Canad. J. Math. Vol. **51** (6), 1999 pp. 1277–1299

Isomorphism Invariants for Projective Configurations

I knew of Donald Coxeter's work on regular polytopes when I was a schoolboy, and this greatly encouraged my love of geometry. I first met him in 1951 when he was external examiner for my doctorate, and I have remained in touch with him ever since. It is therefore with greatest pleasure that I dedicate this paper to him. I believe the subject matter is the sort of geometry that he enjoys.

G. C. Shephard

Abstract. An isomorphism invariant is an expression, defined for a configuration in the projective plane, which takes the same value for all isomorphic configurations. Examples are given as well as a general method (Nehring sequences) for constructing such invariants.

0 Introduction

The purpose of this paper is to define *isomorphism invariants* of configurations in the projective plane and to show how such invariants can be constructed.

To introduce the topic, consider Menelaus' theorem. The classical form of the theorem (Figure 1(a)) concerns a triangle $[V_0, V_1, V_2]$ and a transversal, by which we mean any line which does not pass through a vertex, which cuts the side V_iV_{i+1} in the point W_i. Here all subscripts are reduced modulo 3. Menelaus' theorem states that

$$\frac{\|V_0W_0\|}{\|W_0V_1\|} \cdot \frac{\|V_1W_1\|}{\|W_1V_2\|} \cdot \frac{\|V_2W_2\|}{\|W_2V_0\|} = -1. \tag{1}$$

The double vertical lines indicate *signed* or *directed* lengths of the indicated segments. There is also an n-gonal form of the theorem; an example with $n = 5$ is shown in Figure 1(b). With the corresponding notation (subscripts reduced modulo 5),

$$\prod_{i=0}^{4} \frac{\|V_iW_i\|}{\|W_iV_{i+1}\|} = -1. \tag{2}$$

At first sight it may appear that the n-gonal form is just a straightforward generalisation of the classical theorem, but we claim that it is really of an entirely different character. Since there exists a non-singular projectivity which maps any triangle and transversal into any other triangle and transversal, equation (1) implies that its left side is a *projective invariant*, a fact that is also a consequence of Eves' theorem, see below. However, in general, two

Received by the editors January 8, 1999; revised May 5, 1999.
AMS subject classification: 51N15.

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

pentagons with transversals are *not* projectively equivalent, so the left side of equation (2) clearly has a much stronger invariance property.

Two configurations in the projective plane are said to be *isomorphic* if there exists a bijection which maps the elements (points and lines) of one onto the elements of the other and is compatible with the incidence relation. Any numerical quantity defined for a projective configuration is called an *isomorphism invariant* if it takes the same value for all isomorphic configurations. For example, the left side of (2) is an isomorphism invariant since it takes the same value for every pentagon with transversal. Clearly, since projectively equivalent configurations are isomorphic, all projective invariants are isomorphism invariants, but not conversely. Isomorphism invariants can be constructed using the n-gonal forms of Menelaus' and Ceva's theorems as indicated above, but if $n > 3$, in general, these are *not* projective invariants. A more powerful method is by means of Nehring sequences, which form the topic of Section 1 of this paper.

The rest of this paper is arranged as follows. Section 1 also includes the Main Theorem which is used in constructing invariants. Section 2 contains examples that illustrate the techniques, and the final section is devoted to general comments and historical remarks.

1 Nehring Sequences

Let $b_0, b_1, \dots, b_{n-1}$ be n lines and $P_0, P_1, \dots, P_{n-1}$ be n points in the projective plane such that, for each i, the point P_i does not lie on either of the lines b_i, b_{i+1}. Here, and throughout, all subscripts j are reduced modulo n so they satisfy $0 \le j < n$. Consider a sequence of $2n$ terms in which lines and points alternate:

$$(b_0, P_0, b_1, P_1, b_2, \dots, b_{n-1}, P_{n-1}). \tag{3}$$

Choose any point R_0 on the line b_0, and construct the points $R_1, R_2, \dots, R_n$ as follows: The line R_0P_0 meets b_1 in the point R_1, the line R_1P_1 meets b_2 in the point $R_2, \dots,$ and so on. The line $R_{n-2}P_{n-2}$ meets b_{n-1} in R_{n-1} and finally, $R_{n-1}P_{n-1}$ meets b_0 in R_n. Then if R_0 and R_n *always* coincide, *regardless of the point* R_0 *on* b_0 *from which we started the construction*, then (3) is called a *Nehring sequence* of length n.

At first sight it may seem that the condition that a sequence of points and lines form a Nehring sequence is both unusual and artificial. This is not so, and although they have not been recognised as such, they occur frequently throughout plane projective geometry. Some general methods of constructing Nehring sequences are given in [10], and several examples are described there.

The construction of the sequence of points $R_1, R_2, \dots, R_n$ starting from an arbitrary initial point R_0 on b_0, as described above, will be used many times in this paper, and to avoid repetitions we shall refer to this as the *basic construction*. Also whenever we consider a sequence of alternate lines and points which is either a Nehring sequence, or a candidate for such a sequence, we shall tacitly assume that none of the points lies on either of the two lines which are adjacent to it in the sequence (and P_{n-1} does not lie on either b_{n-1} or b_0).

Since a Nehring sequence is defined in terms of incidences of points and lines, we may apply the principle of duality. It is easy to check that the dual of a Nehring sequence is also a Nehring sequence.

An alternative way to express the condition that a sequence is a Nehring sequence is by means of perspectivities [2, Section 14.5], [4, Section 1.6], [6, Section 6.1]. Clearly (3) is a Nehring sequence if and only if the sequence of perspectivities

$$b_0 \overset{P_0}{\barwedge} b_1 \overset{P_1}{\barwedge} \cdots \overset{P_{n-1}}{\barwedge} b_0 \tag{4}$$

has a product (composition) $\pi\colon b_0 \barwedge b_0$ which is the identity projectivity. This formulation enables perspectivities and projectivities to be used in investigating the properties of Nehring sequences. For example, if we can prove that $R_0 = R_n$ for three distinct positions of R_0 on b_0, then the projectivity π has three self-corresponding points and so must be the identity, that is, $R_0 = R_n$ for all points R_0 on b_0 [2, Section 14.5].

As another example, the interpretation in terms of perspectivities leads immediately to the following (dual) results; details of the proofs are left to the reader.

(i) Let $b_0, b_1, \ldots, b_{n-1}$ be any n concurrent lines, and $P_1, P_2, \ldots, P_{n-1}$ be any $n-1$ points. Then it is possible to find a point P_0 such that (3) is a Nehring sequence.
(ii) Let $B_0, B_1, \ldots, B_{n-1}$ be any n collinear points, and $p_1, p_2, \ldots, p_{n-1}$ be any $n-1$ lines. Then it is possible to find a line p_0 such that $(p_0, B_0, p_1, B_1, \ldots, p_{n-2}, B_{n-2}, p_{n-1}, B_{n-1})$ is a Nehring sequence.

In the Nehring sequence (3) we shall think of the points P_i as fixed and lines $R_0P_0R_1$, $R_1P_1R_2, \ldots$, as "moving" as R_0 varies on b_0, and hence refer to them as *rays* and the points P_i as *pivots*. The lines in the sequence will be called *base lines*, or simply *bases*. In the diagrams the pivots and base lines will be represented by heavy points and lines; other lines in the figure will be lighter and the rays will be represented by dashed lines. Also, in the figures and text, points will be denoted by upper-case letters, and lines by lower-case letters, with or without subscripts. Generally we shall use $P_0, P_1, \ldots$, for the pivots and $b_0, b_1, \ldots$, for the bases.

In each of the examples in the next section we give three parameters, namely n, p and b. The first of these is n, the length of the Nehring sequence, that is, either the number of points or number of lines in it. The letters p and b will represent the number of pivots and number of base lines respectively. Clearly $p \le n$ and $b \le n$, and strict inequalities will occur if some of the pivots or bases are repeated. The determination of all triples (n, p, b) that correspond to Nehring sequences is an open problem. For example, if $p = b = 2$, then it is easy to see that n must be even and must be at least 4.

***Theorem 2* (The Main Theorem)** *Let $(b_0, P_0, b_1, P_1, \ldots, b_{n-1}, P_{n-1})$ be a Nehring sequence, and, starting from a point R_0 on b_0, use the basic construction to determine $R_1, R_2, \ldots, R_{n-1}$. Then, as R_0 varies on b_0,*

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \cdots \cdot \frac{\|R_{n-2}P_{n-2}\|}{\|P_{n-2}R_{n-1}\|} \cdot \frac{\|R_{n-1}P_{n-1}\|}{\|P_{n-1}R_0\|} = \textit{constant}, \tag{5}$$

that is, the value of the left side is independent of the choice of R_0 on b_0 so long as the left side is defined (none of the denominators vanish). Moreover the value of the constant on the right side is $(-1)^n$ if either the base lines $b_0, b_1, \ldots, b_{n-1}$ are concurrent, or the pivots $P_0, P_1, \ldots, P_{n-1}$ are collinear.

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

Before proving the theorem some explanation is necessary. At first sight equation (5) may appear to be not meaningful since it uses directed lengths such as $\|AB\|$ and clearly these are not defined in the projective plane. However, if A, B, C are distinct collinear points, then $\|AB\|/\|BC\|$ is invariant under affine transformations and so this quotient is defined in the affine plane. Moreover, the left side of (5) is what Eves [6, Section 6.1] calls an *h-expression*, which is a product of quotients characterised by the fact that it satisfies the following two conditions:

(1) In each factor the letters that occur (for example, R_0, P_0, R_1 in the first factor) are collinear points, and
(2) If each term such as $\|AB\|$ is replaced by the symbolic product ab, then complete cancellation takes place.

Eves [6, Section 6.1] proves the following remarkable result.

***Theorem 3* (Eves' Theorem)** *Every h-expression is a projective invariant.*

The simplest example of an h-expression is provided by the cross ratio of four collinear points. Thus Eves' theorem represents a substantial generalisation of this most important numerical projective invariant. We feel that Eves' theorem has never been given the recognition it deserves and should be regarded as one of the fundamental results of projective geometry.

We can now explain what is meant by (5). We first embed an affine plane in the projective plane by choosing a suitable "line at infinity". (Here, "suitable" means any line that is not a line in the figure, nor contains any of the points in the figure.) Then the left side of (5) can be evaluated in the affine plane, and by Eves' theorem *its value is independent of the way the affine plane was embedded, and is therefore defined in the projective plane.*

Eves' theorem implies, equivalently, that we may apply any projective transformation to the configuration under consideration, and the left side is unchanged. But notice that the main theorem implies a great deal more; it says that the value of the expression the left side of (4) is unchanged as R_0 varies on b_0. We shall be primarily interested in the situation that arises when such variation does *not* correspond to a projective transformation.

Proof of the Main Theorem Given the Nehring sequence $(b_0, P_0, b_1, P_1, \ldots, b_{n-1}, P_{n-1})$, choose X_0 as any point on b_0, and determine $X_1, X_2, \ldots, X_{n-1}$ by the basic construction, so $X_0 = b_0 \cap P_{n-1}X_{n-1}$. Choose R_0 as any point on b_0 distinct from X_0, and determine $R_1, R_2, \ldots, R_{n-1}$ by the basic construction, so $R_0 = b_0 \cap P_{n-1}R_{n-1}$. For each $i = 0, \ldots, n-1$ consider the rays $X_iP_iX_{i+1}$ and $R_iP_iR_{i+1}$. In Figure 2 we illustrate the four cases that arise according to whether P_i separates or does not separate X_i and X_{i+1}, and separates or does not separate R_i and R_{i+1}.

To begin with, consider all line segments to be positive, that is, unsigned. Then if $\angle R_iP_iX_i = \psi_i$, $\angle P_iX_iR_i = \chi_i$ and $\angle P_iX_{i+1}R_{i+1} = \omega_i$, we see, from elementary geometry, $|R_iP_i|/\sin\chi_i = |R_iX_i|/\sin\psi_i$ and $|P_iR_{i+1}|/\sin\omega_i = |R_{i+1}X_{i+1}|/\sin\psi_i$. Hence, eliminating $\sin\psi_i$,

$$\frac{|R_iP_i|}{|P_iR_{i+1}|} = \frac{|R_iX_i|\sin\chi_i}{|R_{i+1}X_{i+1}|\sin\omega_i}. \tag{6}$$

We wish to convert this relation into one between signed lengths. To do this, assign to each line b_i the positive direction $\overrightarrow{R_iX_i}$; then

$$\frac{\|R_iP_i\|}{\|P_iR_{i+1}\|} = \pm\frac{\sin\chi_i\|R_iX_i\|}{\sin\omega_i\|R_{i+1}X_{i+1}\|} \tag{7}$$

where the sign is positive if P_i separates R_i and R_{i+1} (Figures 2(a) and (d)) and is negative if it does not do so (Figures 2(b) and (c)). Hence

$$\prod_{i=0}^{n-1}\frac{\|R_iP_i\|}{\|P_iR_{i+1}\|} = \pm K\prod_{i=0}^{n-1}\frac{\|R_iX_i\|}{\|R_{i+1}X_{i+1}\|}. \tag{8}$$

The product on the right clearly equals 1, and K is a function of the sines of the angles χ_i and ω_i. These angles are constant as R_i varies on b_i, so K is constant. This proves the first part of the main theorem except for the ambiguity of signs. Clearly the sign will be fixed so long as the separation, or non-separation, of each pair R_i, R_{i+1} by P_i remains the same in the affine plane. In the projective plane this can only change if, as a point R_i varies on b_i, it crosses the line at infinity (for then either the arrangements in Figure 2 (a) and (c) or in (b) and (d) will be interchanged). But exactly two of these changes will take place; the separation of R_i and R_{i+1} by P_i and the separation of R_i and R_{i-1} by P_{i-1} will both change. We deduce that the ambiguous sign in (8) remains the same, so

$$\prod_{i=0}^{n-1}\frac{\|R_iP_i\|}{\|P_iR_{i+1}\|} = \text{constant}$$

and the first statement of the theorem is proved.

For the final statements, we observe that when the base lines are concurrent in a point O, one possible position for the points R_i is that in which they all coincide with O. Each factor on the left side of (5) is then -1 and the value of the constant is $(-1)^n$.

When the pivots are collinear on a line k then we may take all the points X_i as lying on k. In Figure 3 we show four examples of arrangements that can occur. In (8) the constant K is given by $K = \prod_{i=0}^{n-1}(\sin\chi_i/\sin\omega_i)$, but as $\sin\omega_{i-1} = \sin\chi_i$, this takes the value 1 and so $\prod_{i=0}^{n-1}(\|R_iP_i\|/\|P_iR_{i+1}\|) = \pm 1$. To determine the ambiguous sign we observe that $\|R_iP_i\|/\|P_iR_{i+1}\|$ is positive if R_i and R_{i+1} lie on opposite sides of k, and negative if they lie on the same side of k. As R_0 is on the same side of k as R_n (trivially since these points coincide!) we deduce $\prod_{i=0}^{n-1}\|R_iP_i\|/\|P_iR_{i+1}\| = (-1)^n$. This completes the proof of the main theorem. ■

The last two parts of the Main Theorem imply, and are equivalent to, Ceva's and Menelaus' Theorems for n-gons. Hence the Main Theorem may be regarded as a generalisation of both these results.

The examples in the next section illustrate the close relationship between configurations and Nehring sequences. This is not altogether surprising since the existence of many configurations depends on showing that three points are collinear, which is exactly what is required to show a sequence is a Nehring sequence (showing that the points R_{n-1}, P_{n-1},

R_0 are collinear for all positions of R_0 on b_0). Some projective configurations may lead to several Nehring sequences according to which points are selected as pivots, and which lines are selected as bases. In Example 2 below, we show two isomorphism invariants that arise from the Pappus' configuration.

2 Examples of Isomorphism Invariants

Each example begins with the description of a configuration in the projective plane. When a proof of the existence of the configuration is necessary we give either a reference or a hint as to how to construct a proof. This is followed (except in the first part of Example 2, where Menelaus' 6-gonal theorem is used) by an explanation as to how the configuration gives rise to a Nehring sequence, and we conclude with an explicit statement of the isomorphism invariant that arises from application of the Main Theorem.

Example 1 (Complete quadrangle) [3, Section 2.4], [4, Section 1.4], see Figure 4. Let $Q = [V_0, V_1, V_2, V_3]$ be a quadrangle and k be any line that does not pass through any vertex or diagonal point of Q. The quadrangle Q has three pairs of opposite sides. Take one pair as the bases b_0, b_1; let a second pair meet k in P_0, P_2; let the third pair meet k in P_1, P_3. Then

$$(b_0, P_0, b_1, P_1, b_0, P_2, b_1, P_3) \tag{9}$$

is a Nehring sequence with $n = p = 4$ and $b = 2$. As we may interchange P_0 and P_2; P_1 and P_3; b_0 and b_1; and the pair P_0, P_2 with P_1, P_3, (9) represents several Nehring sequences, though, in fact, only two of these are distinct.

The statement that (9) is a Nehring sequence follows immediately from the properties of Q. Notice that if $X_0 = k \cap b_0$ and $X_1 = k \cap b_1$ then the three pairs X_0, X_1; P_0, P_2; P_1, P_3 belong to an involution on k. In fact, the assertion that (9) is a Nehring sequence is equivalent to the fact that the above three pairs of points belong to an involution [2, Section 14.5], [3, Section 4.7], [4, Section 5.3]. We deduce from the Main Theorem,

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_3\|}{\|P_3R_0\|} = 1.$$

The constant is $(-1)^4 = 1$ since the bases are (trivially) concurrent. The left side of the above equality is therefore an isomorphism invariant.

Example 2 (Pappus' configuration) [1, Ex. 1], [3, Section 4.3], [4, Section 4.4], [5, Section 3.5], [6, Section 6.1], see Figure 5. Let X_0, Y_0, Z_0; X_1, Y_1, Z_1 be two sets of three collinear points (Figure 5(a)). Then Pappus' theorem implies that the three points ("cross-joins") $T_0 = X_0Y_1 \cap X_1Y_0$, $T_1 = Y_0Z_1 \cap Y_1Z_0$, $T_2 = Z_0X_1 \cap Z_1X_0$ are collinear on some line k. The 6-gonal form of Menelaus' theorem applied to the hexagon $[X_0, Y_1, Z_0, X_1, Y_0, Z_1]$ with transversal k yields

$$\frac{\|X_0T_0\|}{\|T_0Y_1\|} \cdot \frac{\|Y_1T_1\|}{\|T_1Z_0\|} \cdot \frac{\|Z_0T_2\|}{\|T_2X_1\|} \cdot \frac{\|X_1T_0\|}{\|T_0Y_0\|} \cdot \frac{\|Y_0T_1\|}{\|T_1Z_1\|} \cdot \frac{\|Z_1T_2\|}{\|T_2X_0\|} = (-1)^6 = 1,$$

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

and so the left side is an isomorphism invariant.

Using Nehring sequences, Pappus' theorem gives rise to another isomorphism invariant as follows, see Figure 5(b).

Let P_0, P_1, P_2 be any three distinct points and b_0, b_1, b_2 be three lines such that each of the points lies on just one of the lines, namely P_0, P_1, P_2 lie on b_2, b_0, b_1 respectively. Then

$$(b_0, P_0, b_1, P_1, b_2, P_2, b_0, P_0, b_1, P_1, b_2, P_2)$$

is a Nehring sequence with $n = 6$, $p = b = 3$. We say it is of period 2 since it consists of a subsequence repeated twice.

A proof of this assertion is given in [10, pp. 38, 43] but an alternative proof is as follows. Choose any point R_0 on b_0 and determine $R_1, R_2, \ldots, R_5, R_6$ by the basic construction. Then the fact that R_0 coincides with R_6 will follow if we can show that the rays P_0R_1 and P_2R_5 intersect at a point on b_0. But this is immediate from Pappus' theorem applied to the triples of collinear points P_0, R_5, R_2 and P_2, R_1, R_4.

The Main Theorem implies that as R_0 varies on b_0,

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_0\|}{\|P_0R_4\|} \cdot \frac{\|R_4P_1\|}{\|P_1R_5\|} \cdot \frac{\|R_5P_2\|}{\|P_2R_0\|} = K. \tag{10}$$

As the value of the constant K depends on the original choice of the points P_0, P_1, P_2 and the lines b_0, b_1, b_2, the left side of (10) is *not* an isomorphism invariant. It can be made into one in the following way. Consider the special case in which R_0 coincides with $B = b_0 \cap b_2$. Then $R_1 = R_2 = A = b_1 \cap b_2$, $R_3 = R_4 = C = b_1 \cap b_2$ and $R_5 = B$, and so

$$\frac{\|BP_0\|}{\|P_0A\|} \cdot \frac{\|AP_1\|}{\|P_1A\|} \cdot \frac{\|AP_2\|}{\|P_2C\|} \cdot \frac{\|CP_0\|}{\|P_0C\|} \cdot \frac{\|CP_1\|}{\|P_1A\|} \cdot \frac{\|AP_2\|}{\|P_2A\|} = -\frac{\|BP_0\|}{\|P_0A\|} \cdot \frac{\|AP_2\|}{\|P_2C\|} \cdot \frac{\|CP_1\|}{\|P_1B\|} = K,$$

since the second, fourth and sixth factors in the left product are each equal to -1. Hence

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_0\|}{\|P_0R_4\|} \cdot \frac{\|R_4P_1\|}{\|P_1R_5\|} \cdot \frac{\|R_5P_2\|}{\|P_2R_0\|} \cdot \frac{\|AP_0\|}{\|P_0B\|} \cdot \frac{\|BP_1\|}{\|P_1C\|} \cdot \frac{\|CP_2\|}{\|P_2A\|} = -1,$$

and the left side of this equality is therefore an isomorphism invariant.

Example 3 (Pascal's configuration) [1, Ex. 8], [2, Section 14.7], [3, Section 7.2], [4, Section 9.2], [5, Section 3.8], [6, Section 6.2], see Figure 6. Choose three non-collinear points P_0, P_1, B and two lines b_0, b_1 which do not pass through the chosen points. Define $P_0B \cap b_1 = C$, $P_1B \cap b_0 = D$, $P_0P_1 \cap CD = P_4$, $b_0 \cap b_1 = E$ and $b_2 = BE$. Then

$$(b_0, P_0, b_1, P_1, b_2, P_0, b_0, P_1, b_1, P_4)$$

is a Nehring sequence with n = 5 and p = b = 3.

To prove this, let R_0 be any point on b_0 and determine R_1, R_2, R_3, R_4 by the basic construction. The assertion will be proved if we can show that the points P_4, R_0 and R_4 are collinear. To do this we first observe that $R_1C \cap R_3D = E$, $P_0C \cap P_1D = B$ and $P_1R_1 \cap P_0R_3 = R_2$ are collinear since these three points lie on b_2. These are the "cross-joins" of the two triples of points R_1, D, P_0 and R_3, C, P_1. Therefore, by the converse of Pascal's theorem, these six points lie on a conic section. Rearranging, we see that the triples R_1, D, P_1 and R_3, P_0, C lie on a conic and so their cross-joins $R_1P_0 \cap R_3D = R_0$, $R_1C \cap R_3P_1 = R_4$ and $DC \cap P_0P_1 = P_4$ are collinear, as required.

The Main Theorem implies that for any point R_0 on b_0, since the base lines are concurrent,

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_0\|}{\|P_0R_3\|} \cdot \frac{\|R_3P_1\|}{\|P_1R_4\|} \cdot \frac{\|R_4P_4\|}{\|P_4R_0\|} = (-1)^5 = -1$$

and therefore the left side is an isomorphism invariant.

Example 4 [10, p. 42], see Figure 7. Let $X_0, X_1, \ldots, X_5$ be the vertices of a hexagon of which no three vertices are collinear and no three sides are concurrent. Define $P_0 = X_0X_1 \cap X_3X_4$, $P_1 = X_1X_2 \cap X_4X_5$, $P_2 = X_2X_3 \cap X_5X_0$, $b_0 = X_0X_3$, $b_1 = X_1X_4$, $b_2 = X_2X_5$; then

$$(b_0, P_0, b_1, P_1, b_2, P_2, b_0, P_0, b_1, P_1, b_2, P_2)$$

is a Nehring sequence with $n = 6$, $p = b = 3$, and period 2.

A proof of this fact follows simply from the expression of the Nehring property in terms of perspectivities. Details are given in [10, Theorem 1]. This example can be generalised to any odd $t \geq 3$: starting from a $2t$-gon, the analogous construction (with the pivots as the intersection of opposite sides) yields a Nehring sequence of length $2t$ and period 2.

In the case $t = 3$, the Main Theorem implies that, starting from any point R_0 on b_0, with $R_1, R_2, \ldots, R_5$ determined by the basic construction,

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_0\|}{\|P_0R_4\|} \cdot \frac{\|R_4P_1\|}{\|P_1R_5\|} \cdot \frac{\|R_5P_2\|}{\|P_2R_0\|} = K. \tag{11}$$

The value of the constant K depends on the choice of hexagon originally chosen to define the figure, and so the left side of (11) is *not* an isomorphism invariant. However it can be made into one in the following way. Clearly

$$\frac{\|X_0P_0\|}{\|P_0X_1\|} \cdot \frac{\|X_1P_1\|}{\|P_1X_2\|} \cdot \frac{\|X_2P_2\|}{\|P_2X_3\|} \cdot \frac{\|X_3P_0\|}{\|P_0X_4\|} \cdot \frac{\|X_4P_1\|}{\|P_1X_5\|} \cdot \frac{\|X_5P_2\|}{\|P_2X_0\|} = K,$$

and so

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_0\|}{\|P_0R_4\|} \cdot \frac{\|R_4P_1\|}{\|P_1R_5\|} \cdot \frac{\|R_5P_2\|}{\|P_2R_0\|} \cdot \frac{\|X_1P_0\|}{\|P_0X_0\|} \cdot \frac{\|X_2P_1\|}{\|P_1X_1\|} \cdot \frac{\|X_3P_2\|}{\|P_2X_2\|} \cdot \frac{\|X_4P_0\|}{\|P_0X_3\|} \cdot \frac{\|X_5P_1\|}{\|P_1X_4\|} \cdot \frac{\|X_0P_2\|}{\|P_2X_5\|} = 1$$

and the left side of this equality is an isomorphism invariant.

The next three examples relate to some interesting configurations which are not well-known. In each case the corresponding Nehring sequences have $n = b = 4$ and $p = 3$ or 4.

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

Example 5 (Perspective triangles) [1, Ex 59], see Figure 8. If $[A, B, C]$ and $[D, E, F]$ are two triangles such that AD, BE, CF are concurrent in a point X, they are said to be in perspective from X. If also $[A, B, C]$, $[E, F, D]$ are in perspective from Y, we say they are in *double perspective*. Then, as Chou succinctly puts it, "doubly perspective triangles are, in fact, triply perspective", in other words, under these circumstances, $[A, B, C]$, $[F, D, E]$ are in perspective from a point Z, see Figure 8(a). In fact the configuration is symmetrical in that the three triangles $[A, B, C]$, $[D, E, F]$, $[X, Y, Z]$ are such that, with appropriate ordering of the vertices, every pair are in perspective from each vertex of the third.

This is easy to prove using Pappus' theorem. One way to convert this configuration into a Nehring sequence is to define pivots $P_0 = E$, $P_1 = D$, $P_2 = F$ and bases $b_0 = AP_2$, $b_1 = XP_2$, $b_2 = AP_0$ and $b_3 = XP_0$, see Figure 10(b). Since Z can be chosen as any point on b_0, put $R_0 = Z$ and then $R_1 = C$, $R_2 = Y$, $R_3 = B$ have the required incidences and so

$$(b_0, P_0, b_1, P_1, b_2, P_2, b_3, P_1)$$

is a Nehring sequence with $n = b = 4$ and $p = 3$. By the Main Theorem, as R_0 varies on b_0,

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_1\|}{\|P_1R_0\|} = K. \tag{12}$$

To find the value of the constant K, write $U = b_1 \cap b_2$ $(= EA \cap FX)$ and $G = P_0P_2 \cap XA$ $(= EF \cap XA)$. Consider the special case in which R_0 coincides with A. Then $R_1 = R_2 = U$ and $R_3 = X$, so the left side of (12) becomes

$$\frac{\|AE\|}{\|EU\|} \cdot \frac{\|UD\|}{\|DU\|} \cdot \frac{\|UF\|}{\|FX\|} \cdot \frac{\|XD\|}{\|DA\|}.$$

The second factor is -1 and $(\|AE\|/\|EU\|) \cdot (\|UF\|/\|FX\|) = -\|GA\|/\|XG\|$ by Menelaus' Theorem applied to the triangle $[A, U, X]$ and transversal EF. Hence the value of the constant K in (12) is the cross-ratio $(\|GA\|/\|XG\|) \cdot (\|XD\|/\|DA\|) = \mathrm{cr}(G, D; A, X)$. Consequently

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_1\|}{\|P_1R_0\|} = \frac{\|GA\|}{\|XG\|} \cdot \frac{\|XD\|}{\|DA\|}$$

and so

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_1\|}{\|P_1R_0\|} \cdot \frac{\|XG\|}{\|GA\|} \cdot \frac{\|AD\|}{\|DX\|} = 1$$

and therefore the left side of this equality is an isomorphism invariant.

Example 6 (The double Ceva configuration) [1, Ex. 26], [6, Section 2.3], see Figure 9. Let $T_1 = [X_1, Y_1, Z_1]$ be a triangle and U any point not on a side of T_1. Let X_1U, Y_1U, Z_1U meet the sides Y_1Z_1, Z_1X_1, X_1Y_1 of T_1 the points X_2, Y_2, Z_2 respectively. Let $T_2 = [X_2, Y_2, Z_2]$ and W be any point not on a side of T_2. Let X_2W, Y_2W, Z_2W meet the sides Y_2Z_2, Z_2X_2, X_2Y_2 of T_2 in X_3, Y_3, Z_3 respectively. Then the lines X_1X_3, Y_1Y_3, Z_1Z_3 are concurrent, see Figure 9(a).

To prove this we apply the 4-gonal form of Menelaus' theorem three times, to the quadrangles $[Z_2, Y_2, Z_1, Y_1]$, $[X_2, Z_2, X_1, Z_1]$, $[Y_2, X_2, Y_1, X_1]$, and then Ceva's theorem to the triangles T_1 and T_2. Details are left to the reader.

This configuration gives rise to a Nehring sequence with $n = p = b = 4$ as follows. Let the points Z_2, Y_2, Y_1, Z_1 be the pivots P_0, P_1, P_2, P_3, and define bases $b_0 = X_2P_1$ $(= X_2Y_2)$, $b_1 = X_2X_3$, $b_2 = X_2P_0$ $(= X_2Z_2)$ and $b_3 = X_1X_3$. Then

$$(b_0, P_0, b_1, P_1, b_2, P_2, b_3, P_3)$$

is a Nehring sequence with $n = p = b = 4$, see Figure 9(b).

By the Main Theorem,

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_3\|}{\|P_3R_0\|} = K. \tag{13}$$

To determine the value of the constant K, consider the special case where $R_0 = X_2$. Then clearly $R_1 = R_2 = X_2$ also, and $R_3 = X_4$. Substituting in the above, and simplifying, we obtain $K = \mathrm{cr}(X_2, X_4; Y_1, Z_1)$ as the value of the constant, and so, as in the previous example,

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_3\|}{\|P_3R_0\|} \cdot \frac{\|Z_1X_2\|}{\|X_2Y_1\|} \cdot \frac{\|X_1X_4\|}{\|X_4Z_1\|} = 1$$

and the left side of this equality is an isomorphism invariant.

Example 7 (The double transversal configuration) [1, Ex. 61], see Figure 10. Let $T_1 = [X_1, Y_1, Z_1]$ be a triangle and U any point not on a side of T_1. Let X_1U, Y_1U, Z_1U meet the sides Y_1Z_1, Z_1X_1, X_1Y_1 of T_1 at the points X_2, Y_2, Z_2 respectively. Let an arbitrary line k meet the sides Y_2Z_2, Z_2X_2, X_2Y_2 of T_2 in X_3, Y_3, Z_3 respectively. Define $X_1X_3 \cap Y_1Z_1 = X_4$, $Y_1Y_3 \cap Z_1X_1 = Y_4$ and $Z_1Z_3 \cap X_1Y_1 = Z_4$. Then the points X_4, Y_4, Z_4 are collinear, see Figure 10(a).

This can be proved by applying Menelaus' theorem to the same three quadrangles as in the previous example.

This configuration gives rise to a Nehring sequence with $n = p = b = 4$, in the following way. Let the points Z_1, Y_3, X_1, Y_4 be the pivots P_0, P_1, P_2, P_3 and define $b_0 = X_1Y_1$, $b_1 = Y_2X_3$, $b_2 = Y_2Z_2$ and $b_3 = Y_1Z_1$. Then

$$(b_0, P_0, b_1, P_1, b_2, P_2, b_3, P_3)$$

is a Nehring sequence, see Figure 10(b).

By the Main Theorem equation (13) holds for this new sequence. To determine the value of the constant consider the special case $R_0 = X_1$. Then $R_1 = R_2 = Y_2$ and $R_3 = Z_1$. Substituting in (13) yields $\mathrm{cr}(Y_2, Y_4; X_1, Z_1)$ as the value of the constant and hence, as in the previous example,

$$\frac{\|R_0P_0\|}{\|P_0R_1\|} \cdot \frac{\|R_1P_1\|}{\|P_1R_2\|} \cdot \frac{\|R_2P_2\|}{\|P_2R_3\|} \cdot \frac{\|R_3P_3\|}{\|P_3R_0\|} \cdot \frac{\|Z_1X_2\|}{\|X_2Y_1\|} \cdot \frac{\|X_1Y_4\|}{\|Y_4Z_1\|} = 1$$

and the left side of this equality is an isomorphism invariant.

The final two examples are included to illustrate the fact that Nehring sequences exist with arbitrarily many pivots and bases. Proofs of the existence of these Nehring sequences can be found in [10].

Each construction involves a polygon, and although these are shown as convex in the diagrams, they can be quite general. Vertices may be collinear, edges intersect or overlap, *etc.* The only requirement is that the figure should be well-defined. Thus if a point is defined as the intersection of two lines, then these lines must be distinct—if it is defined as the join of two points, then these points must be distinct, and so on.

Example 8 [10, p. 46], see Figure 11. Let t be any odd integer greater than unity, and put $t = 2s + 1$ ($s \geq 1$). Let $Q = [P_0, P_1, \ldots, P_{t-1}]$ be a t-gon and O any point not on a side of Q. For $i = 0, 1, \ldots, t - 1$ let b_i be the line OP_{i+s}. Then

$$(b_0, P_0, b_1, P_1, \ldots, b_{t-1}, P_{t-1}, b_0, P_0, \ldots, b_{t-1}, P_{t-1})$$

is a Nehring sequence of period 2 with $n = 2t$, $p = b = t$. In Figure 11 we show an example with $t = 5$.

The Main Theorem tells us that as P_0 varies on b_0, with $R_1, R_2, \ldots, R_{2t-1}$ determined by the basic construction,

$$\prod_{i=0}^{2t-1} \frac{\|R_i P_i\|}{\|P_i R_{i+1}\|} = 1$$

where the subscripts of the pivots P_i are reduced modulo t and those of the points R_i are reduced modulo $2t$. The left side of this equality is an isomorphism invariant.

Example 9 [10, p. 45], see Figure 12. Let t be any even integer greater than or equal to 6. Let $Q = [P_0, P_1, \ldots, P_{n-1}]$ be a t-gon and O be any point not on a side of Q. For each $i = 0, 1, \ldots, t - 1$ let b_i be the line joining O to $U_i = P_{i-2}P_{i-1} \cap P_i P_{i+1}$. Then

$$(b_0, P_0, b_1, P_1, \ldots, b_{t-1}, P_{t-1})$$

is a Nehring sequence with $n = p = b = t$. Figure 12 shows an example with $t = 6$.

The Main Theorem tells us that as P_0 varies on b_0,

$$\prod_{i=0}^{n-1} \frac{\|R_i P_i\|}{\|P_i R_{i+1}\|} = 1,$$

where the subscripts of all the points are reduced modulo n. The left side of this equality is therefore an isomorphism invariant.

3 Conclusion

The reason for the name Nehring sequence is as follows. In the nineteen-forties Otto Nehring [7], [8] published several papers describing cyclic incidence properties for polygons. In 1996 W. Reyes [9], apparently unaware of Nehring's work, described a similar property for a triangle. In [10] the theory was unified by the introduction of the concept of a Nehring sequence. Besides their intrinsic interest, Nehring sequences can be used to provide alternative definitions of many familiar concepts in projecive geometry.

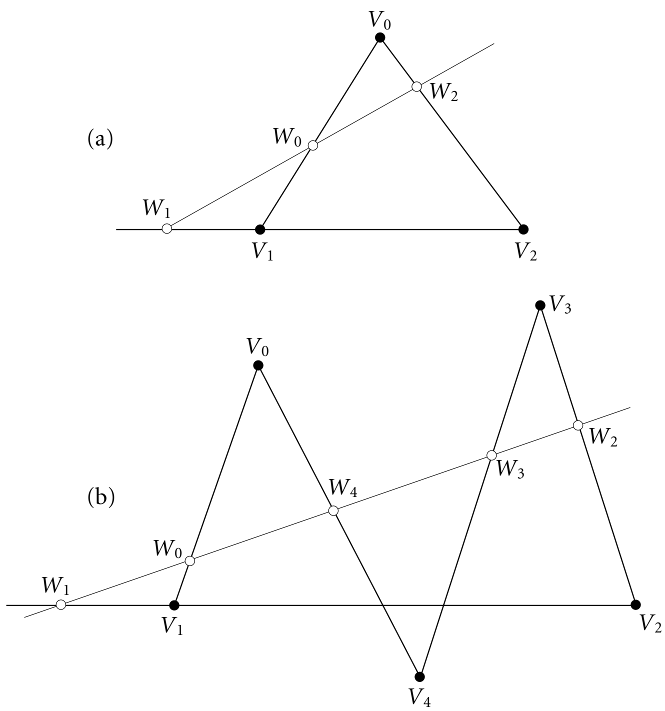


Figure 1

The most interesting feature of the above presentation is probably not in the results themselves, but in the fact that, so far as we are aware, they were not discovered many years ago. This cannot be attributed to the fact that the Main Theorem was unknown, for, in the case of collinear pivots, as we can see from Example 2, similar results can be obtained using Menelaus' Theorem. More likely it is due to the fact that Eves' Theorem had not been discovered (and even today this seems to be known to comparatively few geometers). The left side of (5) is clearly an affine invariant, but until recently it was not realised that it was also a projective invariant. Consequently the inclusion of a term like this was regarded as foreign to the study of projective geometry.

The main consequence of our treatment is that now, instead of just one numerical projective invariant (the cross-ratio), we have many isomorphism invariants, which are also projective invariants, and the means (Nehring sequences) of constructing many more—the list of examples in the previous section could be extended indefinitely. The isolated example of the cross-ratio now seems a "poor relation" to the wealth of invariant material, both in projective geometry and in the study of isomorphic configurations, that is now available.

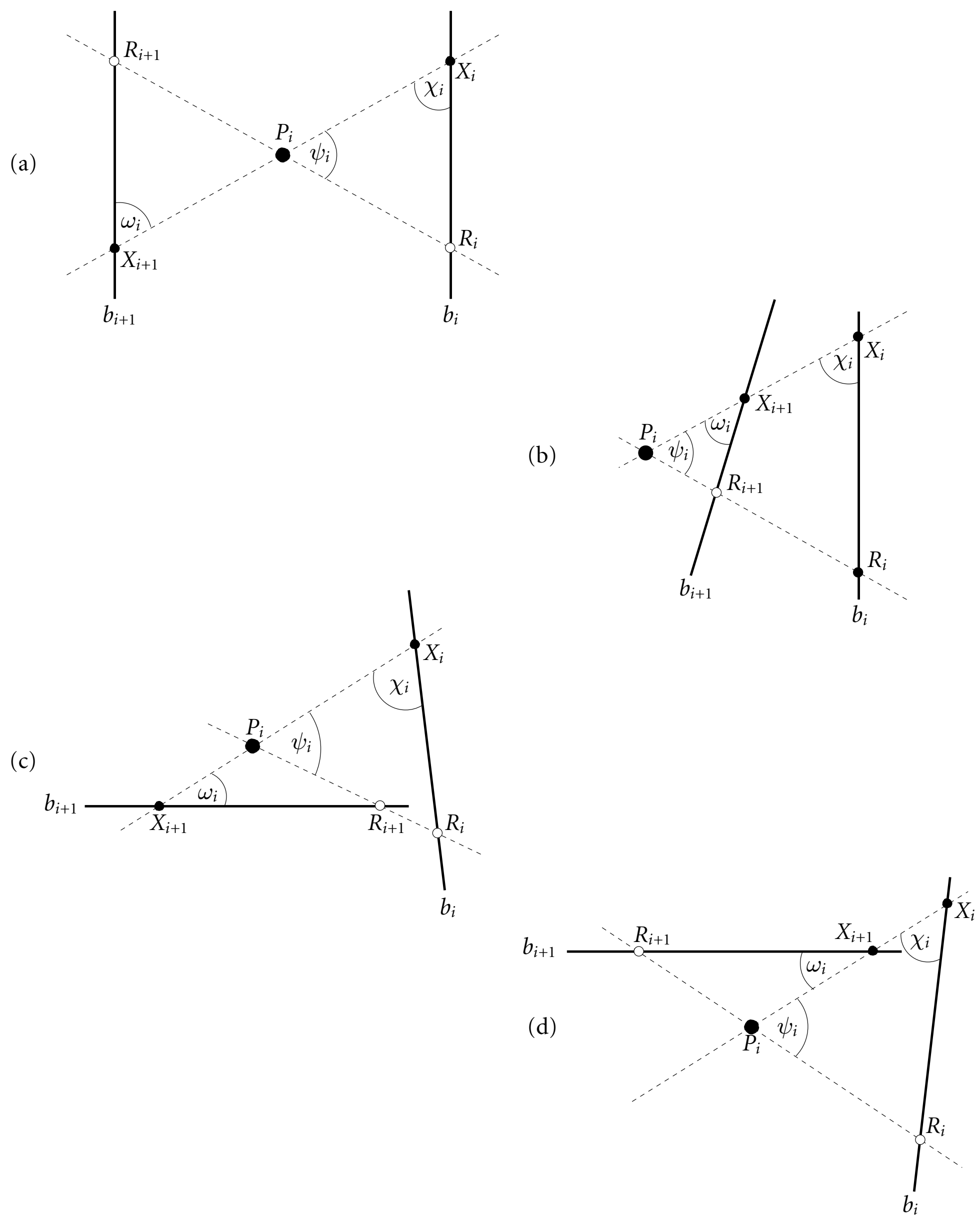


Figure 2

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

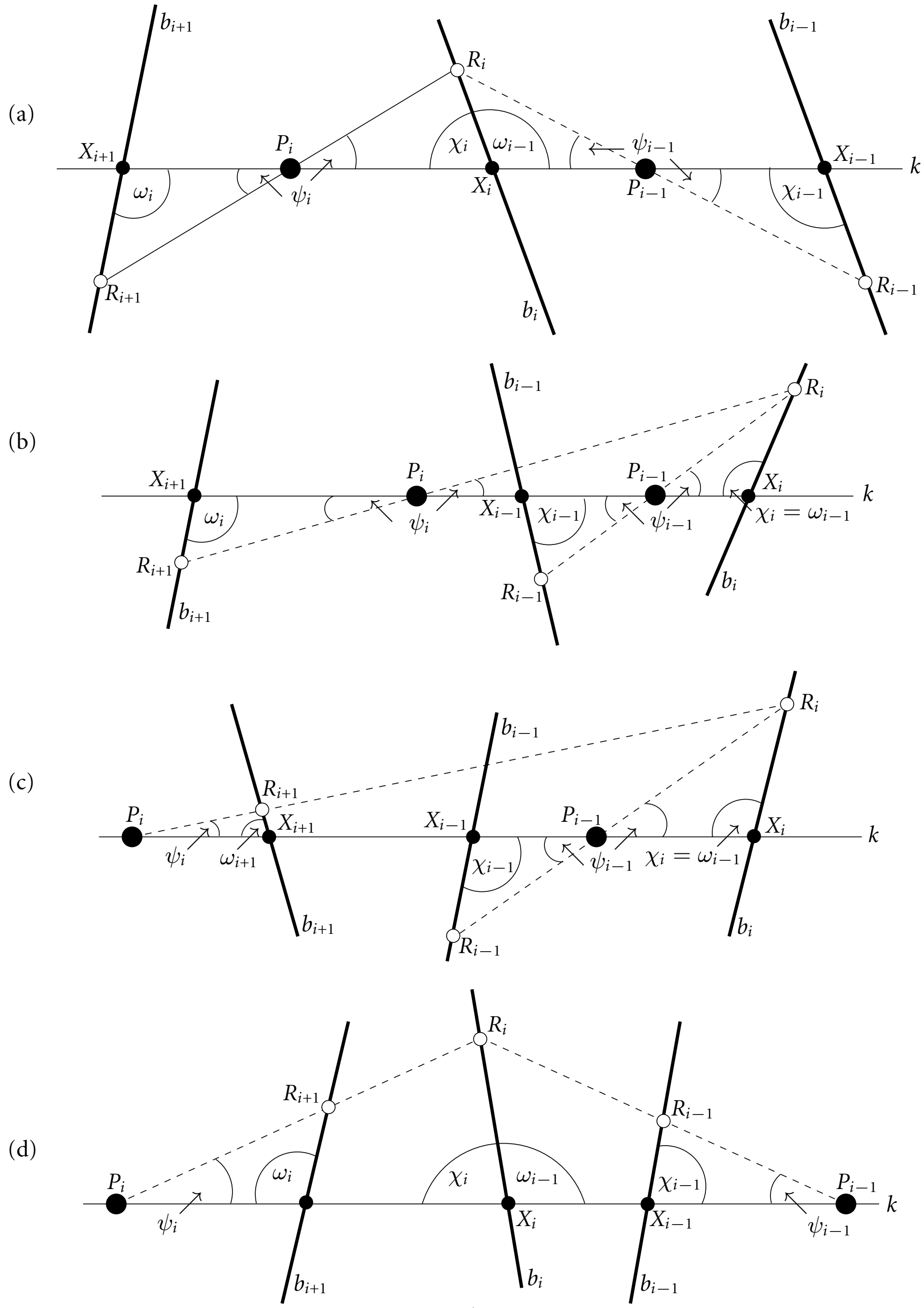


Figure 3

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

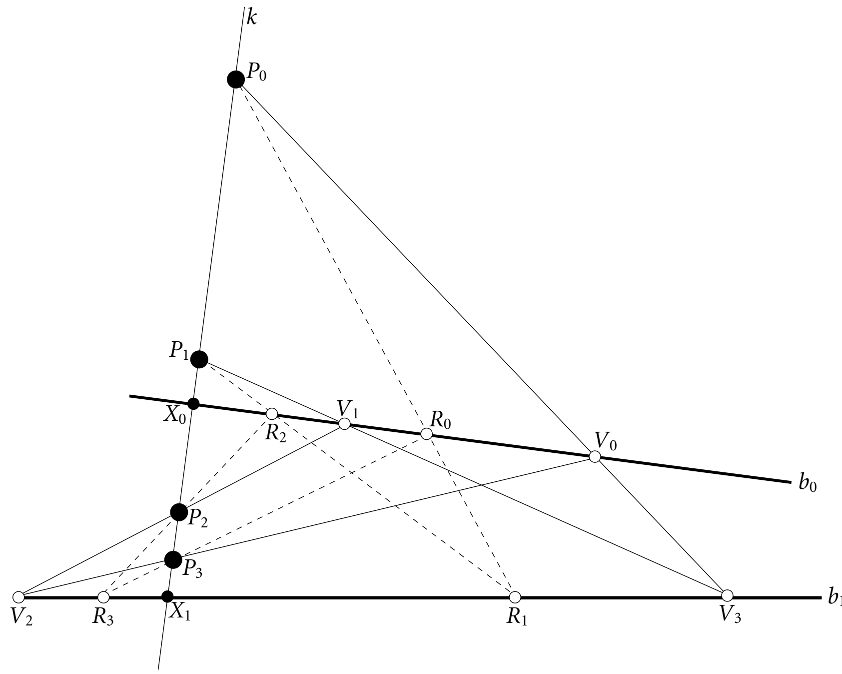


Figure 4

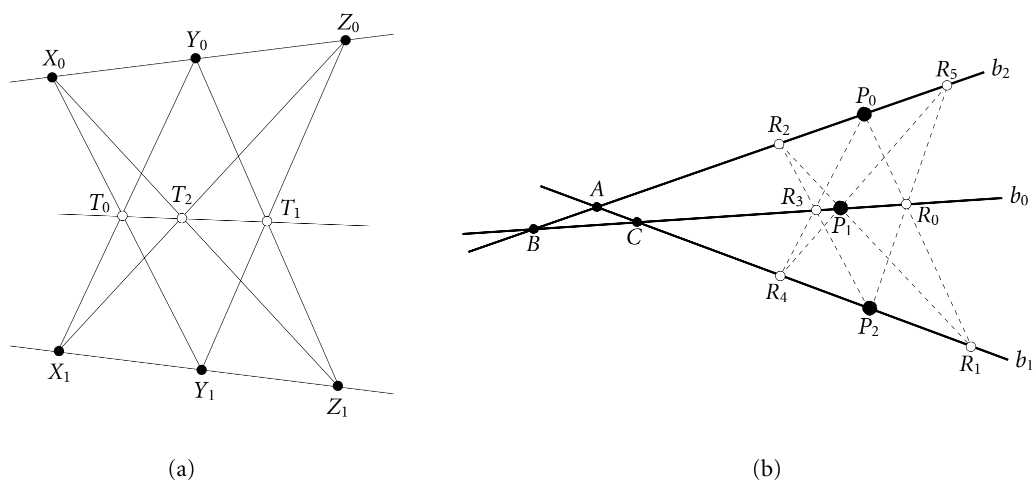


Figure 5

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

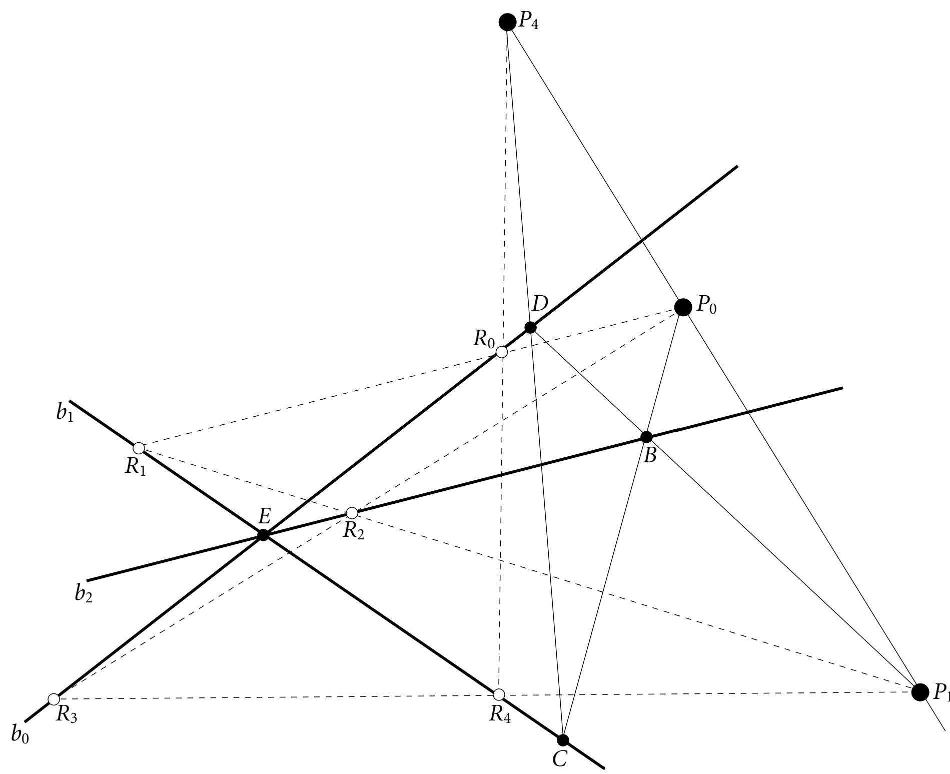


Figure 6

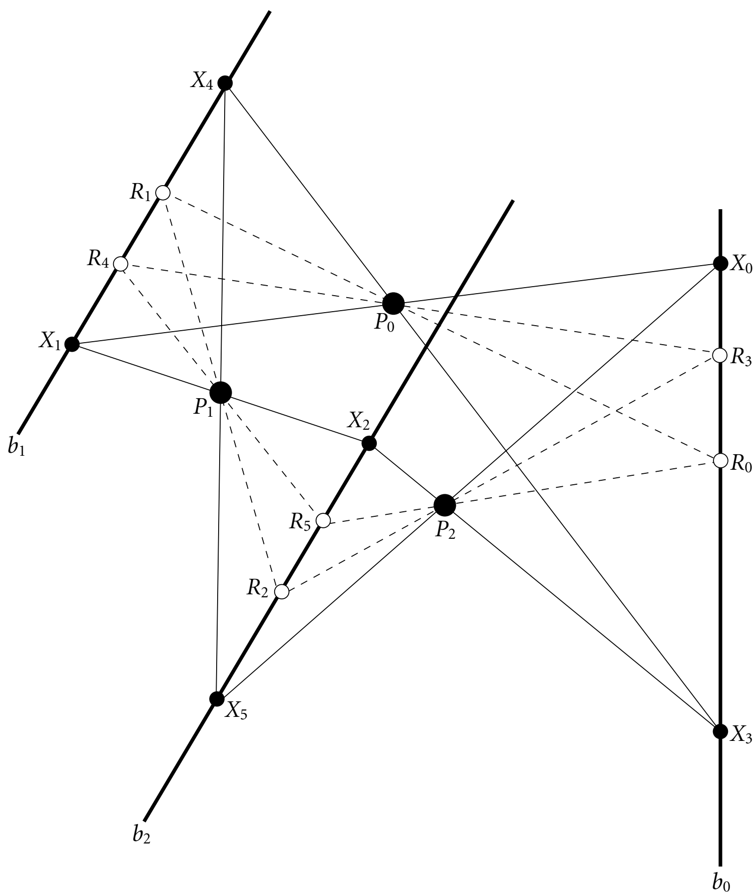
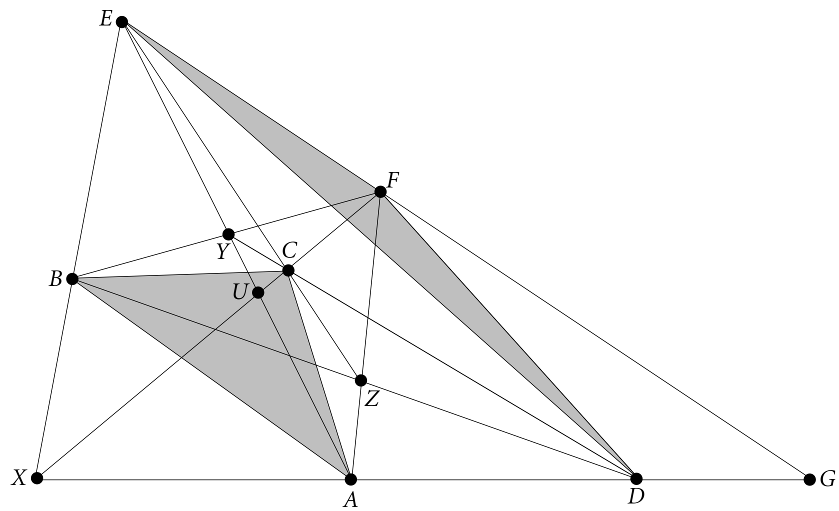
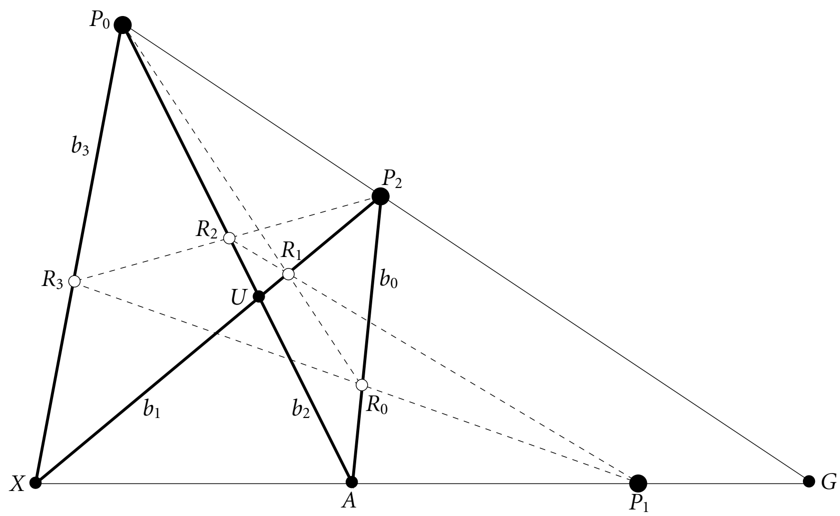


Figure 7



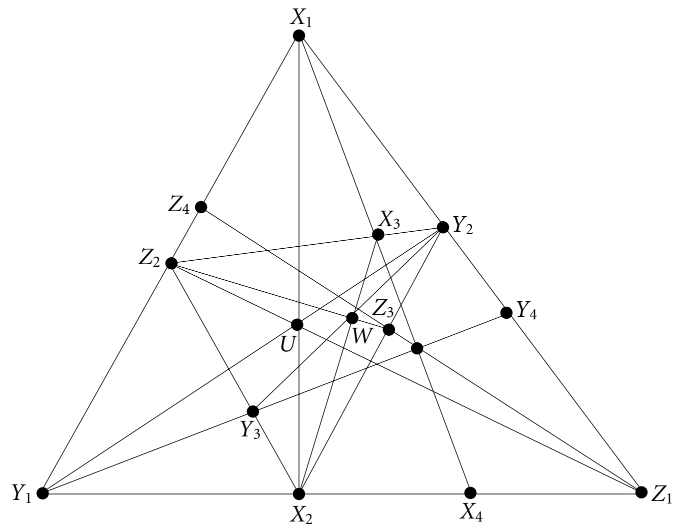
(a)



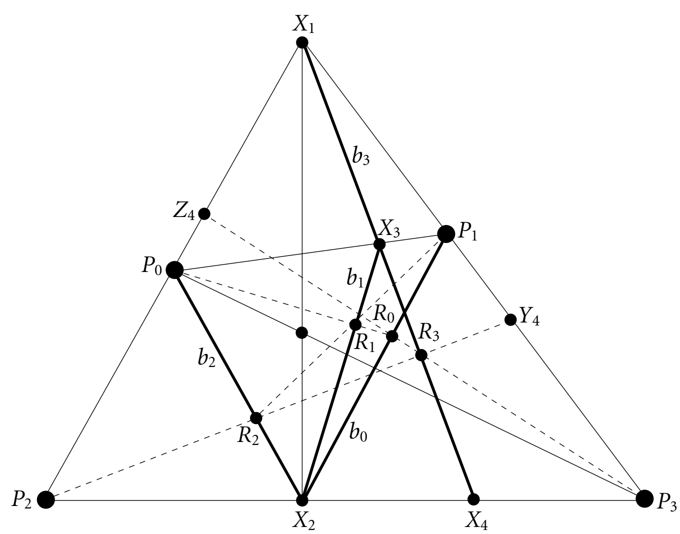
(b)

Figure 8

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press



(a)



(b)

Figure 9

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

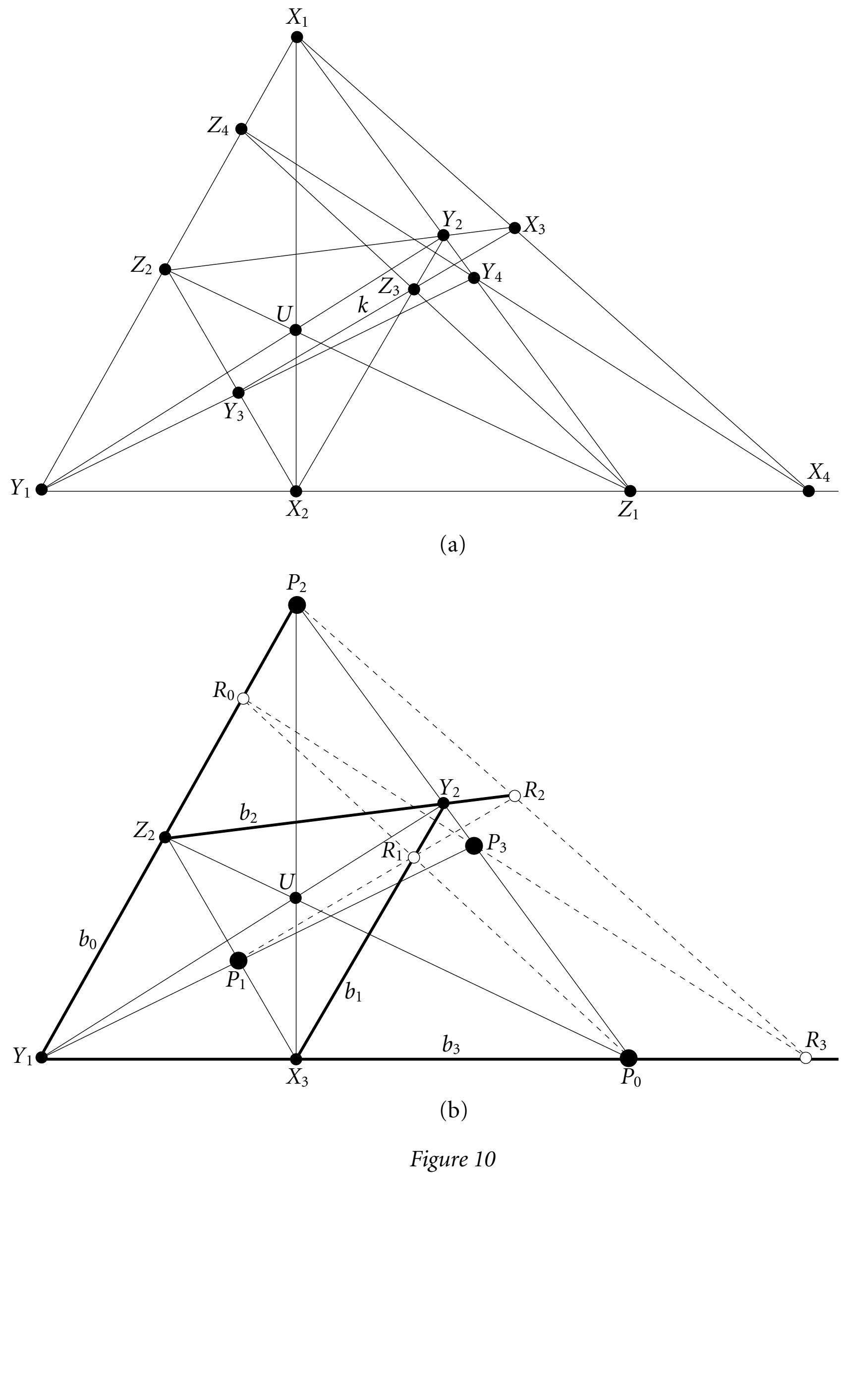


Figure 10

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

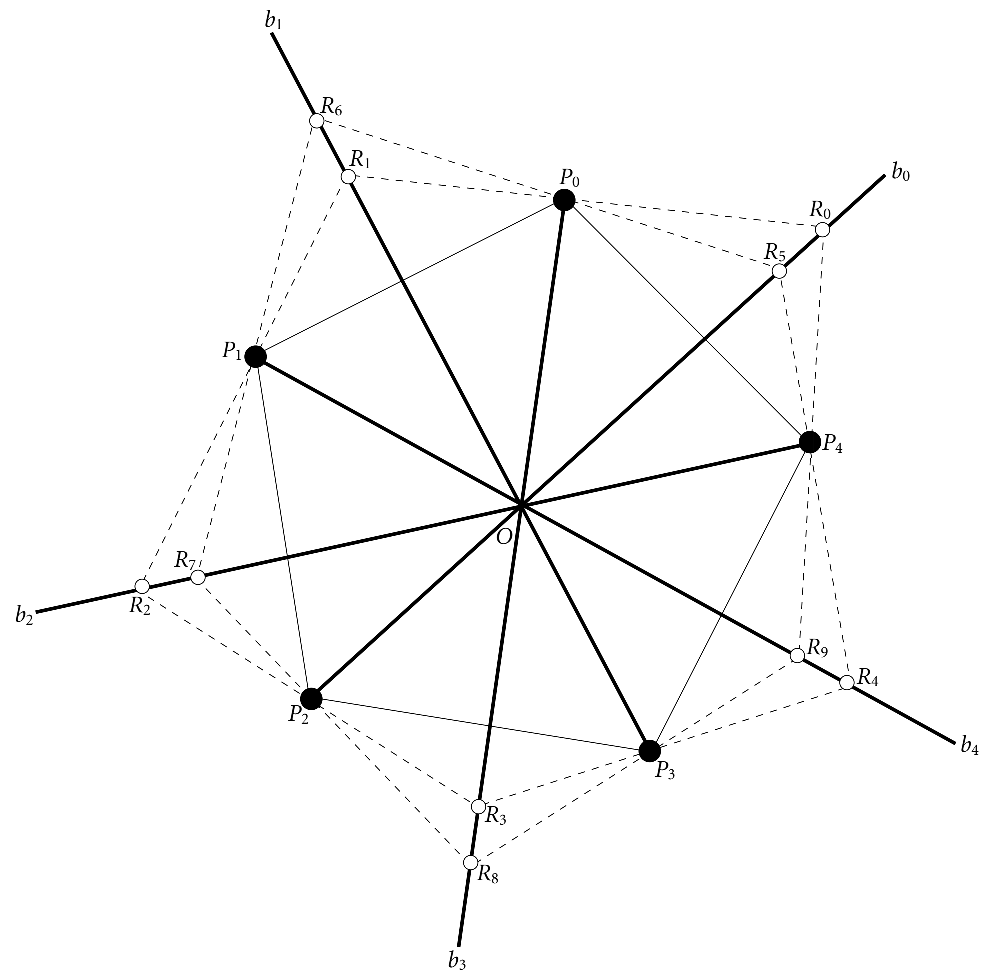


Figure 11

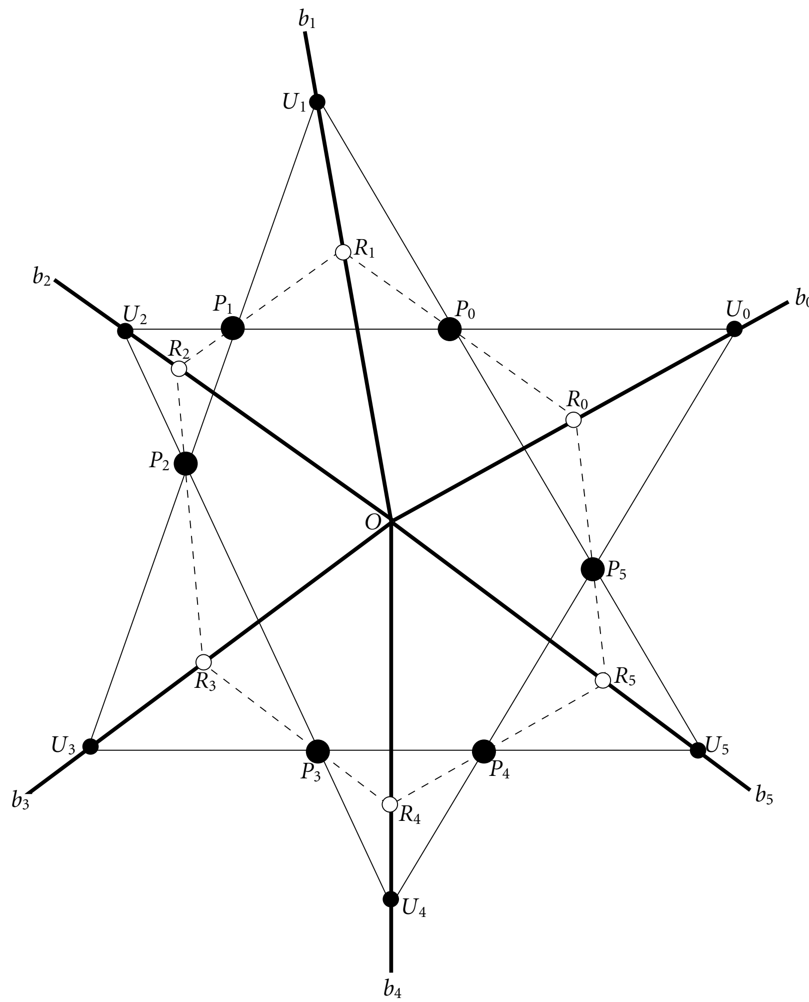


Figure 12

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press

References

[1] S.-C. Chou, *Mechanical Geometry Theorem Proving.* D. Reidel Publishing Company, Dordrecht-Boston-Lancaster-Tokyo, 1988.
[2] H. S. M. Coxeter, *Introduction to Geometry.* 2nd edition, John Wiley and Sons, New York-London-Sydney-Hong Kong, 1969.
[3] ———, *The Real Projective Plane.* 2nd edition, Cambridge University Press, 1955.
[4] ———, *Projective Geometry.* 2nd edition, University of Toronto Press, 1974.
[5] H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited.* Mathematical Association of America, 1967.
[6] Howard W. Eves, *A Survey of Geometry.* Allyn and Bacon Inc., Boston, 1972.
[7] O. Nehring, *Aufgabe 278 und Lösung.* Jahresber. Deutsch. Math.-Verein **49**(1939), 29; **50**(1940), 80–83.
[8] ———, *Zyklische Projectionen in Dreieck.* J. Reine Angew. Math. **184**(1942), 129–137.
[9] W. Reyes, *On a theorem in circle geometry.* Nieuw Arch. Wisk. **14**(1996), 231–233.
[10] G. C. Shephard, *The Nehring-Reyes' theorems for polygons.* Nieuw Arch. Wisk. **16**(1998), 37–56.

School of Mathematics
University of East Anglia
Norwich NR4 7TJ
U.K.
email: g.c.shephard@uea.ac.uk

https://doi.org/10.4153/CJM-1999-058-8 Published online by Cambridge University Press