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STRONG BARRELLEDNESS PROPERTIES IN $B(\Sigma, X)$

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In this paper we show that given a σ-algebra Σ of subsets of a set Ω and a normed space X, then the normed space $B(\Sigma, X)$, endowed with the usual supremum-norm, of the X-valued functions defined on Ω that are the uniform limit of a sequence of Σ-simple X-valued functions on Ω is barrelled of class s if and only if X is barrelled of class s. This extends in the normed case the well known result obtained by Mendoza (1982) for barrelled spaces.

Let Ω be a set and let Σ be a σ-algebra of subsets of Ω. Given a normed space X, $S(\Omega, \Sigma, X)$, or simply $S(\Sigma, X)$, will denote the linear space of the Σ-simple X-valued functions defined on Ω and $B(\Sigma, X)$ the linear space of the X-valued functions defined on Ω that are the uniform limit of a sequence of elements of $S(\Sigma, X)$, these two linear spaces being endowed with the topology defined by the usual supremum-norm $\|f\| = \sup\{\|f(\omega)\| : \omega \in \Omega\}$.

By [**6**], $B(\Sigma, X)$ is barrelled if and only if X is barrelled and, by [**4**], $S(\Sigma, X)$ is barrelled if and only if X is finite-dimensional. In this paper we shall show, without using duality theory, that the normed space $B(\Sigma, X)$ is barrelled of class s if and only if X is barrelled of class s. This extends the results of [**6**] when X is normed and Σ is a σ-algebra since [**6**] deals with an algebra Σ, a barrelled locally convex space such that X'_β has property (B) of Pietsch [**5, 8**], and uses duality theory. Our methods are based upon those of [**1**] and [**3**].

Let us now recall that a space E is Baire-like [**9**] if, given any increasing sequence of closed absolutely convex subsets of E covering E, there is one that is a neighbourhood of the origin; E is *bd* or suprabarrelled [**10, 11**] if, given any increasing sequence of subspaces of E covering E, there is one that is dense and barrelled; and E is totally barrelled [**7, 12**] if, given any sequence of subspaces of E covering E, there is one that is Baire-like. Given $s \in \mathbb{N}$, and considering as $\mathcal{C}_0$ the class of Baire-like spaces, a space E is said to be barrelled of class s [**2**], or briefly $E \in \mathcal{C}_s$, if, given any increasing sequence of subspaces of E covering E, there is one that belongs to $\mathcal{C}_{s-1}$; and E is

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said to be barrelled of class $\aleph_0$ if $E \in \mathcal{C}_s$ for every $s \in \mathbb{N}$. Clearly, $\mathcal{C}_1$ coincides with the class of suprabarrelled spaces.

For every $s \in \mathbb{N}$ we have that

$$\text{Barrelled} \supset \mathcal{C}_{s-1} \supset \mathcal{C}_s \supset \text{Barrelled of class } \aleph_0 \supset \text{Totally barrelled.}$$

In the sequel we shall denote by $B_c(\Sigma, X)$ the dense linear subspace of $B(\Sigma, X)$ formed by those functions that take at most a countable number of different values. Given $A \in \Sigma$, the spaces $S(A, \Sigma|_A, X)$ and $B_c(\Sigma|_A, X)$ will be denoted by $S(A, X)$ and $B_c(A, X)$ respectively. We shall identify these spaces with their natural embeddings into $B(\Sigma, X)$.

If $f \in S(\Sigma, X)$, then obviously there exists a partition $\{A_1, A_2, \ldots, A_n\}$ of Ω formed by non void elements of Σ and some distinct vectors $\{x_1, x_2, \ldots, x_n\}$ of X such that $f(\omega) = x_1$ if $\omega \in A_1$, $1 \leqslant i \leqslant n$. On the other hand, if $f \in B_c(\Sigma, X) \setminus S(\Sigma, X)$ and $\{x_n : n \in \mathbb{N}\}$ is the bounded sequence of the distinct values taken by f, setting $A_n := f^{-1}(\{x_n\})$ for each $n \in \mathbb{N}$, we shall show that $\{A_n : n \in \mathbb{N}\}$ is an infinite partition of Ω formed by nonempty elements of Σ such that $f(\omega) = x_n$ if $\omega \in A_n$, $n \in \mathbb{N}$. In fact, if $\{f_m, m \in \mathbb{N}\}$ is a sequence of elements of $S(\Sigma, X)$ which converges uniformly to f, then define $Y := \langle \{x_n : n \in \mathbb{N}\} \cup \{f_m(\omega), m \in \mathbb{N}, \omega \in \Omega\}\rangle = \langle\{\operatorname{Im} f, \operatorname{Im} f_m, m \in \mathbb{N}\}\rangle$, and choose a weak*-total sequence $\{y_j^*, j \in \mathbb{N}\}$ in Y^*. Now each scalar function $y_j^* f$ is Σ-measurable since it is the pointwise limit of the sequence of Σ-simple functions $\{y_j^* f_m, m \in \mathbb{N}\}$. So, $A_{n,j} := \{\omega \in \Omega, y_j^* f(\omega) = y_j^* x_n\} = \left(y_j^* f\right)^{-1}\left(y_j^* x_n\right) \in \Sigma$ for each $j, n \in \mathbb{N}$. As A_n coincides with $\bigcap\{A_{n,j}, j \in \mathbb{N}\}$, then $A_n \in \Sigma$.

In the following three results, s is any positive integer, $\{E_{m_1} : m_1 \in \mathbb{N}\}$ is an increasing sequence of dense subspaces of $B_c(\Sigma, X)$ covering $B_c(\Sigma, X)$, Σ being a σ-algebra of subsets of a set Ω, and for each $p \in \{2, \ldots, s\}$ and $m_1, \ldots, m_{p-1} \in \mathbb{N}$, $\{E_{m_1 \ldots m_{p-1} m_p} : m_p \in \mathbb{N}\}$ is an increasing sequence of dense subspaces of $E_{m_1 \ldots m_{p-1}}$ covering $E_{m_1 \ldots m_{p-1}}$. For each $m_1, \ldots, m_s \in \mathbb{N}$, suppose $T_{m_1 \ldots m_s}$ is a barrel of $E_{m_1 \ldots m_s}$, $B_{m_1 \ldots m_s}$ is its closure in $B_c(\Sigma, X)$ and $L_{m_1 \ldots m_s} := \langle B_{m_1 \ldots m_s} \rangle$. By decreasing recurrence, for $p = s-1, \ldots, 1$, define the subspaces $F_{m_1 \ldots m_{p+1}} := \bigcap\{L_{m_1 \ldots m_p m} : m \geqslant m_{p+1}\}$, $L_{m_1 \ldots m_p} := \bigcup\{F_{m_1 \ldots m_p m} : m \in \mathbb{N}\}$, and $F_{m_1} := \bigcap\{L_m : m \geqslant m_1\}$. Notice that $\{F_m : m \in \mathbb{N}\}$ and $\{F_{m_1 m_2 \ldots m_p m} : m \in \mathbb{N}\}$ are increasing sequences of subspaces of $B_c(\Sigma, X)$ and $L_{m_1 \ldots m_p}$, respectively covering them, $\forall m_r \in \mathbb{N}$, $1 \leqslant r \leqslant p \leqslant s-1$, and $E_{m_1 \ldots m_p} \subset F_{m_1 \ldots m_p}$, for all $m_r \in \mathbb{N}$, $1 \leqslant r \leqslant p \leqslant s$.

Lemma 1. *If $\{A_n : n \in \mathbb{N}\}$ is a sequence of non void pairwise disjoint elements of Σ, then there exists some $n_0 \in \mathbb{N}$ such that $B_c(\bigcup\{A_n : n > n_0\}, X) \subset F_{n_0}$.*

Proof: Assume the lemma is false and that for each $p \in \mathbb{N}$ there is some $f_p \in$

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$B_c(\bigcup\{A_n : n > p\}, X) \setminus F_p$ such that $\|f_p\| = 1$. Then $\{f_n : n \in \mathbb{N}\}$ is bounded in $B_c(\Sigma, X)$ and if $\xi \in \ell_1$, $\sum_{n=1}^{\infty} \xi_n f_n$ converges in the completion of $B_c(\Sigma, X)$.

Now $\sum_{n=1}^{\infty} \xi_n f_n$ takes at most countably many values in X since if $\omega \in \Omega \setminus \bigcup\{A_n : n \in \mathbb{N}\}$ then $\sum_{n=1}^{\infty} \xi_n f_n(\omega) = 0$ and if $\omega \in \bigcup\{A_n : n \in \mathbb{N}\}$ there is some $q \in \mathbb{N}$ such that $\omega \in A_q$, that is $\omega \notin \bigcup\{A_n : n > q\}$ and, since $\operatorname{supp} f_n \subset \bigcup\{A_i : i > n\}$, $\sum_{n=1}^{\infty} \xi_n f_n(\omega) = \sum_{n=1}^{q} \xi_n f_n(\omega)$. Moreover, the sequence $\{\sum_{n=1}^{m} \xi_n f_n,\ m \in \mathbb{N}\}$ of $B_c(\Sigma, X)$ converges to $\sum_{n=1}^{\infty} \xi_n f_n$ in the completion of $B_c(\Sigma, X)$ and, since each $\sum_{n=1}^{m} \xi_n f_n$ is the uniform limit of Σ-simple functions, $\sum_{n=1}^{\infty} \xi_n f_n$ is the uniform limit of Σ-simple functions.

Hence $D := \{\sum_{n=1}^{\infty} \xi_n f_n : \xi \in B_{\ell_1}\}$ is a Banach disk in $B_c(\Sigma, X)$ and, denoting by E_D the normed space $\langle D \rangle$ whose norm is the gauge of D, there is some $m_1' \in \mathbb{N}$ such that $F_{m_1} \cap E_D$ is a dense Baire subspace of E_D for all $m_1 \geqslant m_1'$. By finite induction, suppose that we have found m_1' and the functions $m_i'(m_1, \dots, m_{i-1})$, $2 \leqslant i \leqslant p \leqslant s-1$, such for any positive integer $m_1 \geqslant m_1'$, $m_i \geqslant m_i'(m_1, \dots, m_{i-1})$, $2 \leqslant i \leqslant p$, $F_{m_1 \dots m_i} \cap E_D$ is a dense Baire subspace of E_D. Then, for any $m_1 \geqslant m_1', \dots, m_p \geqslant m_p'(m_1, \dots, m_{p-1})$, given that $\{F_{m_1 \dots m_p m} : m \in \mathbb{N}\}$ covers $F_{m_1 \dots m_p}$, there is some $m_{p+1}'(m_1, \dots, m_p) \in \mathbb{N}$ such that $F_{m_1 \dots m_{p+1}} \cap E_D$ is a dense Baire subspace of E_D for all $m_{p+1} \geqslant m_{p+1}'(m_1, \dots, m_p)$. Hence $D \subset L_{m_1 \dots m_s}$ if $m_1 \geqslant m_1', \dots, m_s \geqslant m_s'(m_1, \dots, m_{s-1})$, since $B_{m_1 \dots m_s} \cap L_{m_1 \dots m_s} \cap E_D$ is a barrel and consequently a neighbourhood of the origin in the Baire space $L_{m_1 \dots m_s} \cap E_D$ for $m_1 \geqslant m_1', \dots, m_s \geqslant m_s'(m_1, \dots, m_{s-1})$. It follows from this that $D \subset F_{m_1 \dots m_s}$ for $m_1 \geqslant m_1', \dots, m_s \geqslant m_s'(m_1, \dots, m_{s-1})$ and therefore $D \subset L_{m_1 \dots m_{s-1}}$ if $m_1 \geqslant m_1', \dots, m_{s-1} \geqslant m_{s-1}'(m_1, \dots, m_{s-2})$. This implies that $D \subset F_{m_1 \dots m_{s-1}}$ for $m_1 \geqslant m_1', \dots, m_{s-1} \geqslant m_{s-1}'(m_1, \dots, m_{s-2})$. Going on in this way, we obtain that $D \subset F_{m_1}$ for $m_1 \geqslant m_1'$, and, consequently, $f_{m_1} \in F_{m_1}$, a contradiction. ∎

Lemma 2. *If $X \in \mathcal{C}_s$, then there exists some $n_0 \in \mathbb{N}$ such that $S(\Sigma, X) \subset F_{n_0}$.*

Proof: Suppose the lemma is false and there is some $f_1 \in S(\Sigma, X) \setminus F_1$ such that $\|f_1\| = 1$. Let $\{Q_{11}, Q_{12}, \dots, Q_{1k}\}$ be a partition of Ω formed by nonempty elements of Σ such that $f_1(\omega) = x_{1i} \in X$ if $\omega \in Q_{1i}$, $1 \leqslant i \leqslant k_1$, and $x_{1j} \neq x_{1j}$ for $1 \leqslant i < j \leqslant k_1$.

Now given that $S(\Omega, \Sigma, X)$ is the topological direct sum of the subspaces $\{S(Q_{1i}, X) : 1 \leqslant i \leqslant k_1\}$, there must be some $m_1 \in \{1, \dots, k_1\}$ such that $S(Q_{1m_1}, X)$ is not contained in F_n for each $n \in \mathbb{N}$ and, consequently, there is some $f_2 \in$

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$S(Q_{1m_1}, X) \setminus F_2$ so that $\|f_2\| = 1$. Let $\{Q_{21}, Q_{22}, \ldots, Q_{2k_2}\}$ be a partition of Q_{1m_1} formed by nonempty elements of Σ such that $f_2(\omega) = x_{2i} \in X$ if $\omega \in Q_{2i}$, $1 \leqslant i \leqslant k_2$, and $x_{2i} \neq x_{2j}$ for $1 \leqslant i < j \leqslant k_2$. There is an $m_2 \in \{1, \ldots, k_2\}$ such that $S(Q_{2m_2}, X)$ is not contained in F_n for each $n \in \mathbb{N}$.

Assume that we have obtained by induction a sequence $\{f_n : n \in \mathbb{N}\}$ of Σ-simple functions, a sequence of positive integers $\{k_n : n \in \mathbb{N}\}$, and a countable family $\{Q_{ni} : n \in \mathbb{N}, 1 \leqslant i \leqslant k_n\}$ formed by nonempty elements of Σ such that, for each $n \in \mathbb{N}$, $f_n(\omega) = x_{ni} \in X$ if $\omega \in Q_{ni}$, $1 \leqslant i \leqslant k_n$, and $x_{ni} \neq x_{nj}$ for $1 \leqslant i < j \leqslant k_n$, and for each $n \in \mathbb{N}$:

(i) $\|f_n\| = 1$.
(ii) $\operatorname{supp} f_{n+1} \subset Q_{nm_n}$ for some $m_n \in \{1, \ldots, k_n\}$.
(iii) $Q_{n+1, m_{n+1}} \subset Q_{nm_n}$.
(iv) $f_n \notin F_n$.

Set $P := \bigcap\{Q_{nm_n} : n \in \mathbb{N}\}$ and, for each $n \in \mathbb{N}$, define $g_n := f_n$ if $P = \emptyset$ and $g_n := f_n - x_{nm_n} e(P)$ if $P \neq \emptyset$. In this second case the mapping $x \to e(P)x$ of X into $B_c(\Sigma, X)$ is an isometry and since $X \in \mathcal{C}_s$, there must be some $m_1' \in \mathbb{N}$ such that $F_{m_1} \cap e(P)X \in \mathcal{C}_{s-1}$ and is dense in $e(P)X$ for all $m_1 \geqslant m_1'$ and, a fortiori, $L_{m_1} \cap e(P)X \in \mathcal{C}_{s-1}$ and is dense in $e(P)X$ for all $m_1 \geqslant m_1'$. By finite induction, suppose that we have found m_1' and functions $m_i'(m_1, \ldots, m_{i-1})$, $2 \leqslant i \leqslant p \leqslant s-1$, such for any positive integer $m_1 \geqslant m_1'$, $m_i \geqslant m_i'(m_1, \ldots, m_{i-1})$, $2 \leqslant i \leqslant p$, $L_{m_1 \ldots m_i} \cap e(P)X \in \mathcal{C}_{s-i}$ and is dense in $e(P)X$. Then, for any $m_1 \geqslant m_1', \ldots, m_p \geqslant m_p'(m_1, \ldots, m_{p-1})$ given that $\{F_{m_1 \ldots m_p m} : m \in \mathbb{N}\}$ covers $L_{m_1 \ldots m_p}$, there is some $m_{p+1}'(m_1, \ldots, m_p) \in \mathbb{N}$ such that $F_{m_1 \ldots m_{p+1}} \cap e(P)X \in \mathcal{C}_{s-p-1}$ and is dense in $e(P)X$ for all $m_{p+1} \geqslant m_{p+1}'(m_1, \ldots, m_p)$. Consequently, $L_{m_1 \ldots m_{p+1}} \cap e(P)X \in \mathcal{C}_{s-p-1}$ and is dense in $e(P)X$ for all $m_{p+1} \geqslant m_{p+1}'(m_1, \ldots, m_p)$.

Therefore, $e(P)X \subset L_{m_1 \ldots m_s}$ if $m_1 \geqslant m_1', \ldots, m_s \geqslant m_s'(m_1, \ldots, m_{s-1})$ since $L_{m_1 \ldots m_s}$ is the linear hull of $B_{m_1 \ldots m_s}$ and, $L_{m_1 \ldots m_s} \cap e(P)X$ being barrelled, it is closed in $e(P)X$ for $m_1 \geqslant m_1', \ldots, m_s \geqslant m_s'(m_1, \ldots, m_{s-1})$. It follows from this that $e(P)X \subset F_{m_1 \ldots m_s}$ for all $m_1 \geqslant m_1', \ldots, m_s \geqslant m_s'(m_1, \ldots, m_{s-1})$. Thus, $e(P)X \subset L_{m_1 \ldots m_{s-1}}$ if $m_1 \geqslant m_1', \ldots, m_{s-1} \geqslant m_{s-1}'(m_1, \ldots, m_{s-2})$. This implies that $e(P)X \subset F_{m_1 \ldots m_{s-1}}$ for $m_1 \geqslant m_1', \ldots, m_{s-1} \geqslant m_{s-1}'(m_1, \ldots, m_{s-2})$. Going on in this way. we obtain that $e(P)X \subset F_{m_1}$ for all $m_1 \geqslant m_1'$, and, consequently, $e(P)_{jm_j} \in F_{m_1}$ for all $j \in \mathbb{N}$ and for all $m_1 \geqslant m_1'$. Thus, $g_{m_1} \notin F_{m_1}$ for each $m_1 \geqslant m_1'$.

Now, representing by B_{ℓ_1} the unit ball of ℓ_1, from $\bigcap\{\operatorname{supp} g_n : n \in \mathbb{N}\} = \emptyset$ we shall deduce that $D := \{\sum_{n=1}^{\infty} \xi_n g_n : \xi \in B_{\ell_1}\}$ is a Banach disk of the completion of $B_c(\Sigma, X)$ which is contained in $B_c(\Sigma, X)$. In fact, the sequence $\{g_n : n \in \mathbb{N}\}$ is bounded in

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$B_c(\Sigma, X)$ since $\|g_n\| \leqslant \|f_n\| = 1$ for each $n \in \mathbb{N}$. Therefore if $\xi \in \ell_1$, $\sum_{n=1}^{\infty} \xi_n g_n$ converges in the completion of $B_c(\Sigma, X)$ and, besides, takes at most a countable number of values in X. Indeed if $\omega \in P$, $\sum_{n=1}^{\infty} \xi_n g_n(\omega) = 0$, and if $\omega \notin P$, there exists some positive integer m_0 such that $\omega \notin Q_{nm_n}$ for all $n > m_0$ and, consequently, $\sum_{n=1}^{\infty} \xi_n g_n(\omega) = \sum_{n=1}^{\infty} \xi_n f_n(\omega) = \sum_{n=1}^{m_0} \xi_n f_n(\omega) \in X$ since it is a finite sum of elements of X. Moreover, the sequence $\{\sum_{n=1}^{m} \xi_n g_n : m \in \mathbb{N}\}$ of $S(\Sigma, X)$ converges to $\sum_{n=1}^{\infty} \xi_n g_n$ in the completion of $B_c(\Sigma, X)$ and consequently $\sum_{n=1}^{\infty} \xi_n g_n \in B_c(\Sigma, X)$. Hence, taking into account that the linear mapping from ℓ_1 into the completion of $B_c(\Sigma, X)$ where $\xi \to \sum_{n=1}^{\infty} \xi_n g_n$ is continuous, we deduce that $D := \{\sum_{n=1}^{\infty} \xi_n g_n : \xi \in B_{\ell_1}\}$ is a Banach disk in the completion of $B_c(\Sigma, X)$ contained in $B_c(\Sigma, X)$.

Thus, by the Baire category theorem, there must be some positive integer $p \geqslant m_1'$ such that $D \subset F_p$. Hence $g_p \in F_p$, a contradiction. ∎

THEOREM 1. *If $x \in \mathcal{C}_s$, then there exists some $n_0 \in \mathbb{N}$ such that $B_c(\Sigma, X)$ coincides with F_{n_0}.*

PROOF: By Lemma 2 we may assume without loss of generality that $S(\Sigma, X) \subset F_n$ for every $n \in \mathbb{N}$. Suppose that $B_c(\Sigma, X)$ is not contained in any F_n, $n \in \mathbb{N}$. Let $f_1 \in B_c(\Sigma, X) \setminus F_1$ such that $\|f_1\| = 1$ and let $\{Q_{1i}, i \in \mathbb{N}\}$ be the partition of Ω formed by nonempty elements of Σ defined by the function f_1 in such a way that $f_1(\omega) = x_{1i} \in X$ if $\omega \in Q_{1i}$, $i \in \mathbb{N}$, and $x_{1i} \neq x_{1j}$ for $i \neq j$.

Then, by Lemma 1, there exists some positive integer $n_2 > n_1 = 1$ such that $B_c(\bigcup\{Q_{1i} : i > n_2\}, X) \subset F_{n_2}$. Therefore, setting $\Omega_1 := \bigcup\{Q_{1i} : 1 \leqslant i \leqslant n_2\}$, $B_c(\Omega_1, X)$ cannot be contained in any F_n, $n \geqslant n_2$, and there must be some $f_2 \in B_c(\Omega_1, X) \setminus F_{n_2}$ so that $\|f_2\| = 1$. Let $\{Q_{2i}, i \in \mathbb{N}\}$ be the partition of Ω_1 formed by nonempty elements of Σ defined by the function f_2.

Now again, since $\{F_n : n > n_2\}$ covers $B_c(\Omega_1, X)$, by Lemma 1, there exists some positive integer $n_3 > n_2$ such that $B_c(\bigcup\{Q_{2i} : i > n_3\}, X) \subset F_{n_3}$. Setting $\Omega_2 := \bigcup\{Q_{2i} : 1 \leqslant i \leqslant n_3\}$, $B_c(\Omega_2, X)$ cannot be contained in F_n, $n \geqslant n_3$. Take $f_3 \in B_c(\Omega_2, X) \setminus F_{n_3}$ so that $\|f_3\| = 1$.

Assume that, by this means, we have obtained a sequence of positive integers $\{n_i : i \in \mathbb{N}\}$ and a sequence $\{f_n : n \in \mathbb{N}\}$ of functions of $B_c(\Sigma, X)$ which determine a countable family $\{Q_{ni} : n, i \in \mathbb{N}\}$ formed by nonempty elements of Σ such that, for each $n \in \mathbb{N}$, $f_n(\omega) = x_{ni} \in X$ if $\omega \in Q_{ni}$, $i \in \mathbb{N}$, and $x_{ni} \neq x_{nj}$ for $i \neq j$, in such a way that, setting $\Omega_n := \bigcup\{Q_{ni} : 1 \leqslant i \leqslant n_{i+1}\}$ for all $n \in \mathbb{N}$, for each $i \in \mathbb{N}$. we have that,

(i) $\operatorname{supp} f_{i+1} \subset \Omega_1$.
(ii) $e(\Omega_i)f_i \in S(\Omega_i, X) \subset S(\Sigma, X)$.
(iii) $\Omega_{i+1} \subset \Omega_i$.
(iv) $f_i \notin F_{n_i}$.

Let now $g_i := f_i - e(\Omega_i)f_i$ for each $i \in \mathbb{N}$. Then from $S(\Sigma, X) \subset F_n$ for each $n \in \mathbb{N}$, it follows that $g_i \notin F_{n_i}$ for each $i \in \mathbb{N}$. Besides $\operatorname{supp} g_i \cap \operatorname{supp} g_j = \emptyset$ for $i \neq j$. Hence, denoting by G the completion of $B_c(\Sigma, X)$, $\overline{\langle\{g_n : n \in \mathbb{N}\}\rangle}^G$ is a copy of c_0 in G since it can be easily seen that $\{g_n/\|g_n\| : n \in \mathbb{N}\}$ is equivalent to the unit vector basis of c_0.

Moreover, it is easy to see that $\overline{\langle\{g_n : n \in \mathbb{N}\}\rangle}^G \subset B_c(\Sigma, X)$. In fact, $\sum\limits_{n=1}^{\infty} c_n g_n(\Omega) = 0$ if $\omega \notin \bigcup\{\operatorname{supp} g_n : n \in \mathbb{N}\}$, and $\sum\limits_{n=1}^{\infty} c_n g_n(\omega) = c_m g_m(\omega) \in X$ if $\omega \in \operatorname{supp} g_m$, $m \in \mathbb{N}$. So $\sum\limits_{n=1}^{\infty} c_n g_n \in B_c(\Sigma, X)$ since it takes at most countably many values in X and is the uniform limit of Σ-simple functions. Consequently, using the Baire category theorem as above, there must be some $q \in \mathbb{N}$ such that $\{g_n : n \in \mathbb{N}\} \subset F_{n_q}$ for each $k \geqslant n_q$. Hence $g_q \in F_{n_q}$, a contradiction. ∎

Theorem 2. [1, Theorem] *Suppose Σ is a σ-algebra. Then $B_c(\Sigma, X)$ is barrelled if and only if X is barrelled.*

Theorem 3. *Let Σ be a σ-algebra. Given $s \in \mathbb{N}$, then $B_c(\Sigma, X)$ is barrelled of class s if and only if X is barrelled of class s.*

Proof: We have only to show that if $X \in \mathcal{C}_s$ then $B_c(\Sigma, X) \in \mathcal{C}_s$. By Theorem 2, we already know that $B_c(\Sigma, X)$ is a metrisable barrelled space and, consequently, Baire-like.

By recurrence, let $p \in \{1, \ldots, s\}$ and assume $B_c(\Sigma, X) \in \mathcal{C}_{p-1} \setminus \mathcal{C}_p$. Then, if $p - 1 \geqslant 1$, there is an increasing sequence $\{E_{m_1} : m_1 \in \mathbb{N}\}$ of dense subspaces of $B_c(\Sigma, X)$ covering $B_c(\Sigma, X)$ such that each $E_{m_1} \in \mathcal{C}_{p-2} \setminus \mathcal{C}_{p-1}$. Now each E_{m_1} may be covered by an increasing sequence of dense subspaces $\{E_{m_1 m_2} : m_2 \in \mathbb{N}\}$ such that each $E_{m_1 m_2} \in \mathcal{C}_{p-3} \setminus \mathcal{C}_{p-2}$. Continuing in this way we obtain $E_{m_1 \ldots m_{p-2}}$ so that each of them may be covered by an increasing sequence of dense subspaces $\{E_{m_1 \ldots m_{p-2} m_{p-1}} : m_{p-1} \in \mathbb{N}\}$ such that each $E_{m_1 \ldots m_{p-2} m_{p-1}} \in \mathcal{C}_0 \setminus \mathcal{C}_1$. Hence, every $E_{m_1 \ldots m_{p-1}}$, or $B_c(\Sigma, X)$ if $p = 1$, may be covered by an increasing sequence of dense subspaces $\{E_{m_1 \ldots m_{p-1} m_p} : m_p \in \mathbb{N}\}$ that are not barrelled. For each $m_1, \ldots, m_p \in \mathbb{N}$, suppose $T_{m_1 \ldots m_p}$ is a barrel of $E_{m_1 \ldots m_p}$ which is not a neighbourhood of the origin in $E_{m_1 \ldots m_p}$, let $B_{m_1 \ldots m_p}$ be the closure of $T_{m_1 \ldots m_p}$ in $B_c(\Sigma, X)$ and $L_{m_1 \ldots m_p} := \langle B_{m_1 \ldots m_p}\rangle$. By decreasing recurrence, for $i = p - 1, \ldots, 1$, define the subspaces $F_{m_1 \ldots m_{i+1}} := \bigcap\{L_{m_1 \ldots m_i m} : m \geqslant m_{i+1}\}$,

$L_{m_1 \dots m_i} := \bigcup\{F_{m_1 \dots m_i m}\colon m \in \mathbb{N}\}$, and $F_{m_1} := \bigcap\{L_m\colon m \geqslant m_1\}$. Then $\{F_m\colon m \in \mathbb{N}\}$ and $\{F_{m_1 \dots m_i m}\colon \in \mathbb{N}\}$ are increasing sequences of subspaces of $B_c(\Sigma, X)$ and $L_{m_1 \dots m_i}$, respectively covering them, for all $m_r \in \mathbb{N}$, $1 \leqslant r \leqslant i \leqslant p-1$, and $E_{m_1 \dots m_i} \subset F_{1 \dots m_i}$, for all $m_r \in \mathbb{N}$, $1 \leqslant r \leqslant i \leqslant p$.

By Theorem 1, there must be some $m_1 \in \mathbb{N}$ such that $B_c(\Sigma, X)$ coincides with F_{m_1}. Then clearly $L_{m_1} \in \mathcal{C}_{p-1}$ and there is some $m_2 \in \mathbb{N}$ such that $F_{m_1 m_2} \in \mathcal{C}_{p-2}$ and is dense in $B_c(\Sigma, X)$. A fortiori, $L_{m_1 m_2} \in \mathcal{C}_{p-2}$ and is dense in $B_c(\Sigma, X)$. Going on in this way, we are able to find some $F_{m_1 \dots m_p} \in \mathcal{C}_0$. Then $B_{m_1 \dots m_p} \cap F_{m_1 \dots m_p}$ is a neighbourhood of the origin in $F_{m_1 \dots m_p}$ and, consequently, $T_{m_1 \dots m_p}$ is a neighbourhood of the origin in $E_{m_1 \dots m_p}$, which is a contradiction. Thus $B_c(\Sigma, X) \in \mathcal{C}_p$.

Hence $B_c(\Sigma, X) \in \mathcal{C}_s$ and the proof is complete. □

Theorem 4. *Let Σ be a σ-algebra and let s be any positive integer. Then $B(\Sigma, X)$ is barrelled of class s if and only if X is barrelled of class s.*

Proof: This is an obvious consequence of the previous theorem, since $B_c(\Sigma, X)$ is a dense subspace of $B(\Sigma, X)$. □

Corollary. *Let Σ be a σ-algebra. Then $B_c(\Sigma, X)$ $(B(\Sigma, X))$ is barrelled of class $\aleph_0$ if and only if X is barrelled of class $\aleph_0$.*

Open problem: The authors do not know if the analogous result for totally barrelled (unordered Baire-like) spaces is true.

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