11

Beyond Normal Modes

Throughout Part 1, we employed the classical normal mode description of instability growth. All perturbations, $\vec{X}'$, from the equilibrium state were represented by expressions like $\vec{X}' \propto e^{\sigma t + i(kx+\ell y)}$. The goal was to find complex values of σ that signal growth, decay, oscillations/waves, or a combination of these. We also focused on the fastest-growing mode, which is expected to dominate the solution in the long time limit. However, this focus can be too restrictive; other modes can be an important part of the solution. In this chapter we will introduce a different approach to stability analysis that allows for more general temporal structures. We will find that, over limited times, disturbances can grow considerably faster than the fastest-growing normal mode. This has especially important implications for the emergence of turbulence in geophysical flows.

11.1 Instability as an Initial Value Problem

Consider the evolution of a *specific* initial perturbation of some equilibrium state. This is equivalent to an initial value problem where the perturbation is specified at some initial time, $\vec{X}_0 = \vec{X}(t_0)$, and we seek its state at all subsequent times, $\vec{X}(t)$. Previously, we sidestepped consideration of the initial condition by focusing on the fastest-growing mode, which should dominate the solution at late times. At earlier times, however, solutions can exhibit non-intuitive transient behavior, including rapid initial growth. If this initial growth is large enough, it may trigger nonlinear effects such as the transition to turbulence before the fastest-growing normal mode can be established. Also, when unstable conditions last only for a limited time, it is important to know which disturbances will grow the most over that interval.

To fix these ideas, consider a system of coupled, linear, first-order ordinary differential equations,

$$\frac{d\vec{X}}{dt} = \mathsf{A}\vec{X}, \tag{11.1}$$

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which reduces to an eigenvalue problem when the time dependence $\vec{X} \propto e^{\sigma t}$ is assumed. We have often reduced our system of equations to such a form, and it may help the reader to review (2.17, 2.18) from the convection chapter or (3.12) describing shear instability. In the case of convection $\vec{X}(t)$ could represent a concatenation of the vertical velocity and buoyancy discretized in the vertical, or possibly the coefficients of these variables in a Fourier series (Chapter 13). After solving for the eigenvalues, σ_j, and the associated eigenvectors, $\vec{\zeta}_j$, we can write the general solution as

$$\vec{X}(t) = B_1\vec{\zeta}_1 e^{\sigma_1 t} + B_2\vec{\zeta}_2 e^{\sigma_2 t} + \cdots \tag{11.2}$$

The coefficients $B_1, B_2, \ldots$ are then determined by the initial condition:

$$\vec{X}_0 = B_1\vec{\zeta}_1 + B_2\vec{\zeta}_2 + \cdots \tag{11.3}$$

or, more compactly,

$$\vec{X}_0 = \mathsf{Z}\vec{B}, \tag{11.4}$$

where Z is the matrix whose columns are the eigenvectors of A:

$$\mathsf{Z} = [\ \vec{\zeta}_1 \mid \vec{\zeta}_2 \mid \ \ldots],$$

and $\vec{B}$ is a vector of coefficients

$$\vec{B} = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \end{bmatrix}.$$

If we let σ_1 denote the eigenvalue with the maximum real part (which could be negative), then we can see from (11.2) that, as $t \to \infty$, the solution will become dominated by that mode:

$$\vec{X}(t) \to B_1\vec{\zeta}_1 e^{\sigma_1 t}.$$

This simplification is the reason we have focused on the fastest-growing mode in previous sections.

But what if we're interested in growth over a finite time interval? We will see that, in geophysical stability problems, the fastest-growing mode alone is often not sufficient to describe the growth of perturbations over finite times. In fact, over a finite time, a problem like (11.1) can have *all eigenvalues decaying* (i.e., $\sigma_r < 0$), and still exhibit growth! An example is the plane Couette flow that we discussed in section 5.1. That flow has no growing eigenmodes, yet is known to become unstable at sufficiently high Reynolds number (Orszag and Kells, 1980).

11.2 Transient Growth in Simple Linear Systems

As a simple illustration of transient growth, consider a system like (11.1) consisting of only two components, $\vec{X} = (X_1, X_2)$. As a specific example, we choose

$$\mathsf{A} = \begin{bmatrix} -0.1 & 1 \\ 0 & -0.2 \end{bmatrix}. \tag{11.5}$$

Both eigenvalues of A are real and represent exponential decay: $\sigma_1 = -0.1$ and $\sigma_2 = -0.2$. As is conventional, we order the eigenvalues so that $\sigma_1 > \sigma_2$. The eigenvectors, $\vec{\zeta}_1 = (1, 0)$ and $\vec{\zeta}_2 = (-0.9950, 0.0995)$, are not orthogonal; in fact, they are nearly parallel.

To complete the problem, we choose an initial condition that is arbitrary except that it contains substantial contributions from both eigenvectors, as we can tell by solving (11.4) for $\vec{B}$:

$$\vec{X}_0 = (0.1, 1); \quad (B_1, B_2) = (10.1, 10.05).$$

The solution is then

$$\vec{X}(t) = 10.1\vec{\zeta}_1 e^{-0.1t} + 10.05\vec{\zeta}_2 e^{-0.2t}$$

The evolution of this system is shown in Figure 11.1. The left-pointing eigenvector $\vec{\zeta}_2$ decays rapidly, and with it, its contribution to $\vec{X}$. As a result, $\vec{X}$ points

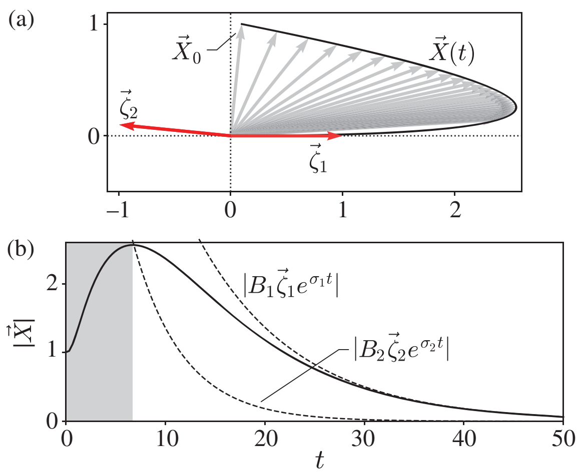

Figure 11.1 (a) Evolution of $\vec{X}$ from its initial value $\vec{X}_0$, with the eigenvectors shown as red arrows (after Schmid, 2007). After an initial period of growth, the evolution of $\vec{X}$ follows that predicted by the eigenvector $\vec{\zeta}_1$ and eigenvalue ($\sigma_1 < 0$) with the largest growth rate. This is also seen in (b), where the amplitude of $|\vec{X}|$ is shown over time (solid line), along with the two different eigenvector terms, $|B_1\vec{\zeta}_1 e^{\sigma_1 t}|$ and $|B_2\vec{\zeta}_2 e^{\sigma_2 t}|$ (dashed lines). The initial growth interval is indicated by gray shading.

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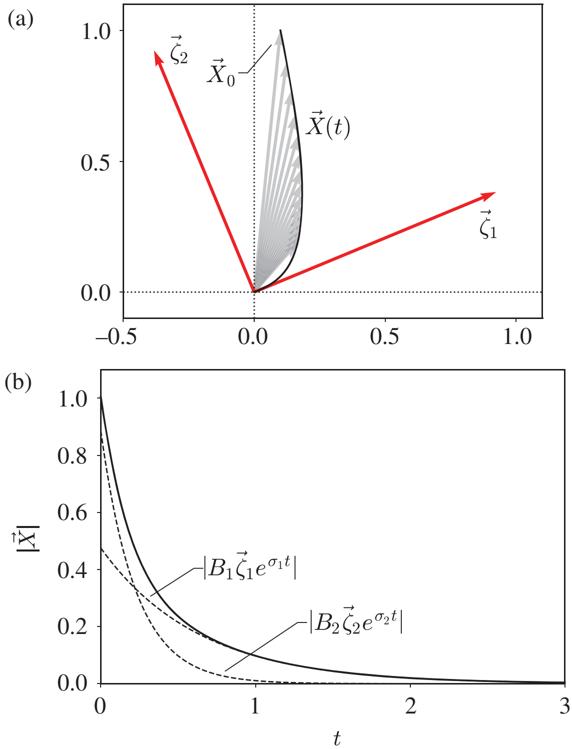

Figure 11.2 As in Figure 11.1, but for the case of orthogonal eigenvectors.

progressively more to the right, and its magnitude increases. At late times, the contribution from $\vec{\zeta}_2$ is negligible. $\vec{X}$ is then nearly parallel to $\vec{\zeta}_1$, and decays with rate σ_1.

As you might anticipate from the last example, a necessary condition for large initial growth is that the eigenvectors be non-orthogonal. In fact, when the eigenvectors are orthogonal, transient growth is not observed. To demonstrate this, let

$$\mathsf{A} = \begin{bmatrix} -2 & 1 \\ 1 & -4 \end{bmatrix}. \tag{11.6}$$

The symmetry of A guarantees that the eigenvectors can be chosen to be orthogonal: $\vec{\zeta}_2 = (-0.38, 0.92)$, and $\vec{\zeta}_1 = (0.92, 0.38)$. The corresponding eigenvalues are again both negative: $\sigma_1 = -1.59$ and $\sigma_2 = -4.41$.

The evolution begins with the same initial condition as in the previous case: $\vec{X}_0 = (0.1, 1)$. The resulting evolution of $\vec{X}(t)$ is shown in Figure 11.2. Once again the contribution from $\vec{\zeta}_2$ decays rapidly so that $\vec{X}$ ultimately becomes parallel to $\vec{\zeta}_1$, but the length of $\vec{X}$ decreases monotonically. This is what we would expect based on our experience with normal modes.

Since transient growth requires that the eigenvectors of the system are non-orthogonal, it would be helpful to have a general rule to assess whether this is the case for a given matrix A. It is a general property of linear algebra that *normal* matrices contain only eigenvectors that are orthogonal,[1] with the converse also holding, i.e., any matrix with orthogonal eigenvectors is normal. A normal matrix, A, has the property that $\mathsf{A}\mathsf{A}^\dagger = \mathsf{A}^\dagger\mathsf{A}$, with the dagger indicating the Hermitian transpose (the transpose of the complex conjugate, also called the adjoint) of A.

11.3 Computing the Optimal Initial Condition

We've seen how an arbitrarily chosen initial state can exhibit transient growth regardless of the signs of the eigenvalues. Under what conditions is this likely to happen? For a given system, can we identify an initial state that is "optimal," in some sense, for transient growth? The meaning of "optimal" can vary depending on the particular behavior we're interested in, but the methods of matrix calculus allow us to explore a wide range of those behaviors.

We begin by writing the general solution to the initial value problem (11.1) as

$$\vec{X}(t) = e^{\mathsf{A}t}\vec{X}_0, \tag{11.7}$$

where $e^{\mathsf{A}t}$ is the matrix exponential function. In case you're unfamiliar, $e^{\mathsf{A}t}$ is a matrix defined by

$$e^{\mathsf{A}t} = \mathsf{I} + \sum_{n=1}^{\infty} \frac{\mathsf{A}^n t^n}{n!}, \tag{11.8}$$

(similar to the infinite series representation of the scalar function e^{at}), where the symbol I is the identity matrix. It has two properties that we'll need:

$$\frac{d}{dt}e^{\mathsf{A}t} = \mathsf{A}e^{\mathsf{A}t} \quad \text{and} \quad e^{\mathsf{A}t}|_{t=0} = \mathsf{I},$$

both of which you can (and should) derive using (11.8). With these you can confirm that $e^{\mathsf{A}t}\vec{X}_0$ is indeed the solution of (11.1).

To generalize the concept of growth beyond the exponential kind, we define an amplification factor (sometimes called the gain) G, at any given time, as

$$\boxed{G(t, \vec{X}_0) = \frac{\langle \vec{X}(t), \vec{X}(t)\rangle}{\langle \vec{X}_0, \vec{X}_0\rangle}.} \tag{11.9}$$

This is the factor by which the squared amplitude of the solution $\vec{X}(t)$ exceeds that of the initial condition, $\vec{X}_0$.

[1] Or can be chosen to be orthogonal.

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Two aspects of (11.9) should be noted:

- We have used $\langle \cdot, \cdot \rangle$ to denote the inner product, which could take different forms. Here we will consider only the Euclidean inner product – equivalent to the vector dot product, $\langle \vec{a}, \vec{b} \rangle = \vec{a}^* \cdot \vec{b}$ for complex vectors $\vec{a}$ and $\vec{b}$ (more on this in section 11.8).
- G depends on (i) the initial state $\vec{X}_0$, (ii) the time t over which the system has evolved, and (iii) the matrix A that controls the evolution of $\vec{X}$. The first two are listed explicitly in (11.9), the third is implicit.

Given the solution (11.7), the amplification factor can be written as

$$G(t, \vec{X}_0) = \frac{\langle e^{\mathsf{A}t} \vec{X}_0, e^{\mathsf{A}t} \vec{X}_0 \rangle}{\langle \vec{X}_0, \vec{X}_0 \rangle} = \frac{\langle e^{\mathsf{A}^\dagger t} e^{\mathsf{A}t} \vec{X}_0, \vec{X}_0 \rangle}{\langle \vec{X}_0, \vec{X}_0 \rangle}. \tag{11.10}$$

The last equality uses the properties

$$\langle \vec{X}, \mathsf{A}\vec{y} \rangle = \langle \mathsf{A}^\dagger \vec{X}, \vec{y} \rangle,$$

and

$$(e^{\mathsf{A}t})^\dagger = e^{\mathsf{A}^\dagger t}.$$

This shows that it is the matrix $e^{\mathsf{A}^\dagger t} e^{\mathsf{A}t}$ that determines the amplification at time t.

Now let us ask, what is the initial condition that optimizes growth over a given time interval $0 \le t \le T$? To answer this, we must maximize $G(T, \vec{X}_0)$ with respect to the initial condition $\vec{X}_0$. For tidiness, we define the matrix

$$\mathsf{E}(T) = e^{\mathsf{A}^\dagger T} e^{\mathsf{A}T}$$

and write the elements of $\vec{X}_0$ as $x_1, x_2, \ldots, x_N$, so that the gain becomes

$$G = \frac{E_{ij} x_i^* x_j}{x_k^* x_k}. \tag{11.11}$$

We now take the derivative of G with respect to a generic element x_ℓ^*, resulting in

$$\frac{\partial G}{\partial x_\ell^*} = \frac{E_{\ell j} x_j}{x_k^* x_k} - \frac{E_{ij} x_i^* x_j}{(x_k^* x_k)^2} x_\ell = \frac{1}{x_k^* x_k} \left[E_{\ell j} x_j - G x_\ell \right] = 0. \tag{11.12}$$

(Variations with respect to x_ℓ and x_ℓ^* are independent. Differentiating with respect to x_ℓ gives the complex conjugate of 11.12.)

The quantity in square brackets must be zero, or

$$\mathsf{E}\vec{X}_0 = G\vec{X}_0. \tag{11.13}$$

This tells us that the optimal gain G is the largest eigenvalue of E, and $\vec{X}_0$ is the corresponding eigenvector. (Because E is Hermitian, the gain G is guaranteed to be real.)

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To sum up, here is the procedure for calculating the maximum growth over time T.

(i) Compute the stability matrix A.
(ii) For a target time T, compute $e^{\mathsf{A}T}$, e.g., in Matlab using the built-in function `expm(A*T)`.
(iii) Multiply $e^{\mathsf{A}T}$ by its Hermitian conjugate to get E.
(iv) Compute the largest eigenvalue of E and its eigenvector $\vec{X}_0$.
(v) If desired, compute the evolution $\vec{X}(t) = e^{\mathsf{A}t}\vec{X}_0$.

The interested student may want to investigate the method of *singular value decomposition*, described in the appendix to this chapter, which facilitates certain aspects of this calculation.

11.4 Optimizing Growth at $t = 0^+$

As we have seen, the optimal initial condition is a function of the target time T. An important special case is the limit $T \to 0$, i.e., the initial condition that grows most rapidly immediately after $t = 0$. This can be found by Taylor expanding $e^{\mathsf{A}^\dagger t}e^{\mathsf{A}t}$ about $t = 0$, i.e.,

$$e^{\mathsf{A}^\dagger t}e^{\mathsf{A}t} = (\mathsf{I} + \mathsf{A}^\dagger t + \cdots)(\mathsf{I} + \mathsf{A}t + \cdots) = \mathsf{I} + (\mathsf{A} + \mathsf{A}^\dagger)t + O(t^2) \tag{11.14}$$

Substituting this into the amplification factor gives

$$G(t, \vec{X}_0) = 1 + \frac{\langle(\mathsf{A} + \mathsf{A}^\dagger)\vec{X}_0, \vec{X}_0\rangle}{\langle\vec{X}_0, \vec{X}_0\rangle}t + O(t^2). \tag{11.15}$$

Recalling the definition (11.9), we express the initial exponential growth rate of $|\vec{X}|$ in terms of G:

$$\frac{1}{2}\frac{dG}{dt} = \frac{|\vec{X}|}{|\vec{X}_0|^2}\frac{d}{dt}|\vec{X}| \to \frac{1}{|\vec{X}|}\frac{d}{dt}|\vec{X}| \tag{11.16}$$

as $t \to 0$, with $|\vec{X}|^2 = \langle\vec{X}, \vec{X}\rangle$. This shows that the initial growth rate is given by $0.5dG/dt$. Therefore, taking the derivative of (11.15) we can write

$$\text{Initial growth rate} = \frac{1}{2}\frac{dG}{dt} = \frac{\langle 0.5(\mathsf{A} + \mathsf{A}^\dagger)\vec{X}_0, \vec{X}_0\rangle}{\langle\vec{X}_0, \vec{X}_0\rangle}. \tag{11.17}$$

Comparing (11.17) with (11.10), the definition of G, we see that the problem of maximizing dG/dt at $t = 0$ is isomorphic to the maximization of G in section 11.3. The same calculation therefore shows that the maximum initial growth rate is the largest eigenvalue of $(\mathsf{A} + \mathsf{A}^\dagger)/2$, and the initial condition $\vec{X}_0$ that achieves that growth rate is the corresponding eigenvector.

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11.5 Growth at Short and Long Times: a Simple Example

To demonstrate the different choices for optimal amplification of a perturbation we look at the system described by the matrix

$$\mathsf{A} = \begin{bmatrix} -1 & -31.8 \\ 0 & -2 \end{bmatrix}. \tag{11.18}$$

This matrix has real decaying eigenvalues $\sigma_{1,2} = -1, -2$, but is non-normal and exhibits transient growth, as shown in Figure 11.3. The evolution of the amplification factor, $G(t)$, is shown for

(i) the optimal perturbation for initial growth (section 11.4),
(ii) the optimal perturbation for the target time $T = 0.1$ (section 11.3), and
(iii) the optimal $G(t)$ possible *at each time*, denoted by $G_{\text{opt}}(t)$, which acts as an upper bound on $G(t)$.

As expected from the preceding discussion, the optimal *initial* growth curve (blue curve on Figure 11.3) is steep initially, exceeding the growth of the optimal for $T = 0.1$ (red curve; see the closeup in the right-hand frame). By $t = 0.03$, however, the optimal curve for $T = 0.1$ has caught up and grows more rapidly from then on.

The overall optimum growth G_{opt} is close to the blue curve near $t = 0$ and close to the red curve near $T = 0.1$. This is not surprising since the initial states for the blue and red curves are optimized for maximum growth over those times.

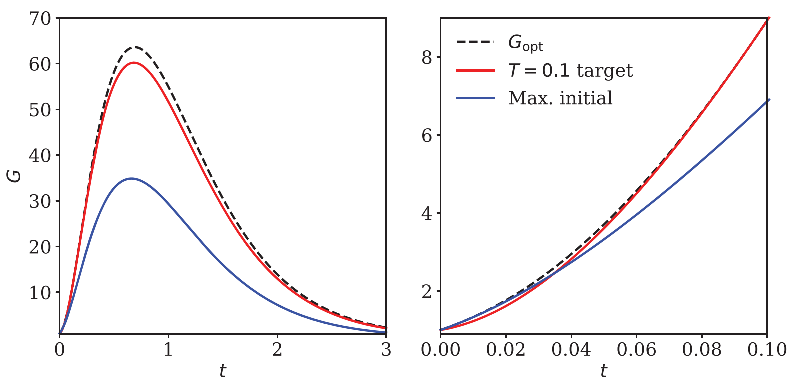

Figure 11.3 Evolution of the amplification factor, $G(t, \vec{X}_0)$, for two different initial conditions in time, corresponding to optimal initial growth (blue), and the optimal amplification at the target time 0.1 (red). The dashed black curve represents the maximum amplification possible at each time, $G_{\text{opt}}(t)$. The right panel is a closeup of the evolution for the time interval $0 \le t \le T$, with $T = 0.1$.

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The final case to consider is the maximum possible amplification over all times. This is referred to as the global optimal, and can be seen to occur at $t = 0.7$ (Figure 11.3, left frame). This initial condition is a good candidate to reach large amplitude and possibly trigger a transition to turbulence.

11.6 Example: The Piecewise Shear Layer

A geophysical example that we are able to solve analytically is the piecewise-linear shear layer of section 3.3. Recall that the solution for the vertical velocity eigenfunction can be written as

$$\hat{w}(z) = B_1 e^{-k|z-h|} + B_2 e^{-k|z+h|} \tag{11.19}$$

with the coefficients B_1, B_2 determined from (3.31). The latter can be reformulated as an eigenvalue problem $\sigma \vec{B} = \mathsf{A}\vec{B}$, where

$$\mathsf{A}_\star = -\iota k_\star \begin{bmatrix} 1 - \frac{1}{2k_\star} & -\frac{e^{-2k_\star}}{2k_\star} \\ \frac{e^{-2k_\star}}{2k_\star} & -1 + \frac{1}{2k_\star} \end{bmatrix}, \tag{11.20}$$

and the shear scaling

$$\hat{w}_\star \equiv \hat{w}/u_0 \,, \quad k_\star \equiv kh \,, \quad \sigma_\star \equiv \sigma h/u_0$$

has been applied.

Note that the dispersion relation for the dimensionless eigenvalue problem, $\det(\mathsf{A}_\star - \sigma_\star \mathsf{I}) = 0$, returns equation (3.32) in dimensionless form, and the results presented in Figure 3.7. The normal modes of the shear layer are unstable for long waves with $0 < k_\star < 0.64$, and are neutral propagating vorticity waves for $k_\star \geqslant 0.64$. Here, we investigate the example $k_\star = 0.2$, for which the eigenvalues are $\sigma_\star = \pm 0.149$. Using the method described in section 11.4 we determine the initial state that grows fastest at $t = 0^+$, then follow its evolution to later times.

The result is shown by the solid curve on Figure 11.4. The dashed lines represent the optimal initial growth rate (steeper line) and the maximum eigenvalue of A (less steep). This exponential growth appears linear because the amplitude is plotted on a log scale. As expected, the perturbation grows at the optimal rate near $t = 0$, then converges to the fastest-growing normal mode as $t \to \infty$.

Note that the optimal perturbation attains the same amplitude as the eigenmode despite its initial amplitude being smaller by about a factor of 2. Conversely, if the two were initialized with the same amplitude, the optimal perturbation would end up bigger by a factor of 2. Along with the change in growth rate comes a change in the structure of the disturbance, which converges over time to the fastest-growing normal mode.

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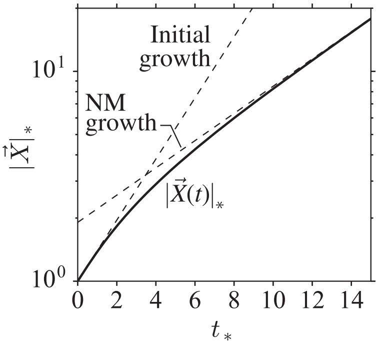

Figure 11.4 Example of large initial growth of a perturbation on the piecewise shear layer with $k_\star = 0.2$. The dashed lines show two curves with different growth rates corresponding to the optimal initial growth and that of the fastest-growing normal mode (NM growth).

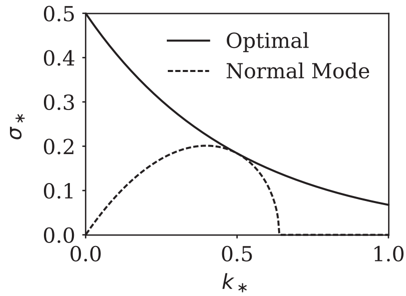

Figure 11.5 Optimal initial growths for the piecewise shear layer (black line). The normal mode growth rates are shown by the dashed line together for comparison. Adapted from Heifetz and Methven (2005).

11.7 Mechanics of Transient Growth in a Shear Layer

Using the method of section 11.4, we may calculate the optimal initial growth for each value of the dimensionless wavenumber, $k_\star$. The result is simply

$$\frac{1}{2}\frac{dG}{dt} = \frac{1}{2}e^{-2k_\star}, \tag{11.21}$$

the right-hand side being the positive eigenvalue of $(\mathsf{A} + \mathsf{A}^\dagger)/2$ with A given by (11.20). As shown in Figure 11.5, this optimal initial growth rate is a monotonically decreasing function of $k_\star$, and exceeds the growth rate of the fastest-growing normal mode for all $k_\star$ except $k_\star = 0.5$.

In section 3.12 we learned of the wave resonance mechanism of shear layer instability. These same ideas can be used to understand the enhanced initial growth found here. In particular, we learned that there is an optimum phase relationship

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between the upper and lower vorticity waves to cause mutual amplification, $\Delta\theta = \pi/2$ (as sketched in Figure 3.23). This optimal phase difference is only found for the special case $k_\star = 0.5$, where the phase speeds of the vorticity waves in isolation from each other are equal. For all other unstable $k_\star$, different $\Delta\theta$ are required so that the waves will phase-lock (i.e., bring each other to a stationary state) and thereby undergo sustained growth.[2]

In the present case, however, we are not concerned about sustained growth; we seek only to maximize *instantaneous* growth at $t = 0$. Therefore, phase-locking is not required. In our discussion of wave resonance we derived the dimensionless growth rate (3.88) in terms of $\Delta\theta$:

$$\sigma_\star = \frac{1}{2}\sin(\Delta\theta)e^{-2k_\star}. \tag{11.22}$$

The factor $e^{-2k_\star}$ quantifies the strength of the interaction between the two waves and is greater when $k_\star$ is small. If $\Delta\theta$ is not constrained by the requirement of phase-locking, we can simply set it to $\pi/2$ for all $k_\star$, in which case (11.22) becomes the optimal *initial* growth rate (11.21). Once again, enhanced instantaneous growth is possible for all $k_\star \neq 0.5$. The initial growth rate is largest in the limit $k_\star \to 0$.

Take note – this result poses a challenge to our "rule of thumb" that says instability on a shear layer has wavelength ~ 8 times the layer thickness (section 3.3.3). Since naturally occurring shear instabilities do not have infinite time to grow, (11.21) suggests that longer wavelengths are more likely to attain visible amplitude. Is this true? We don't know; go find out!

Exercise: Review the other rules of thumb that we derived in Chapter 3 based on the fastest-growing normal mode. How do you think these would change if we considered the optimal initial disturbance instead? Is there an analog of Squire's theorem for optimals? How might three-dimensional disturbances grow?

11.8 Generalizing the Inner Product

Throughout this chapter we have used the simple Euclidean inner product to calculate the magnitude of our solution vector $|\vec{X}(t)|$. However, we have either looked only at arbitrary abstract systems or avoided stating exactly what this quantity corresponds to. Here we mention some different choices that are common, and note that this choice affects the results of the transient growth analysis.

In our featured example of a shear layer, we used the magnitude of the coefficient vector $\vec{X} = (B_1, B_2)$, with the B_j given in (11.19). In this case, our inner

[2] The exact relationship $\Delta\theta(k_\star)$ is plotted in Figure 3.26.

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product $\langle \vec{X}, \vec{X} \rangle = |\vec{X}|^2$ corresponds to the vertical part of the kinetic energy,[3] given explicitly by

$$\int \hat{w}(z,t)^2 dz = \int \hat{w}^* \hat{w} dz \tag{11.24}$$

$$= \int \left[|B_1|^2 \delta(z-h) + |B_2|^2 \delta(z+h) \right] dz \tag{11.25}$$

$$= |B_1|^2 + |B_2|^2. \tag{11.26}$$

Other choices of norms are possible, and these are usually based on physical considerations for each particular problem. Another common choice is an energy norm, which measures the "size" of the perturbation in terms of its total energy. Regardless of the choice of norm, we can switch by a simple transformation so that all of the results in this chapter remain valid.

11.9 Summary

In this chapter we have seen that a more general linear stability analysis is possible when taking into account the evolution of normal modes other than the fastest-growing mode. The results can be summarized as follows.

- Transient growth exceeding the fastest-growing normal mode is possible for systems described by non-normal matrices, which give rise to non-orthogonal eigenvectors. (For normal systems, in contrast, maximum growth is given by the fastest-growing, or least-decaying, normal mode.)
- For any desired target time T, eigen-analysis of the matrix $e^{\mathsf{A}^\dagger T} e^{\mathsf{A} T}$ gives the optimal amplification factor and the initial condition corresponding to it.
- The fastest-growing initial condition and growth rate can be found from an eigen-analysis of the matrix $(\mathsf{A} + \mathsf{A}^\dagger)/2$.
- Transient growth in the piecewise shear layer can be understood intuitively from the wave interaction perspective: the requirement of phase-locking is removed.

[3] As noted previously, these B_j are proportional to vorticity anomalies associated with the vorticity waves on the edges of the shear layer, and can be written as $B_j \propto \hat{q}_j = -\Delta Q_j \hat{\eta}_j$. Therefore, the amplitude factor $G(t) \propto |\hat{q}_1|^2 + |\hat{q}_2|^2$ corresponds to a quantity called *enstrophy*, which is generally defined as

$$\mathcal{E}(t) = \int q'(z,t)^2 dz. \tag{11.23}$$

In this case we say that we are using the *enstrophy norm*. This is a common alternative to energy when examining the transient growth of shear flows.

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11.10 Appendix: Singular Value Decomposition

The procedure for calculating the transient growth of initial conditions can be streamlined through the use of the singular value decomposition (SVD). In fact, this is such a standard method in matrix algebra that it can be accomplished in Matlab with a single command: `svd(A)`. To better understand what is inside the "black box" of the SVD, let us think how we might construct an ideal solution to the problem of optimal growth of $\vec{X}(t)$ without prior knowledge of the SVD.

It is important to keep in mind that we already know a full solution to our problem of determining the time evolution of $\vec{X}(t)$. From (11.2) it is expressed as a linear combination of the eigenvectors, $\vec{\zeta}_j$, of A, along with their amplification factors $e^{\sigma_j t}$, and the coefficients, B_j, required to produce the initial condition $\vec{X}_0$, i.e.,

$$\vec{X}(t) = B_1\vec{\zeta}_1 e^{\sigma_1 t} + B_2\vec{\zeta}_2 e^{\sigma_2 t} + \cdots \tag{11.27}$$

This can all be written compactly in matrix form

$$\vec{X}(t) = \mathsf{Z}e^{\mathsf{D}t}\mathsf{Z}^{-1}\vec{X}_0, \tag{11.28}$$

with the coefficients determined from $\vec{B} = \mathsf{Z}^{-1}\vec{X}_0$.

We have learned that the problem with this representation of the solution, for non-normal A, is that the $\vec{\zeta}_j$ are not orthogonal. This leads to amplification factors, $e^{\sigma_j t}$, that are not representative of the growth of the perturbation. Can we find a different representation of $e^{\mathsf{A}t} = \mathsf{Z}e^{\mathsf{D}t}\mathsf{Z}^{-1}$, so that the eigenvector columns of Z are orthogonal, and therefore the entries in the diagonal matrix are representative of the perturbation growth? It turns out we can, and this representation is the SVD.

To see this, we first note that the different "representation" of $e^{\mathsf{A}t}$ that we seek, is in fact a different set of basis vectors for $\vec{X}(t)$ than the eigenvectors of A. This new set of basis vectors will have the desirable property that they will all be orthogonal to each other – exactly what is missing in the $\vec{\zeta}_j$. Such a representation of $e^{\mathsf{A}t}$ will necessarily have the form of

$$e^{\mathsf{A}t} = \mathsf{U}\Sigma\mathsf{V}^{-1}, \tag{11.29}$$

where the columns, $\vec{u}_j$, of U are the basis vectors we are seeking, Σ is a diagonal matrix with amplification factors, Σ_j, of each basis vector $\vec{u}_j$, and V^{-1} has the job of converting $\vec{X}_0$ to the coordinates of our new basis, i.e., $\mathsf{V}^{-1}\vec{X}_0$ will play an equivalent role as $\vec{B}$.

Since the $\vec{u}_j$ are orthogonal (and normalized to have unit amplitude) we can write

$$\langle \vec{u}_i, \vec{u}_j \rangle = \vec{u}_i^{\dagger} \cdot \vec{u}_j = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases} . \tag{11.30}$$

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In the language of linear algebra we say that U is a *unitary* matrix. An important property of a unitary matrix is that $\mathsf{U}^{-1} = \mathsf{U}^{\dagger}$. It can be shown that V is also a unitary matrix, so that we can write (11.29) exactly as the SVD, i.e.,

$$e^{\mathsf{A}t} = \mathsf{U}\Sigma\mathsf{V}^{\dagger}. \tag{11.31}$$

A useful property of the SVD is that we may order the entries such that the largest amplification factor is first, and given by Σ_1, together with its corresponding basis vector $\vec{u}_1$. It is this combination that we are ultimately after, since we know it provides the maximum amplification (analogous to the fastest-growing mode for a normal matrix). Given what we already know of the SVD, we are now able to come up with a simple recipe to determine all of U, Σ, and V, with the following steps.

(i) Take the right product of (11.31) with V and consider only the first column to give

$$e^{\mathsf{A}t}\vec{v}_1 = \Sigma_1\vec{u}_1. \tag{11.32}$$

(ii) Then find a similar relationship by performing the left product of (11.31) with $\mathsf{U}^{\dagger}$ and then taking the conjugate transpose to give

$$e^{\mathsf{A}^{\dagger}t}\vec{u}_1 = \Sigma_1\vec{v}_1. \tag{11.33}$$

(iii) A formula for $\vec{v}_1$ and Σ_1 can then be found by taking the left product of (11.32) with $e^{\mathsf{A}^{\dagger}t}$ to give

$$e^{\mathsf{A}^{\dagger}t}e^{\mathsf{A}t}\vec{v}_1 = \Sigma_1 e^{\mathsf{A}t}\vec{u}_1 = \Sigma_1^2\vec{v}_1, \tag{11.34}$$

where the last step comes from using (11.33). This shows that Σ_1^2 is the largest eigenvalue of the matrix $e^{\mathsf{A}^{\dagger}t}e^{\mathsf{A}t}$, and $\vec{v}_1$ is its eigenvector.

(iv) Similarly, we can find $\vec{u}_1$ from taking the left product of (11.33) with $e^{\mathsf{A}t}$, and use (11.32) to find

$$e^{\mathsf{A}t}e^{\mathsf{A}^{\dagger}t}\vec{u}_1 = \Sigma_1 e^{\mathsf{A}t}\vec{v}_1 = \Sigma_1^2\vec{u}_1. \tag{11.35}$$

Our recipe to construct the SVD is complete, and we have the ideal form for expressing the solution $\vec{X}(t)$, in terms of orthogonal basis vectors with ordered, real, amplification factors. This also demonstrates that the SVD can be found directly from the eigen-properties of the matrices $e^{\mathsf{A}^{\dagger}t}e^{\mathsf{A}t}$ and $e^{\mathsf{A}t}e^{\mathsf{A}^{\dagger}t}$. Note that an identical statement was arrived at when we considered a formula for $G(t)$ in (11.10).

11.11 Further Reading

Nice overview papers on transient and optimal growth, which this material has been based on, are Farrell (1996), Trefethen et al. (1993), and Schmid (2007). A full treatment of transient growth and wave interactions in the piecewise shear layer is discussed in Heifetz and Methven (2005). More recent advances are described by Kerswell et al. (2014), Kaminski et al. (2014), and Luchini and Bottaro (2014).

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