Given the line u with coordinates (l, m, n), the pole of u with respect to Φ_{12} is (l, m, n) and the polar of this point with respect to f_3 is

$$u_3 \equiv alx + bmy + cnz = 0. \qquad (1)$$

Similarly the pole of u with respect to Φ_{31} is $[\omega(b\omega + c)\,l, (c + a\omega^2)\,m, (a\omega + b)\,n]$ and the polar of this point with respect to f_2 is

$$u_2 \equiv (b\omega + c)\,lx + (c\omega + a)\,my + (a\omega + b)\,nz = 0 \qquad (2)$$

P is the point of intersection of u_2 and u_3. Adding, and remembering that $a + b + c = 0$, we have

$$blx + cmy + anz = 0. \qquad (3)$$

And again using $a + b + c = 0$, equations (1) and (3) are

$$a\,(lx - nz) + b\,(my - nz) = 0 = b\,(lx - nz) + c\,(my - nz)$$

i.e. $$lx = my = nz$$

or, if we prefer, $\sigma x = mn$, $\sigma y = nl$, $\sigma z = lm$ (4)

the standard equations of a quadratic dual transformation the triangle of reference being that self-conjugate to the pencil and so providing the exceptional elements.

Since a, b, c do not enter into these results, *P is uniquely determined by u independently of the choice of f_3.*

In conclusion I should like to express my gratitude to Professor Turnbull who was kind enough to read and provide helpful criticism of the above result.

DUMFRIES ACADEMY.

Collapsible circular sections of quadric surfaces

By H. W. TURNBULL.

Cardboard or wire models of ellipsoids and hyperboloids exist which consist of two sets of circular sections. They cover the quadric surface with curvilinear quadrilaterals, whose sides remain constant in length when the model alters in shape. In fact the models admit of one degree of freedom—they are collapsible—and the angle between the two sets of circular sections can be varied.

Such models, with their explanation, are found at the South

Kensington Museum and elsewhere. A short account of the ellipsoidal model is given by Sommerville,[1] and the present note is but an amplification of this. The mathematical explanation of the collapsible property is simple and interesting, affording a good example of the use of oblique Cartesian coordinates at an early stage in teaching the subject.

Take a set of partially oblique axes $Oxyz$, where Oy is at right angles to both the other axes, which make an angle 2θ with each other. The sets of planes $x =$ constant and $z =$ constant will meet each other in sets of lines parallel to Oy; and evidently the distance apart, in a plane $x = u$, of the parallels $z = v$ and $z = v'$ is $v - v'$, which is independent of θ.

Now consider the quadric surface

$$x^2 + y^2 + z^2 + 2gxz = h.$$

Since the axes of x and y are at right angles the sections $z =$ constant are circles. So too are those of $x =$ constant. Indeed this is the most general equation of a quadric with centre at the origin, for which the x and z sections are circles. The circular section $x = u$ has an equation (in rectangular coordinates y, z)

$$y^2 + (z + gu)^2 = h + u^2 \, (g^2 - 1)$$

whose radius is evidently independent of θ, and so are the arcs intercepted on it by the planes $z = v$, $z = v'$. Hence the quadric surface possesses the collapsible property, and varies its shape as θ varies.

If $O\xi$, $O\zeta$ are the internal and external bisectors of the angle 2θ made by the axes Ox, Oz, and if $O\eta$ coincides with Oy, then we may take completely rectangular axes defined by the relations

$$\xi = (x + z) \cos \theta, \quad \eta = y, \quad \zeta = (z - x) \sin \theta.$$

The new equation of the quadric is

$$\sec^2 \theta \, (1 + g) \, \xi^2 + 2\,\eta^2 + \operatorname{cosec}^2 \theta \, (1 - g) \, \zeta^2 = 2h.$$

Three real cases arise according to the values of g and h. For a real *ellipsoid* we may take $g = \cos 2\alpha$, $h = b^2 > 0$. The semi-axes are therefore a, b, c where

$$a = b \cos \theta \sec \alpha, \quad c = b \sin \theta \operatorname{cosec} \alpha.$$

Every shape is represented, but only those with constant α are

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[1] *Analytical Geometry of Three Dimensions* (Cambridge, 1934), p. 206.

collapsible into one another. When $\theta = 0$ or $\frac{\pi}{2}$ the model flattens into its limiting forms—into ellipses. When $\theta = \alpha$ the shape is spherical, with the circular sections therefore now intersecting at an angle 2α. This serves to define the meaning of the parameter α.

If $\alpha = \frac{\pi}{4}$, then $g = 0$, and the central circular sections pass through umbilics. In this case these central sections are

$$\zeta = \pm \xi \tan \theta$$

and the umbilics are $(\pm b \cos \theta, 0, \pm b \sin \theta)$. The characteristic property of such an ellipsoid is

$$a^2 + c^2 = 2b^2,$$

as may easily be verified.

Secondly, if the surface is a real hyperboloid of one sheet, we may take $g = \cosh 2\alpha$, $h = b^2 > 0$, when the equation becomes

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} - \frac{\zeta^2}{c^2} = 1,$$

where $a = b \cos \theta \operatorname{sech} \alpha$, $c = b \sin \theta \operatorname{cosech} \alpha$.

Thirdly, for the hyperboloid of two sheets we may take $g = \cosh 2\alpha$, $h = -b^2 < 0$, with the same values of a and c.

The general equation of the elliptic paraboloid whose circular sections are $x =$ constant and $z =$ constant, has for its second degree terms

$$x^2 + y^2 + z^2 \pm 2xz.$$

We may take

$$\frac{\xi^2}{a^2} + \frac{\eta^2}{b^2} = \frac{2\zeta}{c},$$

where $a = b \cos \theta$ and c is arbitrary.

The hyperbolic paraboloid has no circular sections, but the network of generators possesses the collapsible property as may be shewn by considering where the planes $z = v$, $z = v'$ and $x = u$ meet the surface $xz = cy$.

A network of circles and rectangular hyperbolas may be collapsible, as, for example, in the case

$$x^2 + y^2 - z^2 = b^2.$$

Indeed the property extends to a network of any two sets of similar conics of any specified eccentricities.

It is well known that all ruled quadrics with real generators possess the collapsible property, where the network of generators replaces that of the circular sections. Such a surface may collapse into a confocal surface by a suitable strain, which in general differs from that required for circular sections. The case of the hyperbolic paraboloid can be regarded from either point of view, if its generators are regarded as circles of infinite radius.

Note on an alternant suggested by statistical theory.

By B. BABINGTON SMITH.

The determinant to be described made its appearance in the course of a search for the frequency distribution of Spearman's rank correlation coefficient (2). This name is given to the form, $\rho = 1 - \dfrac{6\,\Sigma\, d^2}{n^3 - n}$, taken by the product moment correlation coefficient in the special case where the two variables are separate arrangements of the first n natural numbers.

For any value of n the distribution of ρ is completely determined by the values of $\Sigma\, d^2$, the sum of the squares of the differences between the n pairs of numbers.

Consider the array

$$
\begin{array}{c|ccccccc}
 & 1 & 2 & 3 & \cdots\cdots & n-2 & n-1 & n \\
\hline
1 & 0 & 1 & 2 & \cdots\cdots & n-3 & n-2 & n-1 \\
2 & -1 & 0 & 1 & \cdots\cdots & n-4 & n-3 & n-2 \\
\cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\
n-2 & -(n-3) & & & \cdots\cdots & 0 & 1 & 2 \\
n-1 & -(n-2) & & & \cdots\cdots & -1 & 0 & 1 \\
n & -(n-1) & & & \cdots\cdots & -2 & -1 & 0.
\end{array}
$$

Any expression made up of n entries from the body of this array chosen so that no row or column contributes more than one must correspond to a possible pairing of the numbers 1 to n with another arrangement of them. Further there are $n!$ possible ways of choosing such expressions corresponding to the $n!$ possible arrangements.

If now we write

$$
\begin{array}{ccccc}
a^0 & a^{1^2} & a^{2^2} & \cdots & a^{(n-1)^2} \\
a^{1^2} & a^0 & \cdots & \cdots & a^{(n-2)^2} \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
a^{(n-1)^2} & a^{(n-2)^2} & \cdots & \cdots & a^0.
\end{array}
$$

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