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SPACES OF COMMUTING ELEMENTS IN SU(2)

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Abstract Based on the work of Adem and Cohen, this note describes an explicit stable decomposition of the space of commuting n-tuples in SU(2) as a wedge of indecomposable summands.

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1. Introduction

We shall write $\mathbb{Z}^T$ for the free abelian group and $F(U)$ for the free group generated, respectively, by finite sets T and U. Following Adem and Cohen [**1**], we study the stable homotopy type of the space of group homomorphisms $\mathrm{Hom}(\mathbb{Z}^T \times F(U), G)$ from the finitely presented discrete group $\mathbb{Z}^T \times F(U)$ to the compact connected Lie group $G = \mathrm{SU}(2)$. More generally, for a finite family $(U_i)_{i\in I}$ of finite sets we consider the space $\mathrm{Hom}(\prod_{i\in I} F(U_i), G)$ (topologized as a subspace of $\prod_{i\in I} \mathrm{map}(U_i, G)$).

Commutativity in the group SU(2) is easy to understand. The centre $Z(G)$ consists of the two elements ± 1. A non-central element $g \in G$ has distinct eigenvalues α, β (with $\alpha\beta = 1$) and decomposes $\mathbb{C}^2$ as an orthogonal sum of one-dimensional eigenspaces $E_\alpha \oplus E_\beta$, and the centralizer $Z(g) = \{h \in G \mid gh = hg\}$ is the one-dimensional maximal torus consisting of those matrices which preserve E_α (and E_β).

The space of maximal tori in G is naturally identified with the real projective plane $P(\mathfrak{g})$ of the Lie algebra, $\mathfrak{g}$, of G: a circle subgroup L corresponds to its Lie algebra $\mathfrak{l} \in P(\mathfrak{g})$. The Hopf line bundle over $P(\mathfrak{g})$, constructed as a sub-bundle of the trivial bundle $P(\mathfrak{g}) \times \mathfrak{g}$, will be denoted by λ. We write λ^T for the real vector bundle $\lambda^T = \lambda \otimes \mathbb{R}^T$ of dimension $\#T$.

The Thom space of a real vector bundle ξ over a compact Hausdorff space X is written as X^ξ and the sphere bundle is denoted by $S(\xi)$. A superscript '+' is used to indicate the one-point compactification of a locally compact Hausdorff space with basepoint at infinity, and a subscript '+' is used to indicate adjunction of a disjoint basepoint. When we say that two pointed spaces X and Y are stably equivalent, we shall mean that there is a homotopy equivalence $\Sigma^k X \to \Sigma^k Y$ of their k-fold suspensions for some $k \geqslant 0$.

Definition 1.1. Given finite sets T and U, we define $C(T,U)$ to be the homotopy cofibre of the composition

$$f\colon S(\lambda^T)^{p^*\lambda^U} \to P(\mathfrak{g})^{\lambda^U} \hookrightarrow (P(\mathfrak{g}) \times \mathfrak{g}^U)^+ \to (\mathfrak{g}^U)^+$$

of the maps induced, in order, by the projection $p\colon S(\lambda^T) \to P(\mathfrak{g})$, the inclusion of λ^U in the trivial bundle $P(\mathfrak{g}) \times \mathfrak{g}^U$ and the projection to the second factor. In particular, $C(T,\emptyset)$ is the join $S^0 * S(\lambda^T)$ (with basepoint at the basepoint of S^0).

These spaces appear as the wedge summands in our main theorem.

Theorem 1.2. *Let V be a finite set decomposed as a disjoint union $V = T \sqcup U$. There is then a natural stable splitting*

$$\mathrm{Hom}(\mathbb{Z}^T \times F(U), \mathrm{SU}(2))_+ \simeq \bigvee_{W \subseteq V} C(T \cap W, U \cap W).$$

The decomposition is natural in the precise sense that it can be realized as a K-equivariant stable splitting, where $K = \mathrm{Aut}(G) \times \mathfrak{S}(T) \times \mathfrak{S}(U)$ is the product of the group of automorphisms of G $(= \mathrm{SU}(2))$ and the permutation groups of T and U.

It is an elementary exercise to establish the following (non-equivariant) description of the summands in terms of their stably indecomposable factors, which turn out to be suspensions of S^0 and

M, the Thom space of the Hopf line bundle over the real projective line,

N, the Thom space of the tangent bundle of the real projective plane P,

N^*, the Thom space of the double 2λ of the Hopf line bundle λ over P.

The two-dimensional space M is thus a mod 2 Moore space, and the four-dimensional spaces N and N^* are S-dual (by a duality map $N \wedge N^* \to S^6$). (In the traditional notation M is RP^2, N^* is RP^4/RP^2 and ΣN is RP^5/RP^2.) The result will be expressed in terms of spaces P_n defined for $n \geqslant 0$ by

$$P_n = \begin{cases} S^n \vee \Sigma^n M & \text{if } n \equiv 0 \pmod 4, \\ \Sigma^{n-1} M \vee S^{n+2} & \text{if } n \equiv 1 \pmod 4, \\ \Sigma^{n-2} N^* & \text{if } n \equiv 2 \pmod 4, \\ \Sigma^{n-2} N & \text{if } n \equiv 3 \pmod 4. \end{cases}$$

Proposition 1.3. *Let $\#T = l$, $\#U = m$. The space $C(T,U)$ is stably equivalent to*

S^{3m}	*if* $l = 0$,
$(S^{m+1} \vee S^{m+3}) \vee S^{3m}$	*if* $l = 1,\ m > 1$,
$(S^{m+2-\epsilon} \vee S^{m+3-\epsilon} \vee \Sigma^{m+1+\epsilon} M) \vee S^{3m}$	*if* $l = 2,\ m > 1,\ \epsilon = (-1)^m$,
$\Sigma M \vee P_{l+m}$	*if* $l > 2,\ m = 0$ *or* $m = 1$,
$(\Sigma P_m \vee P_{l+m}) \vee S^{3m}$	*if* $l > 2,\ m > 1$,

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and in the remaining low-dimensional cases to

$$\begin{array}{ll} S^3 & \textit{if } (l,m)=(1,0), \\ S^2 \vee \Sigma^2 M & \textit{if } (l,m)=(1,1) \textit{ or } (2,0), \\ \Sigma M \vee \Sigma N & \textit{if } (l,m)=(2,1). \end{array}$$

The existence of stable splittings of this general type was established in Theorems 1.6 and 6.6 in [**1**]. However, the specific splittings described above are inconsistent with the homology calculations stated there as Theorems 1.4 and 4.12. I am grateful to the authors for confirmation that those calculations are incorrect; they have published a correction in [**2**].

Example 1.4. There are stable decompositions

$$\mathrm{Hom}(\mathbb{Z}^2, \mathrm{SU}(2))_+ \simeq S^0 \vee 2(S^3) \vee (S^2 \vee \Sigma^2 RP^2)$$

and

$$\mathrm{Hom}(\mathbb{Z}^3, \mathrm{SU}(2))_+ \simeq S^0 \vee 3(S^3) \vee 3(S^2 \vee \Sigma^2 RP^2) \vee (\Sigma RP^2 \vee RP^5/RP^2).$$

Section 2 contains an account of a general equivariant stable splitting theorem. Theorem 1.2 is proved in § 3, and an outline proof of Proposition 1.3 is given in the final section (§ 4). The methods depend very much on the special feature of the group SU(2), shared trivially with U(1), that any two commuting elements lie in a circle subgroup. The space $\mathrm{Hom}(\mathbb{Z}^T, \mathrm{U}(1))$ is, of course, a torus of rank $l = \#T$ and the stable decomposition of $\mathrm{Hom}(\mathbb{Z}^T, \mathrm{U}(1))_+$ as a wedge of 2^l spheres is elementary. But even in this case the $\mathfrak{S}(T)$-equivariant decomposition is of interest and was used in [**5**] to give a new proof of Miller's stable splitting [**6**] of the unitary group $\mathrm{U}(l)$.

2. An equivariant stable splitting

Consider a finite set V of cardinality $\#V = n$ and a finite set E of 2-element subsets of V: (V, E) is thus a finite graph with n vertices. The product $\mathrm{Aut}(G) \times \mathrm{Aut}(V, E)$ of the automorphism group of G and the subgroup of $\mathfrak{S}(V)$ preserving E will be abbreviated to K. For a subset $W \subseteq V$, we write E_W for the set of edges $\{u, v\}$ in E with both vertices u, v in W. Let $X(V, E)$ be the K-space of maps $x\colon V \to G$ such that $x(u)x(v) = x(v)x(u)$ for all $\{u, v\} \in E$ (topologized as a subspace of the product G^V). The *support* of such a map $x\colon V \to G$ is the subset $\mathrm{supp}(x) = \{v \in V \mid x(v) \neq 1\}$, and the cardinality of the support filters $X(V, E)$ by the closed K-subspaces

$$X^{(k)}(V, E) = \{x \in X(V, E) \mid \#\,\mathrm{supp}(x) \leqslant k\} \quad (0 \leqslant k \leqslant n).$$

The group $G = \mathrm{SU}(2)$ is a real affine subvariety of the four-dimensional affine space $\mathbb{R} \oplus \mathfrak{g}$. Each of the spaces $X^{(k)}(V, E)$ is a real (not necessarily irreducible) subvariety of $(\mathbb{R} \oplus \mathfrak{g})^V$ that is invariant under the linear action of the compact Lie group K and, hence, is a compact K-ENR (Euclidean neighbourhood retract). It follows that each inclusion $X^{(k-1)}(V, E) \subseteq X^{(k)}(V, E)$ is a closed K-cofibration.

Let $A(V,E)$ be the top-filtration open K-subspace $\{x \in X(V,E) \mid \operatorname{supp}(x) = V\}$ of $X(V,E)$. There is a homeomorphism

$$X^{(k)}(V,E)/X^{(k-1)}(V,E) = \bigvee_{W \subseteq V\,:\,\#W=k} A(W,E_W)^+.$$

The non-equivariant version of the following splitting theorem can be found in [**1**] as a special case of Theorem 6.6.

Theorem 2.1. *Let (V,E) be a finite graph. There is an* $\operatorname{Aut}(G) \times \operatorname{Aut}(V,E)$*-equivariant stable splitting*

$$X(V,E)_+ \simeq \bigvee_{W \subseteq V} A(W,E_W)^+.$$

Proof. We outline a variant of the proof in [**1**] to demonstrate K-equivariance. Let

$$Y^{(k)}(V,E) = \bigvee_{W \subseteq V\,:\,\#W \leqslant k} A(W,E_W)^+.$$

The space $Y(V,E) = Y^{(n)}(V,E)$ is filtered by the subspaces $Y^{(k)}(V,E)$, $0 \leqslant k \leqslant n$. It will suffice to construct a filtration-preserving equivariant map

$$\pi\colon (\mathbb{R}^V)^+ \wedge X(V,E)_+ \to (\mathbb{R}^V)^+ \wedge Y(V,E)$$

that is homotopic to the identity on successive quotients

$$X^{(k)}(V,E)_+/X^{(k-1)}(V,E)_+ = Y^{(k)}(V,E)/Y^{(k-1)}(V,E) \quad (1 \leqslant k \leqslant n).$$

For $W \subseteq V$ with $\#W = k$, let

$$\pi_W\colon X(V,E) \to X(W,E_W) \to X^{(k)}(W,E_W)/X^{(k-1)}(W,E_W) = A(W,E_W)^+$$

be the composition of the restriction map and the projection onto the top quotient.

The power set $\mathcal{P}(V)$ is embedded in $\mathbb{R}^V$ by mapping a subset W of V to its characteristic function χ_W. Using this embedding and the Pontryagin–Thom construction we can 'add' the maps π_W $(W \in \mathcal{P}(V))$ to obtain the required K-equivariant map π. To be precise, let $B \subseteq \mathbb{R}^V$ be the open disc of radius $\frac{1}{2}$ (in the Euclidean norm), centred at 0, and let $\kappa\colon B \to \mathbb{R}^V$ be a scaling homeomorphism $a \mapsto \phi(\|a\|)a$, with, for example, $\phi(s) = s/(1-4s^2)^{1/2}$. We define π on $\mathbb{R}^V \times X(V,E)$ by sending $(a + \chi_W, x)$ to $[\kappa(a), \pi_W(x)]$ for $a \in B$, $x \in X(V,E)$ and mapping points outside the tubular neighbourhood $\bigcup_{W \in \mathcal{P}(V)}(B + \chi_W)$ to infinity. On the kth quotient one can use a linear homotopy $t \mapsto (1-t)\chi_W$ $(t \in [0,1])$ on each factor $W \in \mathcal{P}(V)$, $\#W = k$, to deform the quotient map to the identity. □

If we work non-equivariantly, a single suspension is enough to add the maps π_W, and we obtain, as in [**1**], a non-equivariant homotopy decomposition of $\Sigma(X(V,E)_+)$ as a wedge of terms $\Sigma(A(W,E_W)^+)$.

3. Describing the summands

In special cases we can give a concrete description of the terms $A(W, E_W)^+$ occurring in the splitting theorem (Theorem 2.1). This will involve an explicit description of the group $G = \mathrm{SU}(2)$ as a sphere.

Lemma 3.1. *There is an* $\mathrm{Aut}(G)$*-equivariant diffeomorphism*

$$\rho\colon G - \{1\} \to \mathfrak{g},$$

such that $\rho(-1) = 0$ *and* ρ *restricts to a diffeomorphism* $L - \{1\} \to \mathfrak{l}$ *for each circle subgroup* $L \leqslant G$ *with Lie algebra the one-dimensional subspace* $\mathfrak{l} \in P(\mathfrak{g})$ *of* $\mathfrak{g}$.

Proof. The group G is a subspace of the vector space $M_2(\mathbb{C})$ of 2×2-matrices. Identifying $\mathfrak{g}$ with the space of skew-Hermitian matrices of trace zero, we may, for example, define $\rho(g) = (g + 1)(g - 1)^{-1}$. Other choices for ρ are suggested by the identification of G with the unit sphere $S(\mathbb{R} \oplus \mathfrak{g}) \subseteq \mathbb{R} \oplus \mathfrak{g} \subseteq M_2(\mathbb{C})$ in the standard norm: $\|g\|^2 = \mathrm{tr}(gg^*)/2$. □

Proposition 3.2. *Suppose that* $V = T \sqcup U$ *and that* $E = \{\{t, v\} \mid t \in T,\ v \in V - \{t\}\}$. *Then* $A(V, E)^+$ *is* K*-homotopy equivalent to the space* $C(T, U)$ *of Definition 1.1.*

The proof is by inspection: we shall see that $A(V, E)^+$ is actually homeomorphic to the mapping cone of the map f that appears in Definition 1.1. A proof in the special case in which $U = \emptyset$, so that (V, E) is the complete graph on the set V, is implicit in the discussion of cone singularities in [**3**, p. 739]. Let us look informally at this special case: $A(V, E)$ is the space of all maps $x\colon T \to G - \{1\}$ such that $x(t)x(v) = x(v)x(t)$ for all $t, v \in T$. There is one singular element in $A(V, E)$, namely the constant map taking the value -1. For any other $x \in A(V, E)$, there is some $t \in T$ such that $x(t) \neq -1$, and then, as we noted at the beginning of the paper, the centralizer of $x(t)$ is a circle subgroup L and $x(v) \in L$ for all $v \in T$. So the complement of the singular element fibres over the space $P(\mathfrak{g})$ of circle subgroups in G with the fibre over L consisting of the maps $x\colon T \to L - \{1\}$ such that $x(t) \neq -1$ for some $t \in T$. Lemma 3.1 provides a homeomorphism from this fibre to the space of maps $\xi\colon T \to \mathfrak{l}$ such that $\xi(t) \neq 0$ for some t. Using polar coordinates, we can write this space of non-zero vectors in the vector space $\mathfrak{l}^T$ as $(0, \infty) \times S(\mathfrak{l}^T)$. But $\mathfrak{l}$ is the fibre of the Hopf line bundle λ over $P(\mathfrak{g})$. In this way, we see that the complement of the singular point in $A(V, E)$ is homeomorphic to $(0, \infty) \times S(\lambda^T)$. The space $A(V, E)^+$ is obtained by gluing in the point at infinity and the singular point at zero to produce the join $S^0 * S(\lambda^T)$.

Proof. The space $A(V, E)$ of maps $x\colon T \sqcup U \to G - \{1\}$ such that $x(t)x(v) = x(v)x(t)$ for all $t \in T$, $v \in V$, can be written as a union $A_0 \cup B$, where A_0 is the closed subspace of maps $x \in A(V, E)$ such that $x(t) = -1$ for all $t \in T$ and B is the complementary subspace consisting of those maps x such that $x(t) \neq -1$ for some $t \in T$. Using the map ρ of Lemma 3.1 we may identify A_0 with the subspace $\mathfrak{g}^U$ of the vector space $\mathfrak{g}^V = \mathrm{map}(V, G - \{1\})$. If $x \in B$ and $x(t) \neq -1$, then $x(t)$ lies in a unique circle subgroup

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$L \leqslant G$, with Lie algebra $\mathfrak{l} \in P(\mathfrak{g})$, and $x(v) \in L - \{1\}$ for all $v \in V$. So we may identify B, again using the map ρ, with the subspace

$$\bigcup_{\mathfrak{l} \in P(\mathfrak{g})} (\mathfrak{l}^T - \{0\}) \times \mathfrak{l}^U$$

of $\mathfrak{g}^V$. Using the Euclidean norm on $\mathfrak{g}^V$ arising from the standard norm on $\mathfrak{g}$, we have a diffeomorphism $(s, x) \mapsto s(1 - s^2)^{-1/2} x \colon (0, 1) \times S(\mathfrak{l}^T) \to \mathfrak{l}^T - \{0\}$. We thus obtain a homeomorphism $(0, 1) \times D \to B$, where D is the total space of the vector bundle $p^* \lambda^U$ over $S(\lambda^T)$. This homeomorphism extends, in the obvious way, to a continuous map from $[0, 1) \times D$ onto the closure $\bar{B}$ of B in $\mathfrak{g}^V$. Its restriction $D = \{0\} \times D \to A_0 \cap \bar{B} \subseteq A_0 = \mathfrak{g}^U$ compactifies to the attaching map $f \colon D^+ \to (\mathfrak{g}^U)^+$. And so we see that $A(V, E)^+$ is homeomorphic to the mapping cone of f. □

We can now deduce Theorem 1.2. For any subset $W \subseteq V$ we may apply Proposition 3.2 to the restricted graph (W, E_W) to see that $A(W, E_W)^+$ is equivariantly homotopy equivalent to $C(T \cap W, U \cap W)$. The result then follows from Theorem 2.1.

The description of the summands can be extended to cover groups of the more general form $\prod_{i \in I} F(U_i)$. In order to state the result, we note first that the composition of the map $S(\lambda^V) \to P(\mathfrak{g}) \times S(\mathfrak{g}^V)$ induced by the inclusion of the Hopf bundle λ over $P(\mathfrak{g})$ into the trivial bundle $P(\mathfrak{g}) \times \mathfrak{g}$ and the projection to $S(\mathfrak{g}^V)$ gives an embedding

$$j_V \colon S(\lambda^V) \hookrightarrow S(\mathfrak{g}^V),$$

which is a homeomorphism if $\#V = 1$. Now we have a cofibre sequence

$$C(S(\lambda^V)) \cup_{j_V} S(\mathfrak{g}^V) \xrightarrow{k_V} S^0 * S(\lambda^V) \xrightarrow{1 * j_V} S^0 * S(\mathfrak{g}^V) = (\mathfrak{g}^V)^+ \tag{3.1}$$

in which k_V is the map that collapses to a point the subspace $S(\mathfrak{g}^V)$ of the mapping cone of j_V.

Proposition 3.3. *Suppose that $\sim$ is an equivalence relation on the non-empty finite set V with equivalence classes $(U_i)_{i \in I}$ of cardinality $m_i = \#U_i$. Let $E = \{\{u, v\} \mid u \not\sim v\}$. Then $A(V, E)^+$ is K-homotopy equivalent to the homotopy cofibre of the pointed map*

$$\bigvee_{i \in I \colon m_i > 1} (C(S(\lambda^{U_i})) \cup_{j_{U_i}} S(\mathfrak{g}^{U_i})) \to S^0 * S(\lambda^V)$$

given on the ith wedge summand by the composition of k_{U_i} with the inclusion $S^0 * S(\lambda^{U_i}) \hookrightarrow S^0 * S(\lambda^V)$.

Proof. Like Proposition 3.2 this may be verified by inspection. The space $A(V, E)$, regarded as a closed subspace of $\mathfrak{g}^V$, may be written as a union of subspaces A_i $(i \in I)$ and B, where $A_i = \mathfrak{g}^{U_i}$ is the space of maps $x \colon V \to \mathfrak{g}$ such that $x(v) = 0$ for $v \notin U_i$, and the complement B of $\bigcup_i A_i$ fibres as a bundle over $P(\mathfrak{g})$ with fibre at $\mathfrak{l}$ equal to $\mathfrak{l}^V - \bigcup_{i \in I} \mathfrak{l}^{U_i}$.

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Now $A_i \cap S(\mathfrak{g}^V) = S(\mathfrak{g}^{U_i})$ and $B \cap S(\mathfrak{g}^V) = S(\lambda^V) - \bigcup_{i\in I} S(\lambda^{U_i})$. The space $A(V,E)^+$ is thus realized as $S^0 * (A(V,E) \cap S(\mathfrak{g}^V))$, where

$$A(V,E) \cap S(\mathfrak{g}^V) = S(\lambda^V) \cup \bigcup_{i\in I} S(\mathfrak{g}^{U_i}) \quad \text{and} \quad S(\lambda^V) \cap S(\mathfrak{g}^{U_i}) = S(\lambda^{U_i}).$$

If $m_i = 1$, the subspaces $S(\lambda^{U_i})$ and $S(\mathfrak{g}^{U_i})$ coincide. These observations lead to the stated description of $A(V,E)^+$ as a mapping cone. The details are omitted. □

4. The non-equivariant stable homotopy type

We outline the proof of Proposition 1.3. Two observations will deal with most of the cases.

Lemma 4.1. *Let ξ and η be real vector bundles over a finite complex X, and let $p\colon S(\xi) \to X$ be the projection map.*

(i) *There is a homotopy cofibre sequence*

$$X^\xi \xrightarrow{c} S^0 * S(\xi) \xrightarrow{1*p} S^0 * X.$$

If $\dim X < \dim \xi$, then $S^0 * S(\xi)$ is stably homotopy equivalent to $X^\xi \vee (S^0 * X)$.

(ii) *There is a homotopy cofibre sequence*

$$S(\xi)^{p^*\eta} \xrightarrow{p} X^\eta \to X^{\xi\oplus\eta}.$$

If $\dim X < \dim \xi$, then $\Sigma S(\xi)^{p^*\eta}$ is stably equivalent to $\Sigma X^\eta \vee X^{\xi\oplus\eta}$.

Proof. (i) The map c collapses the zero-section $X \subseteq X^\xi$ to a point. The proof that we have a homotopy cofibre sequence generalizes the familiar special case in which ξ is the zero vector bundle. If $\dim X < \dim \xi$, the projection p has a section $X \to S(\xi)$, which gives a null-homotopy of the next stage $S^0 * X \to \Sigma X^\xi$ in the cofibration sequence and so splits $\Sigma(S^0 * S(\xi))$ as $\Sigma X^\xi \vee \Sigma(S^0 * X)$.

(ii) The projection $p\colon S(\xi) \to X$ may be replaced by the inclusion $S(\xi) \hookrightarrow D(\xi)$ of the sphere bundle into the disc bundle. The quotient $D(\xi)/S(\xi)$ is X^ξ. This gives the cofibration sequence. A section of $S(\xi)$ gives a null-homotopy of the inclusion $X^\eta \hookrightarrow X^{\xi\oplus\eta}$ and hence the splitting of the mapping cone. □

Taking $T = \{1,\dots,l\}$ and $U = \{1,\dots,m\}$, we shall write $C_{l,m}$ for the homotopy cofibre of the attaching map $f\colon S(l\lambda)^{p^*(m\lambda)} \to S^{3m}$ in Definition 1.1. The projective plane $P(\mathfrak{g})$ will be written simply as P.

Suppose first of all that $m = 0$, so that $C_{l,0} = S^0 * S(l\lambda)$. The case $l = 1$ is easy: $S(\lambda) = S^2$ and $S^0 * S^2 = S^3$. When $l = 2$, it is convenient to think of the 3-manifold $S(2\lambda) = (S(\mathfrak{g}) \times S(\mathbb{R}^2))/\{\pm 1\}$ as the sphere bundle $S(\mathfrak{g} \otimes \mu)$, where μ (for Möbius) is the Hopf bundle over the real projective line $P(\mathbb{R}^2)$. By Lemma 4.1 (i), $C_{2,0}$ is stably equivalent to $P(\mathbb{R}^2)^{3\mu} \vee S^2$. If $l > 2$, then $C_{l,0}$ splits stably as $P^{l\lambda} \vee (S^0 * P)$, by Lemma 4.1 (i) again.

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The case $m = 1$ follows at once: $C_{l,1}$ is the same as $C_{l+1,0}$, because the graph in each case is the complete graph on $l + 1$ vertices.

Now consider the case for $m > 1$. The factor $P^{m\lambda} \to S^{3m}$ of f is then null-homotopic for dimensional reasons. Hence $C_{l,m}$ is stably equivalent to $\Sigma(S(l\lambda)^{p^*(m\lambda)}) \vee S^{3m}$. We consider the first term $F = \Sigma(S(l\lambda)^{p^*(m\lambda)})$. If $l = 0$, F is a point. If $l = 1$, $S(\lambda) = S^2$ and $p^*\lambda$ is trivial, so that $F = \Sigma^{m+1}(S^2_+)$, which decomposes stably as $S^{m+1} \vee S^{m+3}$. When $l = 2$, it is again easier to think of $S(2\lambda)$ as $S(\mathfrak{g} \otimes \mu)$, and then $F = \Sigma(S(\mathfrak{g} \otimes \mu)^{m\mu})$ is stably equivalent, by Lemma 4.1 (ii), to $\Sigma(P(\mathbb{R}^2)^{m\mu}) \vee P(\mathbb{R}^2)^{(m+3)\mu}$. Of course, 2μ is trivial, so that one factor is a suspension of S^1_+ and the other is a suspension of the Moore space $M = P(\mathbb{R}^2)^{\mu}$. If $l > 2$, we may again use Lemma 4.1 (ii) to split F as $\Sigma P^{m\lambda} \vee P^{(l+m)\lambda}$.

The calculations are completed by noting that $S^0 * P = M$ and checking that the Thom space $P^{n\lambda}$ is stably equivalent to the space called P_n. (The key ingredients are that the multiple 4λ is trivial, that the stable dual of $P^{n\lambda}$ is $\Sigma(P^{-(n+3)\lambda})$ and that P_+ is stably $S^0 \vee M$.)

Remark 4.2. Similar arguments can be used to describe the stable summands in the decomposition of $\mathrm{Hom}(F(U_1) \times F(U_2), \mathrm{SU}(2))_+$. It is sufficient to consider the non-equivariant stable homotopy type of the space $A(V, E)^+$ in Proposition 3.3, for $I = \{1, 2\}$, in the case where m_1 and m_2 are both greater than or equal to 2. For dimensional reasons the map $1 * j_V$ in the sequence (3.1) is non-equivariantly null-homotopic if $\dim V > 1$: $\dim S^0 * S(\lambda^V) = \dim V + 2 < \dim \mathfrak{g}^V = 3 \dim V$. Hence we have stable equivalences

$$C(S(\lambda^{U_i})) \cup_{j_{U_i}} S(\mathfrak{g}^{U_i}) \simeq S^{3m_i - 1} \vee (S^0 * S(m_i\lambda))$$

and

$$A(V, E)^+ \simeq S^{3m_1} \vee S^{3m_2} \vee \Sigma^2(S(m_1\lambda) \times_P S(m_2\lambda))_+.$$

The third summand is obtained as the cofibre of the inclusion map

$$(S^0 * S(\lambda^{U_1})) \vee (S^0 * S(\lambda^{U_2})) \to S^0 * S(\lambda^V),$$

which is the join of the identity on S^0 with the inclusion of the disjoint union

$$S(\lambda^{U_1}) \sqcup S(\lambda^{U_2}) \hookrightarrow S(\lambda^V);$$

the complement is homeomorphic to $S(\lambda^{U_1}) \times_P S(\lambda^{U_2})$. To complete the analysis one has to decompose the spaces $(S(m_1\lambda) \times_P S(m_2\lambda))_+$. For example, there is a stable equivalence

$$(S(2\lambda) \times_P S(2\lambda))_+ \cong (S(2\lambda) \times S^1)_+ \simeq S(2\lambda)_+ \vee \Sigma S(2\lambda)_+$$

because the pullback of the complex line bundle $\mathbb{C} \otimes \lambda$ to $S(\mathbb{C} \otimes \lambda)$ is trivial. The space $S(2\lambda)_+ = S(3\mu)_+$ is stably equivalent to $S^0 \vee S^1 \vee \Sigma M$.

Note added in proof

Some of the results of this paper have been obtained independently by Baird *et al.* [**4**] (see the bibliography for details). I am grateful to the authors for sending me a copy of their preprint.

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