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SUFFICIENCY OF WEIERSTRASS JETS

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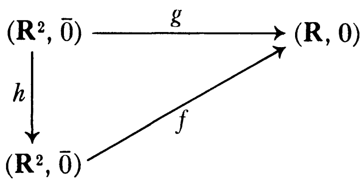
1. Introduction. Let $C^{(r+1)}(2, 1)$ be the set of all $(r + 1)$-time continuously differentiable mappings $f: \mathbf{R}^2 \rightarrow \mathbf{R}$ with $f(\bar{0}) = 0$. Two maps f and $g \in C^{(r+1)}(2, 1)$ are said to be equivalent of order r at $\bar{0}$, if at $\bar{0}$, their Taylor expansions up to and including the terms of degree $\leqq r$ are identical. An r-jet, denoted $j^{(r)}(f)$, is the equivalence class of f with f being called a realization of $j^{(r)}(f)$. The set of all r-jets is denoted $J^r(2, 1)$.

Definition. An r-jet $Z \in J^r(2, 1)$ is called C^0-*sufficient* (in $C^{(r+1)}(2, 1)$), if for any two $C^{(r+1)}(2, 1)$ functions f, g which realize Z, there exists a local homeomorphism $h: \mathbf{R}^2 \rightarrow \mathbf{R}^2$, for which $f(h(x, y)) = g(x, y)$ in a neighborhood of $\bar{0}$. I.e., the following diagram commutes.



This definition is also valid if we replace 2 by n, where n is any positive integer.

The degree of C^0-sufficiency is a useful tool for approximating functions near a singularity. Suppose we would like to approximate a function $f : \mathbf{R}^n \rightarrow \mathbf{R}$ with $f(\bar{0}) = 0$ near $\bar{0}$, an isolated singularity for f. The degree of C^0-sufficiency tells us where to truncate the Taylor series for f so that this polynomial has the same topological type as f.

In this paper we shall improve Kuo's constructive method of determining the degree of C^0-sufficiency for functions of two real variables. Due to the results of Lu [**3**], we can restrict our attention to functions of the form:

$$f(x, y) = x^k + H_{k+1}(x, y) + \ldots + H_r(x, y) + \ldots,$$

where H_i is a homogeneous i-form having no terms involving x^i for $i \geqq k - 1$. We shall say that those functions are in Weierstrass form.

Applying Puiseux's Theorem [**6**] to $f(x, y)$, we will show in Remark 1

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in the appendix that:

$$f(x, y) = \prod_{i=1}^{k} (x - B_i(y)), f_x(x, y) = k \prod_{j=1}^{k-1} (x - p_j(y)) \quad \text{and}$$

$$f_y(x, y) = a(y)h(x, y) \prod_{\alpha=1}^{s} (x - q_\alpha(y))$$

where p_j and q_α are fractional power series in y with order greater than 1, $h(x, y)$ consists of roots of order less than or equal to 1, and $a(y)$ is a function of y alone.

Let $U_j(y)$, $W_\alpha(y)$ denote the real part of p_j, q_α respectively. Define

$$m_j = \min [O(f_x(U_j(y), y)), O(f_y(U_g(y), y))]$$
$$n_\alpha = \min [O(f_x(W_\alpha(y), y)), O(f_y(W_\alpha(y), y))]$$

and let l be the smallest integer such that

$$l > \text{Sup} \{m_1, m_2, \ldots, m_{k-1}, n_1, n_2, \ldots, n_s\}.$$

Kuo's Theorem [**1**] asserts that this l is the degree of C^0-sufficiency of f. We improve this result by showing the following:

THEOREM. *For a function $f(x, y)$ in Weierstrass form,*

$$\text{Sup} \{m_1, m_2, \ldots, m_{k-1}, n_1, \ldots, n_s\} = \text{Sup} \{m_1, \ldots, m_{k-1}\}.$$

It is worth noticing that in finding the Puiseux roots term by term, one has to solve, in each step, a polynomial equality in a complex variable. It can happen that the roots $Z_x = 0$ can be determined completely while those of $Z_y = 0$ cannot. Then there is a question as to how many terms are necessary in W_α to find out what $O(Z_x(W_\alpha, y))$ and $O(Z_y(W_\alpha, y))$ are. In this case, our method is definitely superior to Kuo's. Here are three illustrative examples.

1) Let

$$z = x^3 - 3xy^7$$

then

$$z_x = 3(x - y^{7/2})(x + y^{7/2}) \quad \text{and} \quad z_y = -21xy^6$$
$$m_1 = m_2 = 19/2.$$

The degree of C^0-sufficiency is 10.

2) Let

$$z = \frac{x^7}{7} - \frac{x^5}{5}(y^3 + y^4 + y^5) + \frac{x^3}{3}(y^7 + y^8 + y^9) - xy^{12}$$

then

$$z_x = (x + y^{3/2})(x - y^{3/2})(x + y^2)(x - y^2)(x + y^{5/2})(x - y^{5/2}),$$
$$z_y = \frac{-x^5}{5}(3y^2 + 4y^3 + 5y^4) + \frac{x^3}{3}(7y^6 + 8y^7 + 9y^8) - 12xy^{11},$$
$$U_1 = -y^{3/2},$$
$$U_2 = y^{3/2},$$
$$U_3 = -y^2,$$
$$U_4 = y^2,$$
$$U_5 = -y^{5/2},$$
$$U_6 = y^{5/2},$$
$$m_1 = m_2 = 19/2,\ m_3 = m_4 = 12,\ m_5 = m_6 = 27/2.$$

The degree of C^0-sufficiency is 14.

3) Let

$$z = x^9 + \frac{9x^7}{7}(y^4 + y^6 + y^8 + y^{10}) + \frac{9x^5}{5}(y^{10} + y^{12} + 2y^{14} + y^{16} + y^{18}) + 5x^3(y^{18} + y^{20} + y^{22} + y^{24}) + 9xy^{28} + y^{100}$$
$$z_x = (x + iy^2)(x - iy^2)(x + iy^3)(x - iy^3)(x + iy^4)(x - iy^4) \times (x + iy^5)(x - iy^5).$$

Therefore

$$U_l \equiv 0 \text{ for all } l,$$

implying

$$m_l = \min [O(z_x(0, y)), O(z_y(0, y))] = \min [28, 39] = 28.$$

The degree of C^0-sufficiency is 29.

2. Proof of the theorem. It will suffice to show for each α $(1 \leqq \alpha \leqq s)$,

$$n_\alpha \leqq \max \{m_1, m_2, \dots, m_k\}.$$

Equivalently, it suffices to show for each α there exists a j_0 such that $m_{j_0} \geqq n_\alpha$.

We will first prove the theorem for $\alpha = 1$ by showing there exists an integer j_0 such that:

(1) $O(z_x(W_1(y), y)) \leqq O(z_x(U_{j_0}(y), y))$

(2) $O(z_y(W_1(y), y)) \leqq O(z_y(U_{j_0}(y), y)).$

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Let j_0 be such that

$$\delta = O(W_1 - U_{j_0}) \geqq O(W_1 - U_j) \quad \text{for all } j \quad (1 \leqq j \leqq k-1).$$

This will be the required j_0 for n_1. Also, let α_0 be an integer such that

$$O(W_1 - q_{\alpha_0}) \geqq O(W_1 - q_t) \quad \text{for all } t \quad (1 \leqq t \leqq s).$$

Remark 2 in the appendix shows

(3) $\quad O(W_1 - U_j) \geqq O(W_1 - p_j) \quad$ for all j.

We now have

(4) $\quad \delta = O(W_1 - U_{j_0}) \geqq O(W_1 - U_j) \geqq O(W_1 - p_j) \quad$ for all j.

Proof of (1).

$$\begin{aligned} O(z_x(U_{j_0}(y), y)) &= \sum_{j=1}^{k-1} O(U_{j_0} - p_j) = \sum_{j=1}^{k-1} O(U_{j_0} - W_1 + W_1 - p_j) \\ &\geqq \sum_{j=1}^{k-1} \min\{O(U_{j_0} - W_1), O(W_1 - p_j)\} \\ &= \sum_{j=1}^{k-1} O(W_1 - p_j) \quad \text{by (4)} \\ &= O(z_x(W_1(y), y)). \end{aligned}$$

Proof of (2). Recall

$$z_y(x, y) = a(y)h(x, y) \prod_{\alpha=1}^{s} (x - q_\alpha(y)).$$

Case i). $\delta = O(W_1 - U_{j_0}) \geqq O(W_1 - q_{\alpha_0})$.

$$\begin{aligned} O(z_y(U_{j_0}(y), y)) &= O(a(y)) + O(h(U_{j_0}(y), y)) + \sum_{\alpha=1}^{s} O(U_{j_0} - q_\alpha) \\ &\geqq O(a(y)) + O(h(U_{j_0}(y), y)) \\ &\quad + \sum_{\alpha=1}^{s} \min\{O(U_{j_0} - W_1), O(W_1 - q_\alpha)\} \\ &= O(a(y)) + O(h(W_1(y), y)) + \sum_{\alpha=1}^{s} O(W_1 - q_\alpha) \end{aligned}$$

(see remark 3 in the appendix)

$$= O(z_y(W_1(y), y)).$$

Case ii). $\delta = O(W_1 - U_{j_0}) < O(W_1 - q_{\alpha_0})$.

Let $\bar{W}_1(y)$ be a generic perturbation of $W_1(y)$ of order δ, i.e., $\bar{W}_1 = W_1 + cy^\delta$ where c is picked so that

(5) $\quad O(\bar{W}_1 - p_j) = O(W_1 - p_j) \quad$ for all j and

$\quad\quad O(\bar{W}_1 - q_\alpha) \leqq \delta \quad$ for all α.

By this choice of c we have:

(6) $$O(Z_x(\bar{W}_1(y), y)) = \sum_{j=1}^{k-1} O(\bar{W}_1 - p_j) = \sum_{j=1}^{k-1} O(W_1 - p_j) = O(Z_x(W_1(y), y))$$

and

$$O(U_{j_0} - q_\alpha) \geqq \min\{O(U_{j_0} - \bar{W}_1), O(W_1 - q_\alpha)\} = O(\bar{W}_1 - q_\alpha)$$

or

(7) $$O(U_{j_0} - q_\alpha) \geqq O(\bar{W}_1 - q_\alpha).$$

7 and remark 3 in the appendix yield

(8) $$O(z_y(U_{j_0}(y), y)) = O(a(y)) + O(h(U_{j_0}(y), y)) + \sum_{\alpha=1}^{s} O(U_{j_0} - q_\alpha) \geqq O(a(y)) + O(h(\bar{W}_1(y), y)) + \sum_{\alpha=1}^{s} O(\bar{W}_1 - q_\alpha) = O(z_y(\bar{W}_1(y), y)).$$

If we have

(9) $$O(z_y(\bar{W}_1(y), y)) > O(z_x(\bar{W}_1(y), y))$$

then 6, 8 and 9 prove case ii.

LEMMA. *In the setting of this theorem, given any fractional power series* $\lambda(y) = \sum_{i=1}^{\infty} c_i y^{\delta_i}$ *with* $\delta_1 > 1$ *if there exists an* α_0 *such that for all* j

$$O(\lambda(y) - q_{\alpha_0}(y)) \geqq O(\lambda(y) - p_j(y))$$

then

$$O(z_y(\lambda(y), y)) > O(z_x(\lambda(y), y)).$$

Geometrically, this says if the degree of contact of λ with some q_α is greater than or equal to that of λ with all the p_j's, then λ has more contact with z_y than with z_x.

Being in case ii and using 4 and 5 we obtain

$$O(\bar{W}_1 - q_\alpha) \geqq \min\{O(\bar{W}_1 - W_1), O(W_1 - q_\alpha)\} = \delta > O(\bar{W}_1 - p_j)$$

or

(10) $$O(\bar{W}_1 - q_{\alpha_0}) \geqq O(\bar{W}_1 - p_j) \quad \text{for all } j.$$

10 and the lemma prove 9.

Proof of the lemma. We have

$$O(\lambda(y) - q_{\alpha_0}(y)) \geqq O(\lambda(y) - p_j(y)) \quad \text{for all } j.$$

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Let j_0 be such that

$$O(\lambda(y) - p_{j_0}(y)) = \max_{1\leqq i\leqq k-1} O(\lambda(y) - p_i(y)).$$

If $\lambda(y) = B_i(y)$ (a root for f) then

$$0 \equiv \frac{d}{dy}(z(\lambda(y), y)) = z_x(\lambda(y), y)\lambda'(y) + z_y(\lambda(y), y)$$

implying

$$O(z_y(\lambda(y), y)) = O(z_x(\lambda(y), y)) + O(\lambda'(y)) > O(z_x(\lambda(y), y)).$$

Therefore without loss of generality we will assume $\lambda(y) \neq B_i(y)$ for all i.

Using the technique found in Lemma 3.3 in [**2**] we let

$$X = x - \lambda(y) \qquad Y = y$$

then

$$z(x, y) = z(x + \lambda(Y), Y) \equiv \bar{Z}(X, Y).$$

Since

$$z(x, y) = \prod_{i=1}^{k}(X - B_i(y)) \quad \text{where } O(B_i(y)) \geqq 1,$$

$$Z(X, Y) = \prod_{i=1}^{k}(X - \bar{B}_i(Y)) \quad \text{where } \bar{B}_i(Y) = B_i(Y) - \lambda(Y).$$

$\lambda(y) \neq B_i(y)$ for all i implies $\bar{B}_i(Y) \not\equiv 0$ for all i. So X does not divide $\bar{Z}(X, Y)$ and $\bar{\alpha}_0 \neq \infty$ (see the appendix). We have

$$\bar{Z}_x(X, Y) = z_x(x, y)$$

and

$$\bar{Z}_Y(X, Y) = z_x(x, y)\lambda'(y) + z_y(x, y).$$

This says

$$\bar{Z}_x(X, Y) = \prod_{i=1}^{k-1}(X - \bar{p}_j(Y)) \quad \textit{where } \bar{p}_j(Y) = p_j(Y) - \lambda(Y)$$

implying

$$O(\bar{p}_j(Y)) \geqq \min\{O(p_j(Y)), O(\lambda(Y))\} > 1.$$

Case a. $\max_{1\leqq i\leqq k} O(\bar{B}_i) \geqq \max_{1\leqq j\leqq k-1} O(\bar{p}_j) > 1.$

Lemma 3.2 in [**2**] shows the point $(1, \bar{\alpha}_1)$ lies on the Newton Polygon for $\bar{Z}$. Therefore

$$O(\bar{Z}(0, Y)) - O(\bar{Z}_x(0, Y)) = \bar{\alpha}_0 - \bar{\alpha}_1 = \max_{1\leqq i\leqq k} O(\bar{B}_i) > 1$$

or

$$O(\bar{Z}_Y(0, Y)) + 1 = O(\bar{Z}(0, Y)) > O(\bar{Z}_x(0, Y)) + 1$$

or

$$O[z_x(\lambda(y), y)\lambda'(y) + z_y(\lambda(y), y)] > O(z_x(\lambda(y), y)).$$

Since $O(\lambda'(y)) > 0$ we have

$$O(z_y(\lambda(y), y)) > O(z_x(\lambda(y), y)).$$

Case b. $\max_{1 \leqq i \leqq k} O(\bar{B}_i) < \max_{1 \leqq j \leqq k-1} O(\bar{p}_j)$.

By remark 4 in the appendix, every root $r_s(Y)$ for $Z_Y(X, Y)$ satisfies the following:

$$O(r_s(Y)) \leqq \max_{1 \leqq i \leqq k} O(\bar{B}_i) < \max_{1 \leqq j \leqq k-1} O(\bar{p}_j) \leqq O(\bar{q}_{\alpha 0}).$$

We have

$$(12) \quad O(\bar{Z}_Y(0, Y)) = O(\bar{Z}_Y(\bar{q}_{\alpha 0}(Y), Y)) = O(\bar{Z}_x(q_{\alpha 0}(y), y)\lambda'(y)) = O(\bar{Z}_x(\bar{q}_{\alpha 0}(Y), Y)\lambda'(Y)).$$

Using $O(\bar{q}_{\alpha 0}) \geqq O(\bar{p}_j)$ for all j we get

$$(13) \quad O(\bar{Z}_x(\bar{q}_{\alpha 0}(Y), Y)) \geqq O(\bar{Z}_x(0, Y)).$$

Putting 12 and 13 together we obtain

$$(14) \quad O(\bar{Z}_Y(0, Y)) \geqq O(\bar{Z}_x(0, Y)\lambda'(Y))$$

or

$$O[z_x(\lambda(y), y)\lambda'(y) + z_y(\lambda(y), y)] \geqq O(z_x(\lambda(y), y)\lambda'(y)).$$

So

$$O(z_y(\lambda(y), y)) \geqq O(z_x(\lambda(y), y)\lambda'(y)) > O(z_x(\lambda(y), y))$$

finishing the proof of the lemma.

The same argument can be repeated for each of the α. We replace W_1 with W_α and pick j_0 and α_0 accordingly. Therefore for each α there exists a j_α such that $n_\alpha \leqq m_{j\alpha}$.

Appendix. We will be using the notation found in [**6**]. Given any fractional power series

$$\lambda(y) = \sum_{i=1}^{\infty} c_i y^{\delta_i}, \quad c_1 \neq 0,$$

the order of $\lambda(y)$, denoted $O(\lambda(y))$, equals δ_1. If we write

$$f(x, y) = a_m(y)x^m + \ldots + a_1(y)x + a_0(y),$$

then $O(a_i(y)) \equiv \alpha_i$. In the case $a_1(y) \equiv 0$, α_i is set equal to ∞. Notice:

$$O(f(0, y)) = \alpha_0, \quad O(f_x(0, y)) = \alpha_1,$$

and if $\alpha_0 \neq \infty$,

$$O(f_y(0, y)) = \alpha_0 - 1.$$

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Remark 1. Let $f(x, y)$ be in Weierstrass form. Then

$$f_x(x, y) = k(x - p_1(y)) \dots (x - p_{k-1}(y))$$

where $O(p_j(y)) > 1$ for all j and

$$f_y(x, y) = a(y)h(x, y)(x - q_1(y)) \dots (x - q_s(y))$$

where $O(q_\alpha(y)) > 1$ for all α, $h(x, y)$ contains those roots with order $\leqq 1$, and $a(y)$ is a function of y alone.

Proof. Using Puiseaux's Theorem (see Theorem 3.1 in [**6**]), we can write

$$f(x, y) = \prod_{i=1}^{k} (x - B_i(y)), \quad f_x(x, y) = k \prod_{j=1}^{k-1} (x - p_j(y)) \quad \text{and}$$

$$f_y(x, y) = a(y)h(x, y) \prod_{\alpha=1}^{s} (x - q_\alpha(y))$$

where $B_i(y)$, $p_j(y)$ and $q_\alpha(y)$ are fractional power series. It remains to show $O(p_j(y)) > 1$ for all j. Since

$$f(x, y) = x^k + a_{k-2}(y)x^{k-2} + \dots + a_1(y)x + a_0(y)$$

and

$$f(x, y) = x^k + H_{k+1}(x, y) + \dots + H_r(x, y)$$

we have

$$O(a_l(y)) + l \geqq k + 1 \quad \text{for } 0 \leqq l \leqq k - 2 \quad \text{and}$$
$$f_x(x, y) = kx^{k-1} + (k - 2)a_{k-2}(y)x^{k-3} + \dots + a_1(y).$$

$f_x(p_j(y), y) = 0$ implies

$$(k - 1)O(p_j(y)) \geqq \min_{1 \leqq l \leqq k-2} \{O(a_l(y)) + (l - 1)O(p_j(y))\}$$

for some l. Therefore

$$(k - 1)O(p_j(y)) \geqq (k - l + 1) + (l - 1)O(p_j(y))$$

or

$$(k - l)O(p_j(y)) \geqq k - l + 1$$

implying

$$O(p_j(y)) \geqq \frac{k - l + 1}{k - l} > 1.$$

The same type of argument shows us that $O(B_i(y)) > 1$ for all i.

Remark 2. $O(W_1 - U_j) \geqq O(W_1 - p_j)$ for all j.

Proof.

$$O(W_1 - p_j) \geqq \min \{O(W_1 - U_j), O(-iV_j)\}$$

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(see page 89 in [**6**]). $W_1 - U_j$ is real and $-iV_j$ is complex so it is impossible for any cancellation to occur between them. Therefore,

$$O(W_1 - p_j) = \min\{O(W_1 - U_j), O(V_j)\}.$$

If $O(V_j) < O(W_1 - U_j)$ then

$$O(W_1 - p_j) = O(V_j) < O(W_1 - U_j).$$

If $O(V_j) \geqq O(W_1 - U_j)$ then

$$O(W_1 - p_j) = O(W_1 - U_j).$$

In either case

$$O(W_1 - U_j) \geqq O(W_1 - p_j).$$

Remark 3. If $h(x, y) = \prod_{i=1}^{t} (x - \delta_i(y))$ with $O(\delta_i(y)) \leqq 1$ for all i, then for any fractional power series $\lambda(y)$ with $O(\lambda(y)) > 1$,

$$O(h(\lambda(y), y)) = O(h(0, y)).$$

Proof. $O(\lambda(y) - \delta_i(y)) = O(\delta_i(y))$. This implies

$$O(h(\lambda(y), y)) = \prod_{i=1}^{t} O(\lambda(y) - \delta_i(y)) = \prod_{i=1}^{t} O(\delta_i(y)) = O(h(0, y)).$$

Remark 4. If

$$f(x, y) = \prod_{i=1}^{k} (x - B_i(y)) \quad \text{and}$$

$$f_y(x, y) = a(y)h(x, y)\prod_{\alpha=1}^{s} (x - q_\alpha(y)),$$

then

$$\max_{1\leqq\alpha\leqq s} O(q_\alpha) \leqq \max_{1\leqq i\leqq k} O(B_i).$$

Proof. If $\alpha_0 = \infty$ then x divides f and f_y. This implies

$$\max_{1\leqq\alpha\leqq s} O(q_\alpha) = \infty = \max_{1\leqq i\leqq k} O(B_i).$$

If $\alpha_0 \neq \infty$, let $\delta_1 y^b$, $\delta_2 x^c y^d$ make up the first line segment for the Newton polygon for f_y. Then $b = \alpha_0 - 1$ (α_0 for f), $O(q_{\alpha_0}) = (b - d)/c$, and $\delta_1 y^{b+1}/(b + 1)$ and $\delta_2 x^c y^{d+1}/(d + 1)$ belong to f. The equation for the first line segment for the Newton polygon for f is

$$\nu = -O(B_{i_0})u + \alpha_0.$$

This implies

$$d + 1 \geqq -O(B_{i_0})c + \alpha_0 = -O(B_{i_0})c + b + 1.$$

Therefore

$$O(B_{i_0}) \geqq (b - d)/c = O(q_{\alpha_0}).$$

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