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A CONTINUED FRACTION OF RAMANUJAN

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Abstract

In a manuscript discovered in 1976 by George E. Andrews, Ramanujan states a formula for a certain continued fraction, without proof. In this paper we establish formulae for the convergents to the continued fraction, from which Ramanujan's result is easily deduced.

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In 1976, George E. Andrews discovered a manuscript of Ramanujan (1920?) containing more than six hundred identities. (For the interesting details of this discovery, see Andrews (1979).) One of these identities concerns the curious continued fraction

$$F(a,b,\lambda,q)=1+\frac{aq+\lambda q}{1+}\,\frac{bq+\lambda q^2}{1+}\,\frac{aq^2+\lambda q^3}{1+}\,\frac{bq^2+\lambda q^4}{1+}\cdots. \tag{1.1}$$

Ramanujan states without proof that

THEOREM 1.

$$F(a,b,\lambda,q)=\frac{G(a,b,\lambda)}{G(aq,b,\lambda q)}, \tag{1.2}$$

where

$$G(a,b,\lambda)=\sum_{n\geq 0}\frac{q^{\frac{1}{2}(n^2+n)}(a+\lambda)\dots(a+\lambda q^{n-1})}{(1-q)\dots(1-q^n)(1+bq)\dots(1+bq^n)}. \tag{1.3}$$

Andrews (1979) proves this result directly, though with some difficulty. In this note we establish formulae for the convergents to $F(a,b,\lambda,q)$, from which Theorem 1 follows easily.

Before proving Theorem 1, we note that applying Watson's theorem ('Watson's q-analogue of Whipple's theorem') Watson (1929) to the numerator and denominator of (1.2) yields

THEOREM 2.

$$F(a,b,\lambda,q)=\frac{1+\sum_{r\geq 1}\frac{(1-\lambda q^{2r})}{(1-\lambda q^{r})}\frac{(-\lambda/b)_r}{(-bq)_r}\frac{(-\lambda/a)_r}{(-aq)_r}\frac{(\lambda q)_r}{(q)_r}q^{\frac{1}{2}(3r^2+r)}(-ab)^r}{\sum_{r\geq 0}(1-\lambda q^{2r+1})\frac{(-\lambda q/b)_r}{(-bq)_r}\frac{(-\lambda/a)_r}{(-aq)_{r+1}}\frac{(\lambda q)_r}{(q)_r}q^{\frac{1}{2}(3r^2+3r)}(-ab)^r}. \tag{1.4}$$

Theorem 2 contains as corollaries several elegant continued fractions, all given by Ramanujan in Ramanujan (1920?), some of which have appeared previously in the literature. Thus,

$$1+\frac{q}{1+}\frac{q^2}{1+}\cdots=\prod_{n\geq 0}\frac{(1-q^{5n+2})(1-q^{5n+3})}{(1-q^{5n+1})(1-q^{5n+4})},$$

Rogers (1894), p. 328, Ramanujan (1919),

$$1+\frac{q}{1-}\frac{q-q^2}{1+}\frac{q^3}{1-}\frac{q^2-q^4}{1+}\frac{q^5}{1-}\cdots=1\Big/\sum_{n\geq 0}(-1)^n q^{\frac{1}{2}(n^2+n)},$$

Eisenstein (1844),

$$1+\frac{q+q^2}{1+}\frac{q^2+q^4}{1+}\frac{q^3+q^6}{1+}\cdots=\prod_{n\geq 0}\frac{(1-q^{6n+3})^2}{(1-q^{6n+1})(1-q^{6n+5})},$$

Gordon (1965), p. 742,

$$1+\frac{q}{1+}\frac{q+q^2}{1+}\frac{q^3}{1+}\frac{q^2+q^4}{1+}\frac{q^5}{1+}\cdots=\prod_{n\geq 0}\frac{(1-q^{4n+2})^2}{(1-q^{4n+1})(1-q^{4n+3})},$$

Ramanujan (1920 ?),

$$1+\frac{q+q^2}{1+}\frac{q^4}{1+}\frac{q^3+q^6}{1+}\frac{q^8}{1+}\cdots=\prod_{n\geq 0}\frac{(1-q^{8n+3})(1-q^{8n+5})}{(1-q^{8n+1})(1-q^{8n+7})},$$

Ramanujan (1920 ?),

and

$$1-\frac{q-q^2}{1-}\frac{q^2-q^4}{1-}\frac{q^3-q^6}{1-}\cdots=1\Big/\sum_{n\geq 0}(-1)^n q^{3n^2+2n}(1+q^{2n+1}),$$

Ramanujan (1920 ?).

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Our main result, proved in Section 3, is

THEOREM 3.

$$(2.1)\qquad 1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\cdots\cfrac{}{1+\cfrac{aq^N+\lambda q^{2N-1}}{1}}}} = \frac{P_{2N-1}(a,b,\lambda)}{P_{2N-2}(b,aq,\lambda q)},$$

and

$$1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\cdots\cfrac{}{1+\cfrac{bq^N+\lambda q^{2N}}{1}}}} = \frac{P_{2N}(a,b,\lambda)}{P_{2N-1}(b,aq,\lambda q)},$$

where

$$(2.2)\qquad P_N(a,b,\lambda)=\sum a^s b^t \lambda^u q^{\Delta(s+t)+su+tu+u^2} \times \begin{bmatrix} N+1-s-t-u \\ u \end{bmatrix}\begin{bmatrix} [(N+1)/2]-t-u \\ s \end{bmatrix}\begin{bmatrix} [N/2]-s-u \\ t \end{bmatrix},$$

the sum being taken over all $s,t,u\geqslant 0$ *such that* $s+t+u\leqslant[(N+1)/2]$, *for our present purposes* $\begin{bmatrix} -1 \\ 0 \end{bmatrix}=1$, *and* $\Delta(n)=\frac{1}{2}(n^2+n)$.

Letting $N\to\infty$ in (2.1) and (2.2), we obtain

$$(2.3)\qquad F(a,b,\lambda,q)=\frac{P(a,b,\lambda)}{P(b,aq,\lambda q)},$$

where

$$(2.4)\qquad P(a,b,\lambda)=\sum_{s,t,u\geqslant 0} a^s b^t \lambda^u \frac{q^{\Delta(s+t)+su+tu+u^2}}{(q)_s(q)_t(q)_u}.$$

It is obvious from (2.4) that

$$(2.5)\qquad P(a,b,\lambda)=P(b,a,\lambda).$$

Also

$$(2.6)\qquad P(a,b,\lambda)=\prod_{n\geqslant 1}(1+bq^n).G(a,b,\lambda),$$

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where $G(a,b,\lambda)$ is given by (1.3). For,

$$\begin{aligned}
P(a,b,\lambda) &= \sum_{s,t,u\geqslant 0} a^s b^t \lambda^u \frac{q^{\Delta(s)+\Delta(t)+st+su+tu+u^2}}{(q)_s(q)_t(q)_u} \\
&= \sum_{s,u\geqslant 0} a^s \lambda^u \frac{q^{\Delta(s)+su+u^2}}{(q)_s(q)_u} \sum_{t\geqslant 0} \frac{q^{\Delta(t)}(bq^{s+u})^t}{(q)_t} \\
&= \sum_{s,u\geqslant 0} a^s \lambda^u \frac{q^{\Delta(s)+su+u^2}}{(q)_s(q)_u}(1+bq^{s+u+1})(1+bq^{s+u+2})\dots \\
&= \prod_{n\geqslant 1}(1+bq^n) \sum_{s,u\geqslant 0} a^s \lambda^u \frac{q^{\Delta(s)+su+u^2}}{(q)_s(q)_u(1+bq)\dots(1+bq^{s+u})} \\
&= \prod_{n\geqslant 1}(1+bq^n) \sum_{n\geqslant 0} \frac{q^{\Delta(n)}}{(q)_n(1+bq)\dots(1+bq^n)} \times \sum_{s+u=n} a^s \lambda^u q^{\Delta(u-1)} \begin{bmatrix} n \\ u \end{bmatrix} \\
&= \prod_{n\geqslant 1}(1+bq^n) \sum_{n\geqslant 0} \frac{q^{\Delta(n)}(a+\lambda)\dots(a+\lambda q^{n-1})}{(q)_n(1+bq)\dots(1+bq^n)} \\
&= \prod_{n\geqslant 1}(1+bq^n)\, G(a,b,\lambda).
\end{aligned}$$

From (2.3), (2.5) and (2.6) it follows that

$$F(a,b,\lambda,q) = \frac{P(a,b,\lambda)}{P(aq,b,\lambda q)} = \frac{G(a,b,\lambda)}{G(aq,b,\lambda q)},$$

which is (1.2).

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We establish Theorem 3 by showing that if $P_N(a,b,\lambda)$ is defined by (2.2), then

(3.1) $$P_0 = 1, \quad P_1 = 1+aq+\lambda q$$

and

(3.2) $$P_N(a,b,\lambda) = P_{N-1}(b,aq,\lambda q) + (aq+\lambda q)P_{N-2}(aq,bq,\lambda q^2).$$

We can write (3.2)

(3.3) $$\frac{P_N(a,b,\lambda)}{P_{N-1}(b,aq,\lambda q)} = 1 + \frac{(aq+\lambda q)}{\left(\dfrac{P_{N-1}(b,aq,\lambda q)}{P_{N-2}(aq,bq,\lambda q^2)}\right)}.$$

Theorem 3 follows by iteration of (3.3), together with (3.1).

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PROOF OF (3.2). Write

$$P_N(a, b, \lambda)=\sum a^s b^t \lambda^u q^{f(s,t,u)} c_N(s, t, u), \tag{3.4}$$

where

$$f(s, t, u) = \Delta(s+t)+su+tu+u^2 \tag{3.5}$$

and

$$c_N(s, t, u) = \begin{bmatrix} N+1-s-t-u \\ u \end{bmatrix} \begin{bmatrix} [(N+1)/2]-t-u \\ s \end{bmatrix} \begin{bmatrix} [N/2]-s-u \\ t \end{bmatrix}. \tag{3.6}$$

Then

$$\begin{aligned} f(t, s, u) &= f(s, t, u), \\ s+t+u+f(s-1, t, u) &= f(s, t, u), \\ s+t+2u-1+f(s, t, u-1) &= f(s, \dot{t}, u) \end{aligned} \tag{3.7}$$

and

$$c_{N-2}(s, t, u-1)+q^u(c_{N-2}(s-1, t, u)+q^s c_{N-1}(t, s, u)) = c_N(s, t, u). \tag{3.8}$$

For,

$$\begin{aligned} &c_{N-2}(s-1, t, u)+q^s c_{N-1}(t, s, u) \\ &= \begin{bmatrix} N-s-t-u \\ u \end{bmatrix} \begin{bmatrix} [(N-1)/2]-t-u \\ s-1 \end{bmatrix} \begin{bmatrix} [N/2]-s-u \\ t \end{bmatrix} \\ &\qquad + q^s \begin{bmatrix} N-t-s-u \\ u \end{bmatrix} \begin{bmatrix} [N/2]-s-u \\ t \end{bmatrix} \begin{bmatrix} [(N-1)/2]-t-u \\ s \end{bmatrix} \\ &= \begin{bmatrix} N-s-t-u \\ u \end{bmatrix} \begin{bmatrix} [N/2]-s-u \\ t \end{bmatrix} \left\{ \begin{bmatrix} [(N-1)/2]-t-u \\ s-1 \end{bmatrix} + q^s \begin{bmatrix} [(N-1)/2]-t-u \\ s \end{bmatrix} \right\} \\ &= \begin{bmatrix} N-s-t-u \\ u \end{bmatrix} \begin{bmatrix} [N/2]-s-u \\ t \end{bmatrix} \begin{bmatrix} [(N+1)/2]-t-u \\ s \end{bmatrix} \end{aligned}$$

and so

$$\begin{aligned} &c_{N-2}(s, t, u-1)+q^u(c_{N-2}(s-1, t, u)+q^s c_{N-1}(t, s, u)) \\ &= \begin{bmatrix} N-s-t-u \\ u-1 \end{bmatrix} \begin{bmatrix} [(N+1)/2]-t-u \\ s \end{bmatrix} \begin{bmatrix} [N/2]-s-u \\ t \end{bmatrix} \\ &\qquad + q^u \begin{bmatrix} N-s-t-u \\ u \end{bmatrix} \begin{bmatrix} [N/2]-s-u \\ t \end{bmatrix} \begin{bmatrix} [(N+1)/2]-t-u \\ s \end{bmatrix} \end{aligned}$$

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$$=\begin{bmatrix}[(N+1)/2]-t-u\\ s\end{bmatrix}\begin{bmatrix}[N/2]-s-u\\ t\end{bmatrix}\left\{\begin{bmatrix}N-s-t-u\\ u-1\end{bmatrix}+q^u\begin{bmatrix}N-s-t-u\\ u\end{bmatrix}\right\}$$

$$=\begin{bmatrix}N+1-s-t-u\\ u\end{bmatrix}\begin{bmatrix}[(N+1)/2]-t-u\\ s\end{bmatrix}\begin{bmatrix}[N/2]-s-u\\ t\end{bmatrix}$$

$$=c_N(s,t,u).$$

It follows from (3.4), (3.7) and (3.8) that

$$\begin{aligned}
&P_{N-1}(b,aq,\lambda q)+(aq+\lambda q)P_{N-2}(aq,bq,\lambda q^2)\\
&\quad=\sum b^s a^t \lambda^u q^{t+u+f(s,t,u)}c_{N-1}(s,t,u)\\
&\qquad+a\sum a^s b^t \lambda^u q^{s+t+2u+1+f(s,t,u)}c_{N-2}(s,t,u)\\
&\qquad+\lambda\sum a^s b^t \lambda^u q^{s+t+2u+1+f(s,t,u)}c_{N-2}(s,t,u)\\
&\quad=\sum a^s b^t \lambda^u q^{s+u+f(t,s,u)}c_{N-1}(t,s,u)\\
&\qquad+\sum a^s b^t \lambda^u q^{s+t+2u+f(s-1,t,u)}c_{N-2}(s-1,t,u)\\
&\qquad+\sum a^s b^t \lambda^u q^{s+t+2u-1+f(s,t,u-1)}c_{N-2}(s,t,u-1)\\
&\quad=\sum a^s b^t \lambda^u q^{f(s,t,u)}\\
&\qquad\times\{q^{s+u}c_{N-1}(t,s,u)+q^u c_{N-2}(s-1,t,u)+c_{N-2}(s,t,u-1)\}\\
&\quad=\sum a^s b^t \lambda^u q^{f(s,t,u)}c_N(s,t,u)\\
&\quad=P_N(a,b,\lambda),
\end{aligned}$$

which is (3.2), as required.

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References

G. E. Andrews (1979), 'An introduction to Ramanujan's "lost" notebook', *Amer. Math. Monthly* **86**, 89–108.

G. Eisenstein (1844), 'Transformations remarquables de quelques séries', *Crelle's Journal* **27**, 193–197.

B. Gordon (1965), 'Some continued fractions of the Rogers–Ramanujan type', *Duke Math. J.* **32**, 741–748.

S. Ramanujan (1919), 'Proof of certain identities in combinatory analysis', *Proc. Camb. Phil. Soc.* **19**, 214–216.

S. Ramanujan (1920?), unpublished manuscript.

L. J. Rogers (1894), 'Second memoir on the expansion of certain infinite products', *Proc. London Math. Soc.* **25**, 318–343.

G. N. Watson, 'A new proof of the Rogers–Ramanujan identities', *J. London Math. Soc.* **4**, 4–9.

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