}}$. Finally, since both components of each pair in $\Delta_n \cap (DI_k \times DI_k)$ have rank $k \geqslant \eta_r$, it follows that ρ is transitive.

Now, each of the terms in ρ corresponding to $\eta_1, \ldots, \eta_r$ is a compatible relation on $NI(X)$. Suppose $n \geqslant \aleph_0$ (hence $\eta_r = k$) and let $(\alpha, \beta) \in \Delta_n \cap (DI_k \times DI_k) = \sigma$, say. If $\mu \in NI(X)$ and $r(\alpha\mu) = r(\beta\mu) = k$ then $(\alpha\mu, \beta\mu) \in \sigma$. On the other hand, if $r(\alpha\mu) = k > r(\beta\mu)$ then Lemma 11 gives the contradiction:

$$k = \max\bigl(r(\alpha\mu), r(\beta\mu)\bigr) = \operatorname{dr}(\alpha\mu, \beta\mu) < n < k.$$

Therefore, the only other possibility is that both $r(\alpha\mu)$ and $r(\beta\mu)$ are less than $k = \eta_r$: that is, $(\alpha\mu, \beta\mu) \in I^*_{\eta_r}$ and $\operatorname{dr}(\alpha\mu, \beta\mu) < n \leqslant \xi_{r-1}$, so $(\alpha\mu, \beta\mu) \in \Delta_{\xi_{r-1}} \cap I^*_{\eta_r}$. Similarly, ρ is left compatible on $NI(X)$, and so it is a congruence on $NI(X)$. ∎

We now deduce part of [**4**, Theorem 4.10], and prove a little more.

COROLLARY 2. *Suppose $|X| = k \geqslant \aleph_0$ and write*

$$\Delta_k^+ = \Delta_k \cap \bigl[NI(X) \times NI(X)\bigr].$$

Then Δ_k^+ is the only maximal congruence on $NI(X)$, and hence $NI(X)/\Delta_k^+$ is a congruence-free nilpotent-generated inverse semigroup.

PROOF: Since $NI(X)$ is nilpotent-generated and inverse (by Theorem 1), and Δ_k^+ is a congruence on $NI(X)$, it follows that $NI(X)/\Delta_k^+$ is also nilpotent-generated and inverse.

Suppose $\Delta_k^+ \subseteq \rho$ for some non-universal congruence on $NI(X)$. Now, $\eta(\rho)$ equals the least cardinal greater than $r(\alpha)$ for each $\alpha \in NI(X)$ such that $(\alpha, \emptyset) \in \rho$. But

$(\mathrm{id}_A, \emptyset) \in \Delta_k^+ \subseteq \rho$ for each $A \subsetneq X$ with cardinal less than k (in particular, for infinite A) and so $\eta(\rho) \geqslant \aleph_0$. Therefore, ρ has the form displayed in (7). Clearly, $(\alpha, \emptyset) \in \Delta_k^+ \subseteq \rho$ for each $\alpha \in I_k$, so $\eta_1 = k$. Moreover, if $X = A \dot\cup B \dot\cup C$ where $|A| = |C| = k$ and $|B| < k$, then

$$(\mathrm{id}_{A\cup B}, \mathrm{id}_A) \in \Delta_k \cap [DI_k \times DI_k],$$

and it follows that $n \geqslant k$. Since $I_k^* \subseteq \Delta_k^+$, this implies that each term in (7) is contained in Δ_k^+, hence $\rho \subseteq \Delta_k^+$ and equality follows. Finally, suppose ρ is a maximal congruence on $NI(X)$ for which there exists $(\alpha, \beta) \in \rho$ with $\mathrm{dr}\,(\alpha, \beta) = k$. Then $r(\alpha) = r(\beta) = k$ (by the definition of 'difference rank'). Since such pairs (α, β) do not belong to the congruences described in Theorem 5, we deduce that $\eta(\rho) \geqslant \aleph_0$. However, then (7) implies that $n = k'$, and so we have a contradiction:

$$k' \leqslant \xi_{r-1} < \cdots < \xi_1 \leqslant \eta_1 < \cdots < \eta_r \leqslant k.$$

Thus, $\mathrm{dr}\,(\alpha, \beta) < k$ for all $(\alpha, \beta) \in \rho$, hence $\rho \subseteq \Delta_k^+$, and equality follows by the maximality of ρ and the fact that Δ_k^+ is non-universal. ∎

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