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The local structure theorem for real spherical varieties

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Abstract

Let G be an algebraic real reductive group and Z a real spherical G-variety, that is, it admits an open orbit for a minimal parabolic subgroup P. We prove a local structure theorem for Z. In the simplest case where Z is homogeneous, the theorem provides an isomorphism of the open P-orbit with a bundle $Q \times_L S$. Here Q is a parabolic subgroup with Levi decomposition $L \ltimes U$, and S is a homogeneous space for a quotient $D = L/L_n$ of L, where $L_n \subseteq L$ is normal, such that D is compact modulo center.

1. Introduction

Let $G_{\mathbb{C}}$ be a complex reductive group and $B_{\mathbb{C}} < G_{\mathbb{C}}$ a fixed Borel subgroup. We recall that a normal $G_{\mathbb{C}}$-variety $Z_{\mathbb{C}}$ is called *spherical* provided that $B_{\mathbb{C}}$ admits an open orbit. The local nature of a spherical variety is given in terms of the local structure theorem [BLV86, Kno94]. In its simplest form, namely applied to a homogeneous space $Z_{\mathbb{C}} = G_{\mathbb{C}}/H_{\mathbb{C}}$ for which $B_{\mathbb{C}}H_{\mathbb{C}}$ is open, it asserts that there is a parabolic subgroup $Q_{\mathbb{C}} > B_{\mathbb{C}}$ with Levi decomposition $Q_{\mathbb{C}} = L_{\mathbb{C}} \ltimes U_{\mathbb{C}}$ such that the action of $Q_{\mathbb{C}}$ on $Z_{\mathbb{C}}$ induces an isomorphism of $(L_{\mathbb{C}}/L_{\mathbb{C}} \cap H_{\mathbb{C}}) \times U_{\mathbb{C}}$ onto $B_{\mathbb{C}}H_{\mathbb{C}}$.

The purpose of this paper is to continue the geometric study of real spherical varieties begun in [KS13]. We let G be an algebraic real reductive group and Z a normal real algebraic G-variety. Then Z is called *real spherical* provided a minimal parabolic subgroup $P < G$ has at least one open orbit on Z. With this assumption on Z we prove a local structure theorem analogous to the one above. In particular, when applied to a homogeneous real spherical space $Z = G/H$ with PH open, it yields a parabolic subgroup $Q > P$ with Levi decomposition $Q = L \ltimes U$ such that

$$L_n < Q \cap H < L.$$

Here $L_n \triangleleft L$ denotes the product of all non-compact non-abelian normal factors of L. Furthermore, the action of Q induces a diffeomorphism of $(L/L \cap H) \times U$ onto PH.

Our proof of the real local structure theorem is based on the symplectic approach of [Kno94]. Our investigations also show the number of G-orbits on a real spherical variety is finite. Combined with the main result of [KS13], it implies that the number of P-orbits on a real spherical variety is finite.

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2. Homogeneous spherical spaces

Lie groups in this paper will be denoted by upper-case Latin letters, $A, B \ldots$, and their associated Lie algebras with the corresponding lower- case Gothic letter $\mathfrak{a}$, $\mathfrak{b}$, ...

For a Lie group G we denote by G_0 its connected component containing the identity, by $Z(G)$ the center of G and by $[G, G]$ the commutator subgroup.

On a real reductive Lie algebra $\mathfrak{g}$ we fix a non-degenerate invariant bilinear form $\kappa(\cdot\,,\cdot)$, for example the Cartan–Killing form if $\mathfrak{g}$ is semisimple.

A Lie group G will be called *real reductive* provided that:

- the Lie algebra $\mathfrak{g}$ is reductive;
- there exists a maximal compact subgroup $K < G$ such that we have a homeomorphism (polar decomposition)
$$K \times \mathfrak{s} \to G, \quad (k, X) \mapsto k \exp(X)$$
where $\mathfrak{s} := \mathfrak{k}^{\perp_\kappa}$.

Observe that for a real reductive group the bilinear form κ can (and will) be chosen K-invariant. A real reductive group is called *algebraic* if it is isomorphic to an open subgroup of the group of real points $G_{\mathbb{C}}(\mathbb{R})$ where $G_{\mathbb{C}}$ is a reductive algebraic group which is defined over $\mathbb{R}$.

Now let G be a real reductive group, and let P be a minimal parabolic subgroup. The unipotent part of P is denoted N. If a maximal compact subgroup K as above has been chosen, with associated Cartan involution θ of G, a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$ can also be chosen. These choices then induce an Iwasawa decomposition $G = KAN$ of G and a Langlands decomposition $P = MAN$ of P. Here $M = Z_K(\mathfrak{a})$. However, at present we do not fix K and $\mathfrak{a}$.

Let H be a closed subgroup of G such that H/H_0 is finite. Recall that $Z = G/H$ is said to be *real spherical* if the minimal parabolic subgroup P admits an open orbit on Z. Furthermore, in this case H is called a *spherical subgroup*. Note that H is not necessarily reductive.

Remark 2.1. Here a remark on terminology is in order. Historically, spherical subgroups were first introduced by M. Krämer in the context of compact Lie groups; see [Krä79]. However, as our focus is to investigate non-compact homogeneous spaces we allow a discrepancy between the original definition and the current one. In fact with our definition every closed subgroup of G is spherical if G is compact.

We denote by $z_0 \in Z$ the origin of the homogeneous space $Z = G/H$.

2.1 Semi-invariant functions and the local structure theorem

Let G be a real reductive Lie group.

DEFINITION 2.2. Let $Z = G/H$ with $H \subseteq G$ a closed subgroup.

(1) A finite-dimensional real representation (π, V) of G is called *H-semispherical* provided there is a cyclic vector $v_H \in V$ and a character $\gamma : H \to \mathbb{R}^\times$ such that
$$\pi(h)v_H = \gamma(h)v_H, \quad \forall h \in H.$$

(2) The homogeneous space Z is called *almost algebraic* if there exists an H-semispherical representation (π, V) such that the map
$$Z \to \mathbb{P}(V), \quad g \cdot z_0 \mapsto [\pi(g)v_H]$$
is injective.

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According to a theorem of Chevalley (see [Bor91, Theorem 5.1]), $Z = G/H$ is almost algebraic if G and H are algebraic. In the following we always assume that $Z = G/H$ is almost algebraic.

For a reductive Lie algebra $\mathfrak{g}$ we write $\mathfrak{g}_n$ for the direct sum of the non-compact non-abelian ideals in $[\mathfrak{g}, \mathfrak{g}]$. If $\mathfrak{g}$ is the Lie algebra of G, then G_n denotes the corresponding connected normal subgroup of $[G, G]$.

Theorem 2.3 (Local structure theorem, homogeneous case). *Let $Z = G/H$ be an almost algebraic real spherical space, and let $P \subseteq G$ be a minimal parabolic subgroup such that PH is open. Then there is a parabolic subgroup $Q \supseteq P$ with Levi decomposition $Q = LU$ such that:*

(i) *the map*

$$Q \times_L (L/L \cap H) \to Z, \quad [q, l(L \cap H)] \mapsto ql \cdot z_0$$

is a Q-equivariant diffeomorphism onto $Q \cdot z_0 \subseteq Z$;

(ii) $Q \cap H \subseteq L$;

(iii) $L_n \subseteq H$;

(iv) $(L \cap P)(L \cap H) = L$;

(v) $QH = PH$.

Proof. The proof consists of an iterative procedure in which we construct a strictly decreasing sequence of parabolic subgroups

$$Q_0 \supset Q_1 \supset \cdots \supset P$$

and corresponding Levi subgroups $L_0 \supset L_1 \supset \ldots$, all satisfying (i). Note that (ii) is an immediate consequence of (i). After a finite number of steps a parabolic subgroup is reached which also satisfies (iii)–(v).

Let $Q_0 = G$. It clearly satisfies (i). If $G_n \subseteq H$ then $PH = G$ since P contains both the center of G and every compact normal subgroup of $[G, G]$. Hence in this case $Q = Q_0$ solves (i)–(v). Note also that since $L \cap P$ is a minimal parabolic subgroup of L, the argument just given, but applied to L, shows that (iv) and (v) are consequences of (iii).

Assume now that $G_n \not\subseteq H$. By our general assumption on Z there is a finite-dimensional representation (π, V) of G and a vector $v_H \in V$ satisfying all the properties of Definition 2.2. The assumption on G_n implies that $\pi(g)v_H \notin \mathbb{R}v_H$ for some $g \in G_n$, hence π does not restrict to a multiple of the trivial representation of G_n.

Choose a Cartan involution for G and a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{s}$, but note that these choices may be valid only for the current step of the iteration. Let $v^* \in V^* \backslash \{0\}$ be an extremal weight vector so that the line $\mathbb{R}v^*$ is fixed by AN, say $\pi^*(g)v^* = \chi(g)v^*$ for $g \in AN$ and some character $\chi : AN \to \mathbb{R}^\times$. Now we need the following lemma.

Lemma 2.4. *Let G be a connected semisimple Lie group without compact factors, and with minimal parabolic $P = MAN \subseteq G$. Let V be a non-trivial finite-dimensional irreducible real representation of G. Then $V^{AN} = \{0\}$.*

Proof. Let $\bar{N} = \theta(N)$ be the unipotent part of the parabolic subgroup $\theta(P)$ opposite to P. It follows from the representation theory of $\mathfrak{sl}(2, \mathbb{R})$ that vectors in V^{AN} are also fixed by $\bar{N}$. Since G has no compact factors it is generated by $\bar{N}$ and AN, hence $V^{AN} = V^G = \{0\}$. □

By this lemma and what we have just seen, we can choose v^* such that χ is non-trivial on $G_n \cap A$. The matrix coefficient

$$f(g) := v^*(\pi(g)v_H) \quad (g \in G)$$

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satisfies $f(angh) = \chi(a)^{-1}\gamma(h)f(g)$ for all $g \in G$, $an \in AN$ and $h \in H$. As v_H is cyclic and v^* non-zero, and as PH is open, f is not identically zero on M.

We construct a new function:

$$F(g) := \int_M f(mg)^2 \, dm \quad (g \in G).$$

This function is smooth, G-finite, non-negative, and satisfies

$$F(mangh) = \chi(a)^{-2}\gamma^2(h)F(g) \tag{2.1}$$

for all $g \in G$, $man \in P$ and $h \in H$. Furthermore, $F(e) > 0$.

It follows from the G-finiteness, together with (2.1), that F is a matrix coefficient

$$F(g) = w^*(\rho(g)w_H)$$

of a finite-dimensional representation (ρ, W) of G, with non-zero vectors $w_H \in W$ and $w^* \in W^*$ such that

$$\rho(h)w_H = \gamma(h)^2 w_H, \quad \rho^*(man)w^* = \chi(a)^2 w^*$$

for all $h \in H$ and $man \in P = MAN$. Here, W^* can be chosen to be the span of all left translates of F. Since F is a highest weight vector, W^* and hence W are irreducible. Define $\nu \in \mathfrak{a}^*$ by

$$e^{\nu(X)} = \chi(\exp X)^2.$$

Then ν is the highest $\mathfrak{a}$-weight of ρ^*, and it is dominant with respect to the set $\Sigma(\mathfrak{a}, \mathfrak{n})$ of $\mathfrak{a}$-roots in $\mathfrak{n}$.

Now define a subgroup $Q_1 = Q \subseteq G$ to be the stabilizer of $\mathbb{R}w^*$,

$$Q = \{g \in G \mid \rho^*(g)w^* \in \mathbb{R}w^*\},$$

and define a character $\psi : Q \to \mathbb{R}^\times$ by

$$\rho^*(g)w^* = \psi(g)w^*.$$

In particular, we see that Q is a parabolic subgroup that contains P. Moreover, $\psi : Q \to \mathbb{R}$ extends $\chi^2 : AN \to \mathbb{R}^+$. Let $U \subseteq Q$ be the unipotent radical of Q; its Lie algebra is spanned by the root spaces of the roots $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{n})$ for which $\langle \alpha, \nu \rangle > 0$.

Note that since w_H is cyclic, $\rho^*(g)w^* = cw^*$ if and only if $F(g^{-1}x) = cF(x)$ for all $x \in G$. Hence

$$Q = \{g \in G \mid F(g\,\cdot) \text{ is a multiple of } F\}$$

and $F(q\,\cdot) = \psi(q)F$ for all $q \in Q$. (We use the symbol $F(g\,\cdot)$ for the function $x \mapsto F(gx)$ on G.)

We note that $Q \cap G_n$ is a proper subgroup of G_n, for otherwise ρ^* would be one-dimensional spanned by w^*, and this would contradict the non-triviality of its highest weight $e^\nu = \chi^2$ on $G_n \cap A$.

Set $Z_0 := QH \subseteq Z$. Then Z_0 is open since qPH is open for each $q \in Q$. Following [Kno94, Theorem 2.3], we define a moment-type map

$$\mu : Z_0 \to \mathfrak{g}^*, \quad \mu(z)(X) := \frac{dF(q)(X)}{F(q)} = \frac{d}{dt}\Big|_{t=0} \frac{F(\exp(tX)q)}{F(q)}$$

for $q \in Q$ such that $z = qH \in Z_0$ and $X \in \mathfrak{g}$. Note that this map is well defined: $F(q) \neq 0$ for $q \in Q$, and if $q \cdot z_0 = q' \cdot z_0$ then $q = q'h$ for some $h \in H$.

We let G act on $\mathfrak{g}^*$ via the co-adjoint action and record the following result.

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Lemma 2.5. *The map μ is Q-equivariant.*

Proof. Let $z \in Z_0$, $q \in Q$ and $Y \in \mathfrak{g}$. Then

$$\begin{aligned}\mu(qz)(Y) &= \frac{\frac{d}{dt}\big|_{t=0} F(\exp(tY)qz)}{F(qz)} = \frac{\frac{d}{dt}\big|_{t=0} F(qq^{-1}\exp(tY)qz)}{\psi(q)F(z)} \\ &= \frac{\frac{d}{dt}\big|_{t=0} F(\exp(t\,\mathrm{Ad}(q^{-1})Y)z)}{F(z)} = (\mathrm{Ad}^*(q)\mu(z))(Y). \qquad \square\end{aligned}$$

Note that

$$\mu(z)(X) = d\psi(X), \quad (X \in \mathfrak{q}) \tag{2.2}$$

for all $z \in Z_0$. In particular, $\mu(z_1) - \mu(z_2) \in \mathfrak{q}^\perp \subseteq \mathfrak{g}^*$ for $z_1, z_2 \in Z_0$. Moreover, $\mu(z)(X+Y) = -\nu(X)$ for $X \in \mathfrak{a}$ and $Y \in \mathfrak{m} + \mathfrak{n}$.

We now identify $\mathfrak{g}^*$ with $\mathfrak{g}$ via the invariant non-degenerate form $\kappa(\cdot\,,\cdot)$. Then $\mathfrak{q}^\perp$ is identified with $\mathfrak{u}$, and $(\mathfrak{m}+\mathfrak{n})^\perp$ with $\mathfrak{a}+\mathfrak{n}$. Let

$$X_0 = \mu(z_0) \in \mathfrak{a} + \mathfrak{n}.$$

Then $X_0 \notin \mathfrak{n}$ since $\nu \neq 0$ and hence X_0 is a semisimple element. Write X_s for the $\mathfrak{a}$-part of X_0. Then the eigenvalues of $\mathrm{ad}(X_0)$ on $\mathfrak{n}$ are the $\alpha(X_s)$ where $\alpha \in \Sigma(\mathfrak{a}, \mathfrak{n})$. By the identification of $\mathfrak{g}^*$ with $\mathfrak{g}$, these are the inner products $\langle -\nu, \alpha\rangle$; in particular, they are all non-positive and on $\mathfrak{u}$ they are negative.

We conclude from the above that $\mathrm{im}\,\mu \subseteq X_0 + \mathfrak{u}$. We claim equality:

$$\mathrm{im}\,\mu = X_0 + \mathfrak{u}. \tag{2.3}$$

As μ is Q-equivariant, we have $\mathrm{im}\,\mu = \mathrm{Ad}(Q)X_0$. The lemma below (with $X = -\mathrm{ad}(X_0)$) implies $\mathrm{Ad}(U)X_0 = X_0 + \mathfrak{u}$, and then (2.3) follows.

Lemma 2.6. *Let $\mathfrak{u}$ be a nilpotent Lie algebra and $X : \mathfrak{u} \to \mathfrak{u}$ a derivation which is diagonalizable with non-negative eigenvalues. Then in the solvable Lie algebra $\mathfrak{g} := \mathbb{R}X \ltimes \mathfrak{u}$ the following identity holds:*

$$e^{\mathrm{ad}\,\mathfrak{u}}X = X + [X, \mathfrak{u}]. \tag{2.4}$$

Proof. Note that $[X, \mathfrak{u}] = \mathfrak{u}$ if all eigenvalues are positive. The inclusion $\subseteq$ in (2.4) is easy. The proof of the opposite inclusion is by induction on $\dim \mathfrak{u}$, and the case $\dim \mathfrak{u} = 0$ is trivial. Assume $\dim \mathfrak{u} > 0$ and let $\mathfrak{u} = \sum_{\lambda \geqslant 0} \mathfrak{u}(X, \lambda)$ be the eigenspace decomposition of the operator $X : \mathfrak{u} \to \mathfrak{u}$. Let $\lambda_1 \geqslant 0$ be the smallest eigenvalue and set $\mathfrak{u}_1 := \mathfrak{u}(X, \lambda_1)$ and $\mathfrak{u}_2 := \sum_{\lambda > \lambda_1} \mathfrak{u}(X, \lambda)$. Note that $\mathfrak{u}_2$ is an ideal in $\mathfrak{u}$, and $\mathfrak{u} = \mathfrak{u}_1 + \mathfrak{u}_2$ as vector spaces.

By induction we have $e^{\mathrm{ad}\,\mathfrak{u}_2}X = X + \mathfrak{u}_2$. If $\lambda_1 = 0$ then $[X, \mathfrak{u}] = \mathfrak{u}_2$, and we are done. Otherwise $[\mathfrak{u}_1, \mathfrak{u}_1] \subseteq \mathfrak{u}_2$ and hence

$$e^{\mathrm{ad}\,U}X \in X + \lambda_1 U + \mathfrak{u}_2$$

for $U \in \mathfrak{u}_1$. Note that $e^{\mathrm{ad}\,\mathfrak{u}}$ is a group as $\mathfrak{u}$ is nilpotent. It follows that

$$\begin{aligned} e^{\mathrm{ad}\,\mathfrak{u}}X &\supseteq e^{\mathrm{ad}\,\mathfrak{u}_1}e^{\mathrm{ad}\,\mathfrak{u}_2}X = e^{\mathrm{ad}\,\mathfrak{u}_1}(X + \mathfrak{u}_2) \\ &= \bigcup_{U \in \mathfrak{u}_1} e^{\mathrm{ad}\,U}(X + \mathfrak{u}_2) = \bigcup_{U \in \mathfrak{u}_1} (e^{\mathrm{ad}\,U}X + \mathfrak{u}_2) \\ &= \bigcup_{U \in \mathfrak{u}_1} (X + \lambda_1 U + \mathfrak{u}_2) = X + \mathfrak{u}. \qquad \square \end{aligned}$$

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Continuing with the proof of Theorem 2.3, we conclude that the stabilizer $L \subseteq Q$ of $X_0 \in \mathfrak{q}$ is a reductive Levi subgroup. Let

$$S := \mu^{-1}(X_0) = \{z \in Z_0 \mid \mu(z) = X_0\}.$$

Then for $q \in Q$ we have

$$qz_0 \in S \Leftrightarrow \mu(qz_0) = X_0 \Leftrightarrow qX_0 = X_0 \Leftrightarrow q \in L. \tag{2.5}$$

Hence L acts transitively on S. As μ is submersive, S is a submanifold of Z_0 and we obtain with

$$Q \times_L S \to Z_0 \tag{2.6}$$

a Q-equivariant diffeomorphism. As an L-homogeneous space, S is isomorphic to $L/L \cap H$. Hence (i) is valid.

Note that (2.5) implies that $(L \cap P)H = S \cap (PH)$, which is open in S. Thus $L/L \cap H$ is a real spherical space.

If (iii) is valid, we are done. Otherwise we let $Q_1 = Q$ and consider the real spherical space $Z_1 = L_1/L_1 \cap H$ for $L_1 = L$. Iterating the procedure of before yields a proper parabolic subgroup R of L_1 containing $L_1 \cap P$ and with a Levi subgroup $L_2 \subseteq L_1$ such that

$$(R \cap N) \times L_2/(L_2 \cap H) \to R \cdot z_0 \tag{2.7}$$

is a diffeomorphism. We let $Q_2 = RP = RU_1$, which is a subgroup since R normalizes U_1. Note that (2.7), together with the property (i) for Q_1, implies that this property is valid also for Q_2. We continue iterations until H contains the non-compact semisimple part of some L_i. This will happen eventually since the non-compact semisimple part of a Levi subgroup of P is trivial. $\square$

2.2 Z-adapted parabolics

Definition 2.7. Let $Z = G/H$ be a real spherical space. A parabolic subgroup $Q < G$ will be called *Z-adapted* provided that:

(i) there is a minimal parabolic subgroup $P \subseteq Q$ with PH open;

(ii) there is a Levi decomposition $Q = LU$ such that $Q \cap H \subseteq L$.

(iii) $\mathfrak{l}_n \subseteq \mathfrak{h}$.

A parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$ is called Z-adapted if it is the Lie algebra of a Z-adapted parabolic subgroup Q.

Theorem 2.8. *Let $Z = G/H$ be an almost algebraic real spherical space and P a minimal parabolic subgroup such that PH is open. Then there exists a unique parabolic subgroup $Q \supseteq P$ with unipotent radical U such that $\mathfrak{u}$ is complementary to $\mathfrak{n} \cap \mathfrak{h}$ in $\mathfrak{n}$. Moreover, this parabolic subgroup Q is Z-adapted, and it is the unique parabolic subgroup above P with that property.*

Proof. Note first that if $Q \supseteq P$ and $Q = LU$ is a Levi decomposition then $\mathfrak{n} = (\mathfrak{n} \cap \mathfrak{l}) \oplus \mathfrak{u}$. Assuming in addition (ii) and (iii) above, then $\mathfrak{n} \cap \mathfrak{h} = \mathfrak{n} \cap \mathfrak{l}$, and hence $\mathfrak{n} \cap \mathfrak{h}$ is complementary to $\mathfrak{u}$. Hence every Z-adapted parabolic subgroup $Q \supseteq P$ has this property of complementarity. In particular, this holds then for the parabolic subgroup Q constructed with Theorem 2.3.

It remains to prove that if $Q' \supseteq P$ is another parabolic for which the unipotent radical $\mathfrak{u}'$ is complementary to $\mathfrak{n} \cap \mathfrak{h}$, then $Q' = Q$. Since $\mathfrak{l}_n \subset \mathfrak{h}$ we find

$$\mathfrak{u}' \cap \mathfrak{l} \subseteq \mathfrak{u}' \cap \mathfrak{h} = \{0\}.$$

The lemma below now implies $\mathfrak{u} \supseteq \mathfrak{u}'$. But then $\mathfrak{u} = \mathfrak{u}'$ since both spaces are complementary to $\mathfrak{n} \cap \mathfrak{h}$, and hence $Q = Q'$. $\square$

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Lemma 2.9. *Let $\mathfrak{p}$ be a minimal parabolic subalgebra, and let $\mathfrak{q}, \mathfrak{q}' \supseteq \mathfrak{p}$ be parabolic subalgebras with unipotent radicals $\mathfrak{u}$, $\mathfrak{u}'$. If there exists a Levi decomposition $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ such that $\mathfrak{l} \cap \mathfrak{u}' = \{0\}$, then $\mathfrak{q} \subseteq \mathfrak{q}'$.*

Proof. This follows easily from the standard description of the parabolic subalgebras containing $\mathfrak{p}$ by sets of simple roots. □

2.3 The real rank of Z

Let Q be Z-adapted, with Levi decomposition $Q = LU$ as in Definition 2.7. From the local structure theorem we obtain an isomorphism

$$Q \times_L L/L \cap H \to Q \cdot z_0 = P \cdot z_0.$$

Recall that $\mathfrak{l}_n \subseteq \mathfrak{h}$. We decompose

$$\mathfrak{l} = \mathfrak{z}(\mathfrak{l}) \oplus [\mathfrak{l}, \mathfrak{l}] = \mathfrak{z}(\mathfrak{l}) \oplus \mathfrak{l}_c \oplus \mathfrak{l}_n, \tag{2.8}$$

where $\mathfrak{l}_c$ denotes the sum of all compact simple ideals in $\mathfrak{l}$. Note that $D = L/L_n$ is a Lie group with the Lie algebra $\mathfrak{d} = \mathfrak{z}(\mathfrak{l}) + \mathfrak{l}_c$, which is compact, and that

$$\mathfrak{l} \cap \mathfrak{h} = \mathfrak{c} \oplus \mathfrak{l}_n$$

with $\mathfrak{c} = \mathfrak{d} \cap \mathfrak{h}$. Let $C = (L \cap H)/L_n \subseteq D$; then $L/L \cap H = D/C$, and

$$U \times D/C \to P \cdot z_0 \tag{2.9}$$

is an isomorphism.

Consider the refined version of (2.8),

$$\mathfrak{l} = \mathfrak{z}(\mathfrak{l})_{\mathrm{np}} \oplus \mathfrak{z}(\mathfrak{l})_{\mathrm{cp}} \oplus \mathfrak{l}_c \oplus \mathfrak{l}_n, \tag{2.10}$$

in which $\mathfrak{z}(\mathfrak{l})_{\mathrm{np}}$ and $\mathfrak{z}(\mathfrak{l})_{\mathrm{cp}}$ denote the non-compact and compact parts of $\mathfrak{z}(\mathfrak{l})$. Let $L = K_L A_L (L \cap N)$ be an Iwasawa decomposition of L, and let $G = KAN$ be an Iwasawa decomposition of G which is compatible, that is, $K \supseteq K_L$ and $A = A_L$. Then $\mathfrak{a} = \mathfrak{z}(\mathfrak{l})_{\mathrm{np}} \oplus (\mathfrak{a} \cap \mathfrak{l}_n)$.

Let $\mathfrak{a}_h \subset \mathfrak{z}(\mathfrak{l})_{\mathrm{np}}$ be the image of $\mathfrak{c}$ under the projection $\mathfrak{l} \to \mathfrak{z}(\mathfrak{l})_{\mathrm{np}}$ along (2.10), and let $\mathfrak{a}_Z$ be a subspace of $\mathfrak{z}(\mathfrak{l})_{\mathrm{np}}$, complementary to $\mathfrak{a}_h$. Then

$$\mathfrak{a} = \mathfrak{a}_Z \oplus \mathfrak{a}_h \oplus (\mathfrak{a} \cap \mathfrak{l}_n). \tag{2.11}$$

The number $\dim \mathfrak{a}_Z$ will be called the *real rank* of Z in § 3, where we show (under an additional hypothesis) that it is an invariant of Z (it is independent of the choices of P and L). See Remark 3.5.

2.4 HP-factorizations of a semisimple group

Let $Z = G/H$ be real spherical. In general G/P admits several H-orbits. Here we investigate the simplest case where there is just one orbit.

Proposition 2.10. *Let G be semisimple. Assume that $Z = G/H$ is real spherical and that $\mathfrak{h}$ contains no non-zero ideal of $\mathfrak{g}$. Then $HP = G$ if and only if H is compact.*

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Proof. Assume that $HP = G$. Note that then $HgP = G$ for every $g \in G$ and hence

$$\mathfrak{h} + \mathrm{Ad}(g)(\mathfrak{p}) = \mathfrak{g}$$

for every $g \in G$.

We first reduce to the case where H is reductive in G. Otherwise there exists a non-zero ideal $\mathfrak{h}_u$ in $\mathfrak{h}$ which acts unipotently on $\mathfrak{g}$. By conjugating P if necessary, we may assume that $\mathfrak{h}_u \subseteq \mathfrak{n}$. It then follows from $G = PH$ that $\mathrm{Ad}(g)(\mathfrak{h}_u) \subseteq \mathfrak{n}$ for all $g \in G$, which is absurd.

Assume now that H is reductive and let $H = K_H A_H N_H$ be an Iwasawa decomposition. Let $X \in \mathfrak{a}_H$ be regular dominant with respect to $\mathfrak{n}_H$, and let $\mathfrak{q}$ be the parabolic subalgebra of $\mathfrak{g}$ which is spanned by the non-negative eigenspaces of $\mathrm{ad}\, X$. It follows that $\mathfrak{q} \cap \mathfrak{h}$ is a minimal parabolic subalgebra of $\mathfrak{h}$, and that $\mathfrak{n}_H$ is contained in the unipotent part $\mathfrak{u}$ of $\mathfrak{q}$. As Q contains a conjugate of P we have $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ and hence $\dim(\mathfrak{h}/(\mathfrak{q} \cap \mathfrak{h})) = \dim(\mathfrak{g}/\mathfrak{q})$, from which we deduce that $\mathfrak{n}_H = \mathfrak{u}$. From $\mathfrak{n}_H = \mathfrak{u}$ and $\mathfrak{g} = \mathfrak{h} + \mathfrak{q}$ we deduce that $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$. Let $\mathfrak{h}_n$ be the subalgebra of $\mathfrak{h}$ generated by $\mathfrak{n}_H$ and its opposite $\bar{\mathfrak{n}}_H$ with respect to the Cartan involution of H associated with $H = K_H A_H N_H$. Then $\mathfrak{h}_n$ is $\mathfrak{l}$-invariant and an ideal in $\mathfrak{h}$. With $\mathfrak{g} = \mathfrak{h} + \mathfrak{l}$ we now infer that $\mathfrak{h}_n$ is an ideal in $\mathfrak{g}$, and hence it is zero. It follows that $H = K_H A_H$, where A_H is central in H. We may assume $K_H \subseteq K$ and $A_H \subseteq A$. Then $G = HP$ implies $K = K_H M$, and hence K centralizes A_H. This is impossible unless $A_H = \{1\}$ and then H is compact.

Conversely, if H is compact then the open H-orbit on G/P is closed, and since G/P is connected it follows that $HP = G$. □

3. Real spherical varieties

All complex varieties $Z_\mathbb{C}$ in this section will be defined over $\mathbb{R}$. Typically we denote by Z the set of real points of $Z_\mathbb{C}$. If Z is Zariski dense in $Z_\mathbb{C}$, then we call Z a *real (algebraic) variety*.

We say that a subset $U \subset Z$ is (quasi-)affine if there exists a (quasi-)affine subset $U_\mathbb{C} \subset Z_\mathbb{C}$ such that $U = U_\mathbb{C} \cap Z$.

Remark 3.1. Even if $Z_\mathbb{C}$ is irreducible it might happen that Z has several connected components with respect to the Euclidean topology. However, by Whitney's theorem, the number of connected components is always finite. Take, for example, $Z = \mathrm{GL}(n, \mathbb{R})$ and $Z_\mathbb{C} = \mathrm{GL}(n, \mathbb{C})$. Here Z breaks into two connected components $\mathrm{GL}(n, \mathbb{R})_+$ and $\mathrm{GL}(n, \mathbb{R})_-$ characterized by the sign of the determinant; certainly it would be meaningful to call $\mathrm{GL}(n, \mathbb{R})_+$ a real algebraic variety as well.

Let $Z_1 \amalg \ldots \amalg Z_n$ be the decomposition of Z into connected components (with respect to the Euclidean topology). A more general notion of *real variety* would be to allow arbitrary unions of those Z_j which are Zariski dense in $Z_\mathbb{C}$. In fact, all the statements derived in this section for real varieties are valid in this more general setup.

In this section we let G be a real algebraic reductive group and $G_\mathbb{C} \supseteq G$ its complexification. Furthermore, P is a minimal parabolic subgroup of G and $P = MAN$ a Langlands decomposition of it.

By a *real G-variety* Z we understand a real variety Z endowed with a real algebraic G-action. A real G-variety will be called *linearizable* provided there is a finite-dimensional real G-module V such that Z is realized as real subvariety of $\mathbb{P}(V)$.

An algebraic real reductive group G is called *elementary* if $G \cong M \times A$ with M compact and $A = (\mathbb{R}^+)^l$. This is equivalent to $G = P$. A real G-variety Z will then be called *elementary* if G/J is elementary where J is the kernel of the action on Z.

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Definition 3.2. A linearizable real G-variety Z will be called *real spherical* provided that:

– $Z_{\mathbb{C}}$ is irreducible,
– Z admits an open P-orbit.

Remark 3.3. (a) In the definition of a (complex) spherical variety one requires in particular that the variety is normal. We now explain how this is related to our notion of real spherical.

Assume that $Z_{\mathbb{C}}$ is normal. Then it follows from a theorem of Sumihiro [KKLV89, p. 64] that every every point $z \in Z_{\mathbb{C}}$ has a $G_{\mathbb{C}}$-invariant open neighborhood U which can be equivariantly embedded into $\mathbb{P}(V_{\mathbb{C}})$ where $V_{\mathbb{C}}$ is a finite-dimensional representation of $G_{\mathbb{C}}$. It follows that if $z \in Z$ then $U_0 := (U \cap \overline{U}) \cap Z$ is a linearizable open neighborhood of z. Observe that there is always a normalization map $\nu : \tilde{Z} \to Z$ where $\tilde{Z}$ is a normal G-variety and ν is proper, finite to one, and invertible over an open dense subset of Z.

(b) If Z is a real spherical variety, then the number of open P-orbits is finite: As $Z_{\mathbb{C}}$ is irreducible, there is exactly one open $P_{\mathbb{C}}$-orbit on $Z_{\mathbb{C}}$ and the real points of this open $P_{\mathbb{C}}$-orbit decompose into finitely many P-orbits. We conclude in particular that there are only finitely many open G-orbits in Z. Let $\mathcal{O} \simeq G/H$ be one of them. Then G/H is a real spherical algebraic homogeneous space which we considered before.

(c) Let Z be an elementary real spherical variety. If $G = A$, then Z consists of the real points of a toric variety defined over $\mathbb{R}$.

(d) Let $G = M \times A$ be an elementary algebraic real reductive group and $Z = G/H$ a homogeneous real spherical G-variety. Since there are no algebraic homomorphisms between a split torus and a compact group, the group H splits as $H = M_0 \times A_0$ with $M_0 \subseteq M$ and $A_0 \subseteq A$. Thus $Z = M/M_0 \times A/A_0$.

3.1 Some general facts about real G-varieties

Let Z be an irreducible real variety. We denote by $\mathbb{C}[Z]$ (respectively, $\mathbb{C}(Z)$) the ring of *regular* (respectively, *rational*) functions on Z, that is, $\mathbb{C}[Z]$ consists of the restrictions of the regular functions on $Z_{\mathbb{C}}$ to Z, and likewise for $\mathbb{C}(Z)$.

As Z is Zariski dense we observe that the restriction mapping $\mathrm{Res} : \mathbb{C}(Z_{\mathbb{C}}) \to \mathbb{C}(Z)$ is bijective. Next we note that both $\mathbb{C}(Z)$ and $\mathbb{C}[Z]$ are invariant under complex conjugation $f \mapsto \overline{f}$. In particular with $f \in \mathbb{C}[Z]$ (respectively, $\mathbb{C}(Z)$), we also have that $\mathrm{Re}\, f$ and $\mathrm{Im}\, f$ belong to $\mathbb{C}[Z]$ (respectively, $\mathbb{C}(Z)$).

If a compact real algebraic group M acts on Z, then the M-average

$$f \mapsto f^M; \quad f^M(z) := \int_M f(m \cdot z)\, dm \quad (z \in Z)$$

preserves $\mathbb{C}[Z]$. This follows from the fact that the G-action on $\mathbb{C}[Z]$ is locally finite. Put together, we conclude

$$f \in \mathbb{C}[Z] \Rightarrow (|f|^2)^M \in \mathbb{C}[Z]^M \quad \text{with } f \neq 0 \Rightarrow (|f|^2)^M \neq 0. \tag{3.1}$$

Let us denote by $\widehat{P}$ the set of real algebraic characters $\chi : P \to \mathbb{R}^\times$ such that $MN \subseteq \ker \chi$. Note that the subgroup MN of P, and hence $\widehat{P}$, is independent of the choice of a Langlands decomposition of P. However, when that has been chosen, there is a natural identification of $\widehat{P}$ with a lattice $\Lambda \subseteq \mathfrak{a}^*$.

For the rest of this subsection we let Z be a real G-variety. We denote by $\mathbb{C}(Z)^{(P)}$ the set of P-semi-invariant functions, i.e. the rational functions $f \in \mathbb{C}(Z)\backslash\{0\}$ for which there is a $\chi \in \widehat{P}$ such that $f(p^{-1}z) = \chi(p)f(z)$ for all $p \in P$, $z \in Z$ for which both sides are defined. We denote

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by $\mathbb{C}(Z)^P$ the set of P-invariants in $\mathbb{C}(Z)$. Likewise we define $\mathbb{C}[Z]^{(P)}$ and $\mathbb{C}[Z]^P$. Further, we denote by $\mathbb{R}(Z)$ and $\mathbb{R}[Z]$ the real-valued functions in $\mathbb{C}(Z)$ and $\mathbb{C}[Z]$.

Lemma 3.4. *Let Z be a quasi-affine real G-variety. Then for all non-zero $f \in \mathbb{R}(Z)^P$ there exist $f_1, f_2 \in \mathbb{R}[Z]^{(P)}$ such that $f = f_1/f_2$.*

Proof. Let $f \in \mathbb{R}(Z)^P$. As Z is quasi-affine, we find regular functions $h_1, h_2 \in \mathbb{C}[Z]$, $h_2 \neq 0$ such that $f = h_1/h_2$. Consider the ideal

$$I := \{h \in \mathbb{C}[Z] \mid hf \in \mathbb{C}[Z]\}.$$

Note that:

- $I \neq \{0\}$ as $h_2 \in I$;
- $I = \overline{I}$ as f is real;
- I is P-invariant as f is P-fixed.

The action of P on $\mathbb{C}[Z]$ is algebraic, hence locally finite, and thus we find an element $0 \neq h \in I$ which is an eigenvector for the solvable group AN. We use (3.1) to obtain with $f_2 = (|h|^2)^M$ a non-zero element of $I \cap \mathbb{R}[Z]^{(P)}$. Now we put $f_1 = f_2 f \in \mathbb{R}[Z]^{(P)}$. □

For $\chi \in \widehat{P} = \Lambda$ we let

$$\mathbb{C}[Z]_\chi := \{f \in \mathbb{C}[Z] \mid (\forall p \in P, z \in Z)\ f(p^{-1}z) = \chi(p)f(z)\},$$

and define $\mathbb{C}(Z)_\chi$ likewise. We define a sub-lattice of Λ by

$$\Lambda_Z := \{\chi \in \widehat{P} \mid \mathbb{C}(Z)_\chi \neq \{0\}\}.$$

With that we declare the *real rank* of Z by

$$\mathrm{rk}_{\mathbb{R}}(Z) := \dim_{\mathbb{Q}}(\Lambda_Z \otimes_{\mathbb{Z}} \mathbb{Q}). \tag{3.2}$$

It is easily seen that $\mathrm{rk}_{\mathbb{R}}(Z)$ is independent of the choice of minimal parabolic subgroup P.

Remark 3.5. Let $Z = G/H$ be homogeneous. Then $\mathrm{rk}_{\mathbb{R}}(Z) = \dim \mathfrak{a}_Z$ where $\mathfrak{a}_Z$ is defined by (2.11). In fact, as a Q-variety, an open subset of Z is isomorphic to $U \times L/L\cap H$. Thus $\mathbb{R}(Z)^{(P)} = \mathbb{R}(L/L\cap H)^{(L\cap P)}$. Since H contains L_n the variety $L/L\cap H$ is elementary. By Remark 3.3(d), we have $\mathbb{R}(L/L\cap H)^{(L\cap P)} = \mathbb{R}(A/A_0)^{(A)}$ which implies the claim, as $A/A_0 \simeq \mathfrak{a}_Z$.

Lemma 3.6. *Let Z be a linearizable irreducible real G-variety and $Y \subseteq X$ a Zariski closed G-invariant subvariety. Then there exists a P-stable affine open subset $Z_0 \subseteq Z$ which meets Y and such that the restriction mapping*

$$\mathbb{R}[Z_0]^{(P)} \to \mathbb{R}[Z_0 \cap Y]^{(P)}$$

is onto.

Proof. If G is complex, then this is the real-points version of [Bri97, Proposition 1.1]. Further, with P replaced by AN, one can literally copy the proof of [Bri97]. Finally, the additional M-invariance when moving from AN to P is obtained from (3.1). □

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Denote by $\Lambda^+ \subseteq \Lambda$ the semigroup of elements dominant with respect to P. For all $\lambda \in \Lambda^+$ we set

$$m(\lambda) := \dim_{\mathbb{C}} \mathbb{C}[Z]_\lambda.$$

If we identify Λ^+ with a subset of the irreducible finite-dimensional representations of G, then $m(\lambda)$ is the multiplicity of the irreducible representation λ occurring in the locally finite G-module $\mathbb{C}[Z]$. The following proposition is a real analogue of the Vinberg–Kimel'feld theorem [VK78].

Proposition 3.7. *Let Z be a quasi-affine irreducible G-variety. Then the following assertions are equivalent:*

(i) *Z is real spherical;*
(ii) *$m(\lambda) \leqslant 1$ for all $\lambda \in \Lambda^+$.*

Proof. (i) $\Rightarrow$ (ii) Let $z \in Z$ such that $P \cdot z$ is open in Z. Then two P-semi-invariant functions f_1 and f_2 with respect to the same character $\lambda \in \widehat{P}$ satisfy $f_1|_{P\cdot z} = cf_2|_{P\cdot z}$ for some constant $c \in \mathbb{C}$. As $Z_{\mathbb{C}}$ is irreducible we conclude that $f_1 = cf_2$.

(ii) $\Rightarrow$ (i) We recall that there is an open P-orbit on Z if and only if $\mathbb{C}(Z)^P = \mathbb{C}\mathbf{1}$. This follows from Rosenlicht's theorem [Spr89, p. 23], applied to $Z_{\mathbb{C}}$. Now let $f \in \mathbb{C}(Z)^P$. According to Lemma 3.4, there exist $f_1, f_2 \in \mathbb{C}[Z]^{(P)}$ such that $f = f_1/f_2$. Clearly f_1 and f_2 correspond to the same character $\lambda \in \widehat{P}$. As $m(\lambda) \leqslant 1$, we conclude that f_1 is a multiple of f_2. □

Corollary 3.8. *Let Z be a real spherical variety and $Y \subseteq Z$ a closed G-invariant irreducible subvariety. Then Y is real spherical.*

Proof. If Z is quasi-affine, then this is immediate from the previous proposition as the restriction mapping $\mathbb{C}[Z] \to \mathbb{C}[Y]$ is onto. The more general case is reduced to that by considering the affine cone over Z. Recall that $Z \subseteq \mathbb{P}(V)$. The preimage of Z in $V\backslash\{0\}$ will be denoted by $\widehat{Z}$. Note that $\widehat{Z}$ is quasi-affine. Moreover, Z is real spherical if and only if $\widehat{Z}$ is real spherical for the enlarged reductive group $G_1 = G \times \mathbb{R}^\times$. □

Corollary 3.9. *Let Z be a real spherical variety. Then the number of G-orbits on Z is finite and each G-orbit is spherical.*

Proof. In view of the preceding corollary we only need to show that there are finitely many G-orbits. Suppose that there are infinitely many G-orbits. We let $Y \subseteq Z$ be a closed irreducible G-subvariety of minimal dimension which admits infinitely many G-orbits. By Corollary 3.8, Y is spherical. In particular, Y admits open G-orbits. After deleting the finitely many open G-orbits from Y, we obtain a G-invariant subvariety $Y_1 \subseteq Y$ with infinitely many G-orbits. As $\dim Y_1 < \dim Y$ we reach a contradiction. □

The main result of [KS13] was that every homogeneous real spherical space admits only finitely many P-orbits. With Corollary 3.9 we then deduce the following result.

Theorem 3.10. *Let Z be a real spherical variety. Then the number of P-orbits on Z is finite.*

3.2 The local structure theorem

Let Z be a real spherical variety and $Y \subseteq Z$ a G-invariant closed subvariety. Our goal is to find a P-invariant coordinate chart Z_0 for Z which meets Y. For that we may assume that Z is Zariski closed in $\mathbb{P}(V)$, where V is a finite-dimensional G-module. Moreover, we may assume that $Y \subseteq Z$

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is a closed G-orbit. In particular, Y is real spherical by Corollary 3.9, and we let $Q_Y < G$ be a Y-adapted parabolic.

Under these assumption on Y and Z there is the following immediate generalization of Lemma 3.6.

Lemma 3.11. *Let Z be real spherical variety, closed in $\mathbb{P}(V)$, and $Y \subseteq Z$ a closed G-orbit. Then there exists a Q_Y-stable affine open subset $Z_0 \subseteq Z$ which meets Y and such that the restriction mapping*

$$\mathbb{R}[Z_0]^{(Q_Y)} \to \mathbb{R}[Z_0 \cap Y]^{(Q_Y)}$$

is onto.

Proof. The proof is analogous to that of Lemma 3.6. We obtain that Z_0 is the non-vanishing locus of a Q_Y-semi-invariant homogeneous polynomial function on V. ☐

Corollary 3.12. *Let $Z \subset \mathbb{P}(V)$ be a closed real spherical variety and Y an elementary closed subvariety. Then there exists a G-stable affine open subset $Z_0 \subset Z$ such that $Z_0 \cap Y \neq \emptyset$.*

Proof. One has $Q_Y = G$. ☐

We now start with the construction of Z_0. If Y is elementary, Z_0 is given by Corollary 3.12. So let us assume that Y is not elementary, i.e. G_n does not act trivially on Y. Let $\bar{P} = MA\bar{N}$ be opposite to P. As $Y \subseteq \mathbb{P}(V)$ is closed, we can find a vector $y_0 \in V$ such that $[y_0] \in Y$ is $A\bar{N}$-fixed, and such that A acts by a non-trivial character on y_0. This can be seen as follows. Assume for simplicity that V is irreducible. Then Y contains a vector y of which the A-weight decomposition has a non-trivial component y_0 in the lowest weight space of V. Compression of y by A^+ then exhibits a non-zero multiple of y_0 as a limit of elements from Y.

Next we choose $v_0^* \in V^*$ such that $[v_0^*]$ is AN-fixed and $v_0^*(y_0) = 1$. Let $\chi : A \to \mathbb{R}^+$ be the character defined by $a \cdot v_0^* = \chi(a) v_0^*$.

Consider the function

$$F : V \to \mathbb{R}, \quad v \mapsto \int_M v_0^*(m \cdot v)^2 \, dm$$

and note that

$$F(man \cdot v) = \psi(a) F(v)$$

for all $man \in MAN$ and $v \in V$, where $\psi = \chi^{-2}$. Further, F is real algebraic and homogeneous of degree 2. Thus $\{[v] \in \mathbb{P}(V) \mid F(v) \neq 0\}$ defines an affine open set in $\mathbb{P}(V)$ and the intersection with Z yields an affine open set Z_0. Note that F is not constant and hence Z_0 is a proper subvariety. We define $Q \supseteq P$ to be the parabolic subgroup which fixes the line $\mathbb{R}F|_{Z_0}$, that is, $Q = \{g \in G \mid gZ_0 = Z_0\}$.

As before, we define on Z_0 a moment-type map

$$\mu : Z_0 \to \mathfrak{g}^*, \quad \mu(z)(X) := \frac{dF(v)(X)}{F(v)}$$

for $z = [v] \in Z \subseteq \mathbb{P}(V)$. This map is algebraic and Q-equivariant. Let $U < Q$ be the unipotent radical.

We claim that $\operatorname{im} \mu$ is a Q-orbit. In fact for $X \in \mathfrak{q}$ we have $\mu(z)(X) = d\psi(X)$ for all $z \in Z$, and after identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ we obtain, as in the previous section, that

$$\operatorname{im} \mu = \operatorname{Ad}(Q) X_0 = X_0 + \mathfrak{u}$$

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with $X_0 = \mu([y_0])$. The stabilizer of X_0 determines a Levi subgroup $L < Q$. Then $S := \mu^{-1}(X_0)$ is an L-stable affine subvariety of Z_0 and we obtain an algebraic isomorphism

$$Q \times_L S \to Z_0.$$

The affine L-variety S is real spherical and meets Y. We continue the procedure with $(L, S, S \cap Y)$ instead of (G, Z, Y). The procedure will stop at the moment when $S \cap Y$ is fixed under L_n. We have thus shown the following result.

Theorem 3.13 (Local structure theorem, general case). *Let Z be a real spherical variety and $Y \subseteq Z$ a closed G-invariant subvariety. Then there is parabolic subgroup $Q \supseteq P$ with Levi decomposition $Q = LU$ with the properties that there is a Q-invariant affine open piece $Z_0 \subseteq Z$ meeting Y and an L-invariant closed spherical subvariety $S \subseteq Z_0$ such that:*

(i) *there is a Q-equivariant isomorphism*

$$Q \times_L S \to Z_0;$$

(ii) *$S \cap Y$ is an elementary spherical L-variety.*

4. The normalizer of a spherical subalgebra

As in the preceding section, we assume that G is algebraic and let $\mathfrak{h}$ be the Lie algebra of a spherical subgroup $H < G$. We denote by $\tilde{\mathfrak{h}} := \mathfrak{n}_{\mathfrak{g}}(\mathfrak{h})$ the normalizer of $\mathfrak{h}$ in $\mathfrak{g}$ and by $\tilde{H}$ the normalizer in G. Note that $\mathfrak{h} \lhd \tilde{\mathfrak{h}}$ is an ideal. Let $\mathfrak{p}$ be a minimal parabolic subalgebra such that $\mathfrak{p} + \mathfrak{h} = \mathfrak{g}$ and let $\mathfrak{q}$ denote the unique parabolic subalgebra above $\mathfrak{p}$, which is Z-adapted. Let $\tilde{Z} = G/\tilde{H}$.

Lemma 4.1. *The parabolic subalgebra $\mathfrak{q}$ is also $\tilde{Z}$-adapted.*

Proof. We write $\tilde{\mathfrak{q}}$ for the unique $\tilde{Z}$-adapted parabolic above $\mathfrak{p}$ and $\tilde{\mathfrak{u}}$ for its unipotent radical. Then

$$\mathfrak{n} = (\mathfrak{n} \cap \mathfrak{h}) \oplus \mathfrak{u} = (\mathfrak{n} \cap \tilde{\mathfrak{h}}) \oplus \tilde{\mathfrak{u}}.$$

It follows that $\tilde{\mathfrak{u}} \subseteq \mathfrak{u}$ and $\mathfrak{q} \subseteq \tilde{\mathfrak{q}}$. To obtain a contradiction we assume that $\mathfrak{q} \subsetneq \tilde{\mathfrak{q}}$. Then $\tilde{\mathfrak{u}} \subsetneq \mathfrak{u}$ and $\mathfrak{n} \cap \mathfrak{h} \subsetneq \mathfrak{n} \cap \tilde{\mathfrak{h}}$. In particular, the Lie algebra $\tilde{\mathfrak{h}}/\mathfrak{h}$ cannot be compact.

To conclude the proof we now show that $\tilde{\mathfrak{h}}/\mathfrak{h}$ is compact. Suppose first that Z is quasi-affine and let $\mathbb{C}[Z] = \bigoplus_{\pi \in \widehat{G}} \mathbb{C}[Z]_\pi$ be the decomposition of the G-module $\mathbb{C}[Z]$ into G-isotypical components. For each π we choose a model space V_π and let $\mathcal{M}_\pi := \mathrm{Hom}_G(V_\pi, \mathbb{C}[Z])$ be the corresponding multiplicity space. Note that $\mathcal{M}_\pi$ is finite dimensional as there is a natural identification of $\mathcal{M}_\pi$ with the space of H-fixed elements in V_π^*.

Let $C := \tilde{H}/H$. Note that C acts from the right on $\mathbb{C}[Z]$ and preserves each $\mathbb{C}[Z]_\pi$, thus inducing an action on $\mathcal{M}_\pi$. Since Z is quasi-affine we can choose finitely many $\pi_1, \ldots, \pi_k$ so that we obtain a faithful representation of C on the sum $\mathcal{M} := \bigoplus_{j=1}^k \mathcal{M}_{\pi_j}$.

Let $B < G_{\mathbb{C}}$ be a Borel subgroup contained in $P_{\mathbb{C}}$. For every π we let v_π be a B-highest weight vector in V_π. To every $\eta \in \mathcal{M}_\pi$ we associate the function $f_\eta(g) = \eta(\pi(g^{-1})v_\pi)$ and define an inner product on $\mathcal{M}_\pi$ by

$$\langle \eta, \eta \rangle_\pi := (|f_\eta|^2)^M(z_0)$$

with the notation of (3.1). As $(|f_\eta|^2)^M$ is a matrix coefficient of a representation in Λ, and as multiplicities for these are at most one by Proposition 3.7, we obtain that there is a real character $\chi_\pi : C \to \mathbb{R}^\times$ such that

$$\langle h \cdot \eta, h \cdot \eta \rangle_\pi = \chi_\pi(h) \langle \eta, \eta \rangle_\pi.$$

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The group $C_1 := \bigcap_{j=1}^{k} \ker \chi_{\pi_j}$ acts unitarily and faithfully on $\mathcal{M}$, hence is compact. By definition $C/C_1 < (\mathbb{R}^\times)^k$, hence the Lie algebra of C is compact.

Finally, we reduce to the quasi-affine case using the affine cone over $\mathbb{P}(V)$ as before; see the proof of Corollary 3.8. $\square$

Let $Q = LU$ be a Levi decomposition as in Definition 2.7 and recall the decomposition (2.10).

Proposition 4.2. *The normalizer* $\tilde{\mathfrak{h}}$ *of* $\mathfrak{h}$ *is of the form*

$$\tilde{\mathfrak{h}} = \mathfrak{h} \oplus \tilde{\mathfrak{c}} \tag{4.1}$$

with $\tilde{\mathfrak{c}}$ *a subalgebra of the form* $\tilde{\mathfrak{c}} = \tilde{\mathfrak{a}} \oplus \tilde{\mathfrak{m}}$ *where* $\tilde{\mathfrak{a}} < \mathfrak{z}(\mathfrak{l})_{\mathrm{np}}$ *and* $\tilde{\mathfrak{m}} < \mathfrak{z}(\mathfrak{l})_{\mathrm{cp}} + \mathfrak{l}_{\mathrm{c}}$.

Proof. From Lemma 4.1 we conclude that $\tilde{\mathfrak{h}} = \mathfrak{h} + \tilde{\mathfrak{h}} \cap \mathfrak{l}$, and we obtain (4.1) with a subspace $\tilde{\mathfrak{c}}$ of $\mathfrak{z}(\mathfrak{l}) + \mathfrak{l}_{\mathrm{c}}$. It is a subalgebra because $\mathfrak{z}(\mathfrak{l}) + \mathfrak{l}_{\mathrm{c}}$ is reductive and $\mathfrak{h}$ is an ideal in $\tilde{\mathfrak{h}}$.

Write $\tilde{\mathfrak{a}}$ for the orthogonal projection of $\tilde{\mathfrak{c}}$ to $\mathfrak{z}(\mathfrak{l})_{\mathrm{np}}$ and $\tilde{\mathfrak{m}}$ for the orthogonal projection of $\tilde{\mathfrak{c}}$ to $\mathfrak{z}(\mathfrak{l})_{\mathrm{cp}} + \mathfrak{l}_{\mathrm{c}}$. Then $\tilde{\mathfrak{c}} \subseteq \tilde{\mathfrak{a}} + \tilde{\mathfrak{m}}$, and it remains to show equality. This will follow if we can show that both $\tilde{\mathfrak{a}}$ and $\tilde{\mathfrak{m}}$ normalize $\mathfrak{h}$. For that we decompose $X \in \tilde{\mathfrak{c}}$ as $X = X_a + X_m$ with $X_a \in \tilde{\mathfrak{a}}$ and $X_m \in \tilde{\mathfrak{m}}$. Observe that $\operatorname{ad} X_a$ commutes with $\operatorname{ad} X_m$. Both operators are diagonalizable with real (respectively, imaginary) spectrum. As $\operatorname{ad} X$ preserves $\mathfrak{h}$ we therefore conclude that $\operatorname{ad} X_a$ and $\operatorname{ad} X_m$ preserve $\mathfrak{h}$ as well. $\square$

Corollary 4.3. *Let* $H \subseteq G$ *be real spherical. Then* $N_G(H)/H$ *is an elementary group.*

Corollary 4.4. *The normalizer* $\tilde{\mathfrak{h}}$ *is its own normalizer:* $\tilde{\tilde{\mathfrak{h}}} = \tilde{\mathfrak{h}}$.

Proof. It suffices to show that the normalizer $\tilde{\tilde{\mathfrak{h}}}$ of $\tilde{\mathfrak{h}}$ normalizes $\mathfrak{h}$ as well. Let $\tilde{H} = N_G(\mathfrak{h})$. Observe that $\tilde{H}/H$ is an elementary real algebraic group; in particular, it is reductive. Thus, $\tilde{\mathfrak{h}}_u = \mathfrak{h}_u$ for the nilpotent radicals. This implies that $\tilde{\tilde{\mathfrak{h}}}$ normalizes $\mathfrak{h}_u$ and that $\tilde{H}/H_u$ is a reductive real algebraic group. A connected group, which acts by algebraic automorphisms on a reductive Lie group, acts by inner automorphisms, hence fixes every ideal. Thus $\mathfrak{h}/\mathfrak{h}_u \subseteq \tilde{\mathfrak{h}}/\mathfrak{h}_u$ is normalized by $\tilde{\tilde{\mathfrak{h}}}$ as well. $\square$

Remark 4.5. On the group level, the statement is wrong. For example, let $G = GL(2, \mathbb{R})$ and $H = \left(\begin{smallmatrix} * & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Then $N_G(H) = T = \left(\begin{smallmatrix} * & 0 \\ 0 & * \end{smallmatrix}\right)$. Thus $N_G(N_G(H)) = N_G(T)$ is strictly larger than $N_G(H) = T$.

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