J. Austral. Math. Soc. (*Series A*) **26** (1978), 227–240.

INFINITE τ_T PRODUCTS OF DISTRIBUTION FUNCTIONS

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(Received 21 September 1976)

Communicated by J. Gani

Abstract

Let T be a continuous *t-norm* (a suitable binary operation on $[0,1]$) and Δ^+ the space of distribution functions which are concentrated on $[0,\infty)$. The τ_T product of any F, G in Δ^+ is defined at any real x by

$$\tau_T(F,G)(x) = \sup_{u+v=x} T(F(u),G(v)),$$

and the pair (Δ^+, τ_T) forms a semigroup. Thus, given a sequence $\{F_i\}$ in Δ^+, the n-fold product $\tau_T(F_1,\cdots,F_n)$ is well-defined for each n. Moreover, the resulting sequence $\{\tau_T(F_1,\cdots,F_n)\}$ is pointwise non-increasing and hence has a weak limit. This paper establishes a convergence theorem which yields a representation for this weak limit. In addition, we prove the Zero-One law that, for Archimedean t-norms, the weak limit is either identically zero or has supremum 1.

Subject classification (*Amer. Math. Soc.* (*MOS*) 1970): 60F99, 60B99.

1. Introduction

If T is a *t-norm*, that is, a suitable binary operation on $[0,1]$, and Δ^+ is the space of one dimensional distribution functions which are concentrated on $[0,\infty)$, then the τ_T product of F, G in Δ^+ is defined at any x by

$$\tau_T(F,G)(x) = \sup_{u+v=x} T(F(u),G(v)). \tag{1.1}$$

If the t-norm T is left-continuous as a two place function then the operation τ_T is a mapping from $\Delta^+\times\Delta^+$ into Δ^+ and the pair (Δ^+,τ_T) is a semigroup, called a τ_T *semigroup.* The τ_T operations are quite distinct from the operation of convolution of distribution functions [Schweizer and Sklar (1974)] and τ_T semigroups play a prominent role in the theory of probabilistic metric spaces [Schweizer (1967, 1975)].

Since the τ_T operations are associative, for any sequence $\{F_i\}$ in Δ^+, the

n-fold τ_T product $\tau_T(F_1, \cdots, F_n)$ is well defined for each n. Moreover, the induced sequence of distribution functions $\{\tau_T(F_1, \cdots, F_n)\}$ is pointwise non-increasing [Moynihan (1978)] and hence has a unique weak limit in Δ^+. We call this weak limit *the infinite* τ_T *product of the sequence* $\{F_i\}$. Two naturally arising problems in this situation are to determine when an infinite τ_T product is non-trivial (that is, not identically zero) and to give a representation for it. The first question was partially solved in Moynihan (1978), where, using the concept of the *T-conjugate transform* on a given τ_T semigroup [Moynihan (1977)], we established:

THEOREM 1.1. *Given an Archimedean t-norm T and a sequence* $\{F_i\}$ *in* Δ^+, *the sequence of* τ_T *products* $\{\tau_T(F_1, \cdots, F_n)\}$ *has a non-trivial weak limit in* Δ^+ *if and only if there exists a sequence of positive numbers* $\{a_i\}$ *such that* $\sum_{i=1}^{\infty} a_i < \infty$ *and* $\lim_{n\to\infty} T(F_1(a_1), \cdots, F_n(a_n)) > 0$.

In this paper we greatly improve on the above result by showing in Section 2 that, for any continuous t-norm T, if G is the infinite τ_T product of the sequence $\{F_i\}$ in Δ^+, then, for any x,

$$G(x) = \sup\left\{\lim_{n\to\infty} T(F_1(a_1), \cdots, F_n(a_n)) \,\middle|\, \sum_{i=1}^{\infty} a_i = x\right\}. \tag{1.2}$$

Note that, for any integer n, (1.1) implies that

$$\tau_T(F_1, \cdots, F_n)(x) = \sup\left\{T(F_1(a_1), \cdots, F_n(a_n)) \,\middle|\, \sum_{i=1}^{n} a_i = x\right\}. \tag{1.3}$$

Thus (1.2) asserts that the limit and sup operations may be interchanged (for continuity points) and thus we obtain a convergence theorem for infinite τ_T products. Clearly (1.2) shows that Theorem 1.1 holds for any continuous t-norm.

However, as will be seen, Theorem 1.1 is a necessary and key tool used in establishing the results in this paper.

In Section 3 we show that, for an Archimedean t-norm T, if $G \in \Delta^+$ is the infinite τ_T product of a sequence $\{F_i\}$ in Δ^+, then, if G is non-trivial,

$$\sup_x G(x) = \lim_{n\to\infty} T\left(\sup_x F_1(x), \cdots, \sup_x F_n(x)\right).$$

In particular, it then follows that if each F_i is non-defective (that is, has supremum 1) then the supremum of the infinite τ_T product of the sequence $\{F_i\}$ is either 0 or 1, that is, the limit function is either identically zero or has *supremum 1. Finally, for a sequence of non-defective distribution functions* $\{F_i\}$, we show that the corresponding infinite τ_T product is non-trivial for

T = Product exactly when it is non-trivial for $T = T_m$, where $T_m(a, b) = \max\{a + b - 1, 0\}$.

Before we present our results, we state some definitions and known facts: The spaces of distribution functions which we will consider are

$$\Delta^+ = \{F : \mathbb{R} \to [0, 1] \mid F \text{ is left-continuous, non-decreasing and } F(0) = 0\}$$

and

$$\mathcal{D}^+ = \left\{F \in \Delta^+ \,\middle|\, \sup_x F(x) = 1\right\}.$$

In particular ε_0 and ε_∞ in Δ^+ are defined by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leqq 0, \\ 1, & x > 0; \end{cases} \quad \text{and } \varepsilon_\infty(x) = 0 \text{ for all } x.$$

A *t-norm* is a two-place function $T : [0, 1] \times [0, 1] \to [0, 1]$ which is symmetric, associative, non-decreasing in each place and has 1 as a unit and 0 as a null element. We say that a t-norm is *Archimedean* if T is continuous and satisfies $T(a, a) < a$ for all $a \in (0, 1)$; and *strict* if T is continuous on the closed unit square and is strictly increasing in each place on $(0, 1] \times (0, 1]$. Note that a strict t-norm must also be Archimedean.

From Aczél (1966), Ling (1965) we have the following important characterization of t-norms: The t-norm T is Archimedean if and only if there exists a continuous and increasing function $h : [0, 1] \to [0, 1]$ with $h(1) = 1$ such that T is representable in the form

$$T(x, y) = h^{[-1]}(h(x) \cdot h(y)), \tag{1.4}$$

where $h^{[-1]}$ is the *pseudo-inverse* of h, that is,

$$h^{[-1]}(x) = \begin{cases} 0, & 0 \leqq x \leqq h(0), \\ h^{-1}(x) & h(0) \leqq x \leqq 1; \end{cases} \tag{1.5}$$

where h^{-1} is the usual inverse of h on $[h(0), 1]$. The function h of (1.4) is called a *multiplicative generator* of the Archimedean t-norm T.

Finally, if $\{F_n\}$ is a sequence in Δ^+ then we say $\{F_n\}$ *converges weakly* to F in Δ^+, written $F_n \xrightarrow{w} F$, if $F_n(x) \to F(x)$ for all continuity points x of the limit function F.

2. A convergence theorem for infinite τ_T products

In this section we establish the identity (1.2) for infinite τ_T products for any continuous t-norm T.

First note that, since any t-norm T is associative, it naturally induces a well-defined n-place operation on $[0, 1]$. Thus, for any sequence $\{a_i\}$ in $[0, 1]$, we define, recursively,

$$T(a_1, \cdots, a_n) = \mathop{T}_{i=1}^{n} a_i = T\left(\mathop{T}_{i=1}^{n-1} a_i, a_n\right). \tag{2.1}$$

Also, we let

$$\mathop{T}_{i=1}^{\infty} a_i = \lim_{n\to\infty} \mathop{T}_{i=1}^{n} a_i, \tag{2.2}$$

where the sequence $\{T_{i=1}^{n} a_i\}$ is non-increasing and hence its limit always exists.

The τ_T operations given by (1.1) are examples of *triangle functions* [Schweizer (1975)] on Δ^+. For any triangle function τ and sequence $\{F_i\}$ in Δ^+, we also define, recursively,

$$\tau(F_1, \cdots, F_n) = \tau(\tau(F_1, \cdots, F_{n-1}), F_n) \tag{2.3}$$

and let $\tau_{i=1}^{\infty} F_i$ denote the *weak* limit in Δ^+ of the sequence $\{\tau(F_1, \cdots, F_n)\}$.

Our first step toward establishing (1.2) is:

LEMMA 2.1. *Let T be a continuous t-norm and let $\tau = \tau_T$. Then, for any sequence $\{F_i\}$ in Δ^+ and any x, we have that*

$$\left(\mathop{\tau}_{i=1}^{\infty} F_i\right)(x) \geqq \sup\left\{ \mathop{T}_{i=1}^{\infty} F_i(a_i) \,\middle|\, \sum_{i=1}^{\infty} a_i = x\right\}. \tag{2.4}$$

PROOF. For any x, choose $\{a_i\}$ so that $\Sigma_{i=1}^{\infty} a_i = x$ and $a_i > 0$ for all i. Then, for any n,

$$\begin{aligned}\tau_T(F_1, \cdots, F_n)(x) &\geqq \tau_T(F_1, \cdots, F_n)\left(\sum_{i=1}^{n} a_i\right)\\ &\geqq \mathop{T}_{i=1}^{n} F_i(a_i) \geqq \mathop{T}_{i=1}^{\infty} F_i(a_i)\end{aligned} \tag{2.5}$$

Also note that if any $a_i \leqq 0$, then the last term in (2.5) is zero. Thus, since the right hand side of (2.4) is easily shown to be left-continuous, if we let $n \to \infty$ in (2.5), then our desired result is obtained.

Next we prove (1.2) for Archimedean t-norms.

LEMMA 2.2. *Let T be an Archimedean t-norm and let $\tau = \tau_T$. Then, for any sequence $\{F_i\}$ in Δ^+ and any x, we have that*

$$\left(\mathop{\tau}_{i=1}^{\infty} F_i\right)(x) = \sup\left\{ \mathop{T}_{i=1}^{\infty} F_i(a_i) \,\middle|\, \sum_{i=1}^{\infty} a_i = x\right\}. \tag{2.6}$$

https://doi.org/10.1017/S1446788700011721 Published online by Cambridge University Press

PROOF. If $\tau_{i=1}^{\infty} F_i = \varepsilon_\infty$ then by Lemma 2.1 we are done. So assume otherwise, so that by Theorem 1.1 there exists a sequence of positive numbers $\{a_i\}$ such that

$$\sum_{i=1}^{\infty} a_i < \infty \quad \text{and} \quad \mathop{T}_{i=1}^{\infty} F_i(a_i) > 0. \tag{2.7}$$

Now choose any x and let $\varepsilon > 0$ be arbitrary. By the uniform continuity of T there exists a $\delta > 0$ so that

$$T(b, 1-\delta) > b - \frac{\varepsilon}{4} \quad \text{for any} \quad b \in [0,1]. \tag{2.8}$$

Next, using Moynihan (1978), Lemma 3.1, if h is the multiplicative generator of T then we have from (2.7) that

$$h^{[-1]}\left(\prod_{i=1}^{\infty} hF_i(a_i)\right) = \mathop{T}_{i=1}^{\infty} F_i(a_i) > 0, \tag{2.9}$$

whence, by (1.5), $\Pi_{i=1}^{\infty} hF_i(a_i) > h(0) \geqq 0$. Thus, since $h^{[-1]}$ is continuous with $h^{[-1]}(1) = 1$, we have, for some integer $M > 0$, that $\Pi_{i=M}^{\infty} hF_i(a_i)$ is sufficiently close to 1 to insure that

$$\mathop{T}_{i=M}^{\infty} F_i(a_i) = h^{[-1]}\left(\prod_{i=M}^{\infty} hF_i(a_i)\right) > 1 - \delta. \tag{2.10}$$

Now by left-continuity there exists a continuity point y of $\tau_{i=1}^{\infty} F_i$ with $y < x$ so that

$$\left(\mathop{\tau}_{i=1}^{\infty} F_i\right)(y) > \left(\mathop{\tau}_{i=1}^{\infty} F_i\right)(x) - \frac{\varepsilon}{4}; \tag{2.11}$$

and, by weak convergence, we have, for some $N > 0$, that for $n \geqq N$

$$\left| \tau_T(F_1, \cdots, F_n)(y) - \left(\mathop{\tau}_{i=1}^{\infty} F_i\right)(y) \right| < \frac{\varepsilon}{4}. \tag{2.12}$$

Now choose $n \geqq \max\{M, N\}$ and also sufficiently large so that $\Sigma_{i=n+1}^{\infty} a_i < x - y$. Then by (1.3) there exist $\{b_1, \cdots, b_n\}$ such that $\Sigma_{i=1}^{n} b_i = y$ and

$$T(F_1(b_1), \cdots, F_n(b_n)) \geqq \tau_T(F_1, \cdots, F_n)(y) - \frac{\varepsilon}{4}. \tag{2.13}$$

Letting $b_i = a_i$ for $i > n$, we then have that $\Sigma_{i=1}^{\infty} b_i < x$ and, from (2.8) through (2.13),

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$$
\begin{aligned}
\mathop{T}_{i=1}^{\infty} F_i(b_i) &= T\left(\mathop{T}_{i=1}^{n} F_i(b_i),\ \mathop{T}_{i=n+1}^{\infty} F_i(a_i)\right) \\
&\geqq T\left(\tau_T(F_1,\cdots,F_n)(y)-\frac{\varepsilon}{4},\ \mathop{T}_{i=M}^{\infty} F_i(a_i)\right) \\
&> \tau_T(F_1,\cdots,F_n)(y)-\frac{\varepsilon}{2} \qquad (2.14)\\
&> \left(\mathop{\tau}_{i=1}^{\infty} F_i\right)(y)-3\frac{\varepsilon}{4} \\
&> \left(\mathop{\tau}_{i=1}^{\infty} F_i\right)(x)-\varepsilon .
\end{aligned}
$$

Clearly, if we let $c_1 = b_1 + (x - \Sigma_{i=1}^{\infty} b_i)$ and $c_i = b_i$ for $i > 1$, then $\Sigma_{i=1}^{\infty} c_i = x$ and $T_{i=1}^{\infty} F_i(c_i) \geqq T_{i=1}^{\infty} F_i(b_i)$, whence, since $\varepsilon > 0$ was arbitrary, (2.14) establishes the reverse inequality to (2.4), completing the proof.

We will also need:

LEMMA 2.3. *Let T be a continuous t-norm, let $\tau = \tau_T$ and let $\{F_i\}$ be a sequence in Δ^+. Then, for any $\varepsilon > 0$, if $(\tau_{i=1}^{\infty} F_i)(x) \geqq \varepsilon$ for some $x > 0$, then there exists a sequence of non-negative numbers $\{a_i\}$ such that*

$$\sum_{i=1}^{\infty} a_i < \infty \quad \textit{and} \quad \inf_i \{F_i(a_i)\} \geqq \varepsilon \qquad (2.15)$$

PROOF. Suppose that (2.15) does not hold for some $\varepsilon > 0$. Let

$$a_i = \sup\{x \mid F_i(x) < \varepsilon\} \quad \text{for all } i.$$

Then $a_i \geqq 0$ for each i. Also, if $a_k = \infty$ for any integer k, then it follows, since τ_T is non-decreasing and $F_i \leqq \varepsilon_0$ for each i, that $(\tau_{i=1}^{\infty} F_i)(x) \leqq F_k(x) < \varepsilon$ for all x. Otherwise, $\inf_i \{F_i(a_i + 2^{-i})\} \geqq \varepsilon$, whence, necessarily,

$$\sum_{i=1}^{\infty} (a_i + 2^{-i}) = \left(\sum_{i=1}^{\infty} a_i\right) + 1 = \infty .$$

Now choose any $x > 0$. Then, for some $N > 0$, we have $\Sigma_{i=1}^{N} a_i > x$. If we let $\delta = (\Sigma_{i=1}^{N} a_i) - x$, then, for any $\{b_1, \cdots, b_N\}$ with $\Sigma_{i=1}^{N} b_i = x$, we must have $b_k \leqq a_k - \delta/N$ for some integer k with $1 \leqq k \leqq N$. Thus, since Min is the strongest t-norm [Schweizer (1975)], that is, $T(u, v) \leqq \mathrm{Min}(u, v)$ for all $u, v \in [0, 1]$, it follows that

$$\left(\overset{\infty}{\underset{i=1}{\tau}} F_i\right)(x) \leqq \tau_T(F_1, \cdots, F_N)(x) = \sup\left\{ \overset{N}{\underset{i=1}{T}} F_i(b_i) \,\middle|\, \sum_{i=1}^{N} b_i = x \right\}$$
$$\leqq \sup\left\{ \operatorname{Min}\{F_1(b_1), \cdots, F_N(b_N)\} \,\middle|\, \sum_{i=1}^{N} b_i = x \right\}$$
$$\leqq \operatorname{Max}\left\{ F_1\left(a_1 - \frac{\delta}{N}\right), \cdots, F_N\left(a_N - \frac{\delta}{N}\right)\right\} < \varepsilon,$$

completing the proof.

We can now establish:

THEOREM 2.1. *Let T be any continuous t-norm, let $\tau = \tau_T$ and let $\{F_i\}$ be a sequence in Δ^+. Then, for any x,*

$$\left(\overset{\infty}{\underset{i=1}{\tau}} F_i\right)(x) = \sup\left\{ \overset{\infty}{\underset{i=1}{T}} F_i(a_i) \,\middle|\, \sum_{i=1}^{\infty} a_i = x \right\}.$$

PROOF. We have from Paalman-de Miranda (1964), Theorem 2.5.4, p. 87 that T is an "ordinal sum" of Archimedean t-norms and the t-norm Min; that is, if

$$E = \{x \in [0,1] \mid T(x, x) = x\}$$

then $[0,1]\backslash E = \bigcup_{i \in J}(d_i, e_i)$ where $\{(d_i, e_i) \mid i \in J\}$ is a finite or countable collection of disjoint open intervals. Furthermore, if T_i denotes T restricted to $[d_i, e_i] \times [d_i, e_i]$, then $([d_i, e_i], T_i)$ is a semigroup with unit e_i and null element d_i. (Note $T_i(x, x) < x$ for all $x \in (d_i, e_i)$.) In other words, T consists of Archimedean "blocks" along the diagonal of the unit square and $T = \operatorname{Min}$ outside of these blocks, that is, $T(x, y) = \operatorname{Min}(x, y)$ if $(x, y) \notin [d_i, e_i] \times [d_i, e_i]$ for any $i \in J$.

Let (d, e) be any one of these open intervals and, for any $F \in \Delta^+$, define $F^* \in \Delta^+$ by

$$F^*(x) = \begin{cases} 0, & F(x) \leqq d, \\ F(x), & d < F(x) \leqq e, \\ e, & F(x) > e. \end{cases}$$

Then, for any $F, G \in \Delta^+$ and real x, we claim that:

(2.16) If $\tau_T(F, G)(x) \in (d, e]$ then $\tau_T(F, G)(x) = \tau_T(F^*, G^*)(x)$.

To prove (2.16) we first note that if the first part of (2.16) holds, then we can evaluate $\tau_T(F, G)(x)$ by restricting the supremum in (1.1) to those pairs u, v where $T(F(u), G(v)) \in (d, e]$. Now, using the ordinal sum above, this can happen only if either (i) both $F(u), G(v) \geqq e$ and $T(F(u), G(v)) = e$; or (ii) $F(u) \in (d, e)$ and $G(v) \geqq e$, so that $T(F(u), G(v)) = F(u)$; or (iii)

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$G(v) \in (d, e)$ and $F(u) \geqq e$, so that $T(F(u), G(v)) = G(v)$; or (iv) both $F(u), G(v) \in (d, e)$. But in all of these cases $T(F(u), G(v)) = T(F^*(u), G^*(v))$. Since clearly $T(F(u), G(v)) \geqq T(F^*(u), G^*(v))$ for all other pairs u, v, (2.16) then follows.

In addition, we can easily extend (2.16) inductively to obtain that if $\tau_T(F_1, \cdots, F_n)(x) \in (d, e]$ then

$$\tau_T(F_1, \cdots, F_n)(x) = \tau_T(F_1^*, \cdots, F_n^*)(x). \tag{2.17}$$

Thus, if x is a continuity point of $\tau_{i=1}^{\infty} F_i$ and $(\tau_{i=1}^{\infty} F_i)(x) \in (d, e)$, then $\tau_T(F_1, \cdots, F_n)(x) \in (d, e)$ for all n sufficiently large, whence

$$\tau_T(F_1^*, \cdots, F_n^*)(x) \to \left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(x). \tag{2.18}$$

Next define the operation T_A on $[0, 1] \times [0, 1]$ by

$$T_A(w, y) = \frac{T(d + w(e-d), d + y(e-d)) - d}{e - d}. \tag{2.19}$$

Then it is clear that T_A is an Archimedean t-norm. Furthermore, for any i, if we define

$$G_i(u) = \begin{cases} 0, & F_i^*(u) = 0, \\ \dfrac{F_i^*(u) - d}{e - d}, & \text{otherwise}; \end{cases} \tag{2.20}$$

then $G_i \in \Delta^+$ for all i and, for all u, v, if $F_1^*(u) > 0$ and $F_2^*(v) > 0$ then

$$T_A(G_1(u), G_2(v)) = (T(F_1^*(u), F_2^*(v)) - d)(e - d)^{-1}.$$

An easy induction step then yields that for any integer n, if $F_i^*(u_i) > 0$ for all i, then

$$T_A(G_1(u_1), \cdots, G_n(u_n)) = (T(F_1^*(u_1), \cdots, F_n^*(u_n)) - d)(e - d)^{-1} \tag{2.21}$$

Now if any $F_i^*(u_i) = 0$ then $T(F_1^*(u_1), \cdots, F_n^*(u_n)) = 0$. Thus, using (1.3) and (2.21), we have, for any y such that $\tau_T(F_1^*, \cdots, F_n^*)(y) > 0$, that

$$\tau_{T_A}(G_1, \cdots, G_n)(y) = (\tau_T(F_1^*, \cdots, F_n^*)(y) - d)(e - d)^{-1}.$$

In particular then, if G denotes the weak limit in Δ^+ of the sequence $\{\tau_{T_A}(G_1, \cdots, G_n)\}$ and x is as in (2.18) then

$$G(x) = \left(\left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(x) - d \right)(e - d)^{-1}. \tag{2.22}$$

Hence, using Lemma 2.2 and the fact that (2.21) holds whenever its left-hand side is non-zero, we have that

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$$
(2.23) \qquad \begin{aligned} G(x) &= \sup\left\{ \mathop{T_A}_{i=1}^{\infty} G_i(a_i) \,\middle|\, \sum_{i=1}^{\infty} a_i = x \right\} \\ &= \left[\sup\left\{ \mathop{T}_{i=1}^{\infty} F_i^*(a_i) \,\middle|\, \sum_{i=1}^{\infty} a_i = x \right\} - d \right] (e-d)^{-1}, \end{aligned}
$$

Since $F_i \geqq F_i^*$ for each i, (2.22) and (2.23) then yield that

$$
\sup\left\{ \mathop{T}_{i=1}^{\infty} F_i(a_i) \,\middle|\, \sum_{i=1}^{\infty} a_i = x \right\} \geqq \sup\left\{ \mathop{T}_{i=1}^{\infty} F_i^*(a_i) \,\middle|\, \sum_{i=1}^{\infty} a_i = x \right\} = \left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(x),
$$

whence, using Lemma 2.1, we have that (2.6) holds.

To complete our proof suppose, for a given x, that $(\tau_{i=1}^{\infty} F_i)(x) \not\in (d_i, e_i)$ for any i, that is, suppose $(\tau_{i=1}^{\infty} F_i)(x) = c \in E$ so that $T(c, c) = c$. Then by Lemma 2.3 there exists a sequence of non-negative numbers $\{a_i\}$ such that

$$
\sum_{i=1}^{\infty} a_i < \infty \quad \text{and} \quad \inf_i \{F_i(a_i)\} \geqq c.
$$

Let $\varepsilon > 0$ be arbitrary. Now by left-continuity there exists a continuity point y of $\tau_{i=1}^{\infty} F_i$ with $y < x$ such that

$$
\left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(y) > c - \frac{\varepsilon}{2}.
$$

We can then find an integer N sufficiently large so that we have both $\sum_{i=N+1}^{\infty} a_i \leqq x - y$ and

$$
\left| \tau_T(F_1, \cdots, F_N)(y) - \left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(y) \right| < \frac{\varepsilon}{4}.
$$

Next by (1.3) there exist $\{b_1, \cdots, b_N\}$ so that $\sum_{i=1}^{N} b_i = y$ and

$$
T(F_1(b_1), \cdots, F_N(b_N)) \geqq \tau_T(F_1, \cdots, F_N)(y) - \frac{\varepsilon}{4}.
$$

Thus if we let $b_i = a_i$ for $i > N$ then $\sum_{i=1}^{\infty} b_i \leqq x$ and, combining the above results and using the given facts about ordinal sums, we have that

$$
\begin{aligned} \mathop{T}_{i=1}^{\infty} F_i(b_i) &= T\left(\mathop{T}_{i=1}^{N} F_i(b_i), \quad \mathop{T}_{i=N+1}^{\infty} F_i(a_i) \right) \\ &\geqq T\left(\left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(y) - \frac{\varepsilon}{2}, c \right) \geqq T(c - \varepsilon, c) = c - \varepsilon. \end{aligned}
$$

As in the end of the proof of Lemma 2.2, this yields equality in (2.6), at least for continuity points of $\tau_{i=1}^{\infty} F_i$. But, since both sides of (2.6) are left-continuous, the result then follows for all x.

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REMARK. The pointwise limit of the sequence $\{\tau_T(F_1,\cdots,F_n)\}$ may not be left-continuous, and hence may not equal the right-hand side of (2.6). For example, for each integer n, let $F_n(x) = \varepsilon_0(x - 2^{-n})$ for all x. Then, for any t-norm T,

$$\tau_T(F_1,\cdots,F_n)(x) = \varepsilon_0(x - (1 - 2^{-n}))$$

Thus $\tau_T(F_1,\cdots,F_n)(1) = 1$ for all n, but $\tau_T(F_1,\cdots,F_n)(x) \to 0$ for all $x < 1$.

3. Supremums of infinite τ_T products

In general, for supremums of weak limits of infinite τ_T products, the most we can say is:

THEOREM 3.1. *Let T be a continuous t-norm and let $\tau = \tau_T$. Then, for any sequence $\{F_i\}$ in Δ^+, we have that*

$$\sup_x \left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(x) \leqq \mathop{T}_{i=1}^{\infty} \sup_x F_i(x). \tag{3.1}$$

PROOF. Using Theorem 2.1, for any y, we have

$$\left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(y) = \sup\left\{ \mathop{T}_{i=1}^{\infty} F_i(a_i) \;\middle|\; \sum_{i=1}^{\infty} a_i = y \right\} \leqq \mathop{T}_{i=1}^{\infty} \sup_x F_i(x).$$

Letting $y \to \infty$ then yields our result.

In the Archimedean t-norm case we obtain the following improvement to Theorem 3.1:

THEOREM 3.2. *Let T be Archimedean and let $\tau = \tau_T$. Then, for any sequence $\{F_i\}$ in Δ^+, we have that if $\tau_{i=1}^{\infty} F_i \neq \varepsilon_\infty$ then*

$$\sup_x \left(\mathop{\tau}_{i=1}^{\infty} F_i \right)(x) = \mathop{T}_{i=1}^{\infty} \sup_x F_i(x). \tag{3.2}$$

PROOF. In view of Theorem 3.1, we need only establish the reverse inequality to (3.1). This is easily done by using part of the proof of Lemma 2.2.

First, let $\varepsilon > 0$ be arbitrary and let $\delta > 0$ be such that (2.8), in which $\varepsilon/4$ is replaced by ε, holds. Then, since $\tau_{i=1}^{\infty} F_i \neq \varepsilon_\infty$, there exists a sequence of positive numbers $\{a_i\}$ such that (2.7) holds.

Also, we can again find an integer $M > 0$ so that (2.10) holds.

Hence, combining (2.8) and (2.10) with (2.6) we have that

$$\sup_x \left(\mathop{\tau}_{i=1}^{\infty} F_i\right)(x) = \lim_{x\to\infty} \left(\mathop{\tau}_{i=1}^{\infty} F_i\right)\left(x + \sum_{i=M+1}^{\infty} a_i\right)$$

$$\geqq \lim_{x\to\infty} T\left(\mathop{T}_{i=1}^{M} F_i(x/M), \mathop{T}_{i=M+1}^{\infty} F_i(a_i)\right)$$

$$\geqq T\left(\mathop{T}_{i=1}^{M}\left(\sup_x F_i(x)\right), 1-\delta\right) \tag{3.3}$$

$$> \mathop{T}_{i=1}^{M}\left(\sup_x F_i(x)\right) - \varepsilon$$

$$\geqq \mathop{T}_{i=1}^{\infty}\left(\sup_x F_i(x)\right) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this completes the proof.

For sequences of non-defective (that is, supremum 1) distribution functions, Theorem 3.2 yields the following *Zero-One Law for infinite* τ_T *products*:

THEOREM 3.3. *Let T be Archimedean and let $\tau = \tau_T$. Then, for any sequence $\{F_i\}$ in $\mathcal{D}^+$, the supremum of $\tau_{i=1}^{\infty} F_i$ is either* 0 *or* 1.

EXAMPLE. Theorem 3.3 (and hence also Theorem 3.2) does not hold for (continuous) non-Archimedean t-norms. For suppose the t-norm T satisfies $T(c, c) = c$ for some c with $0 < c < 1$. Then if we let $F_n \in \mathcal{D}^+$ be given by

$$F_n(x) = \begin{cases} 0, & x \leqq 0, \\ c, & 0 < x \leqq 1, \\ 1, & 1 < x; \end{cases} \qquad \text{for all } n,$$

then it is easily shown that $\sup_x (\tau_{i=1}^{\infty} F_i)(x) = c$.

The method of Theorem 3.2 can also be used to establish:

THEOREM 3.4. *Let T be a strict t-norm and let $\tau = \tau_T$. Let $\{F_i\}$ be a sequence in Δ^+ so that $F_i(x) > 0$ for all $x > 0$ and all i. Then either $\tau_{i=1}^{\infty} F_i = \varepsilon_\infty$ (so that $(\tau_{i=1}^{\infty} F_i)(x) = 0$ for all $x > 0$) or $(\tau_{i=1}^{\infty} F_i)(x) > 0$ for all $x > 0$.*

Thus if $\tau_T(F_1, \cdots, F_n)(y) \to 0$ for any $y > 0$, then $\tau_T(F_1, \cdots, F_n)(x) \to 0$ for all x.

PROOF. If $(\tau_{i=1}^{\infty} F_i) \neq \varepsilon_\infty$ then again by Theorem 1.1 there exists a sequence of positive numbers $\{a_i\}$ such that (2.7) holds. Then for any $x > 0$ there exists an integer $M > 0$ so that $\Sigma_{i=M+1}^{\infty} a_i < x/2$. Thus, using (2.6) and the fact that T is strict (so that $T(\varepsilon, \delta) > 0$ for any $\varepsilon, \delta > 0$), we have that

https://doi.org/10.1017/S1446788700011721 Published online by Cambridge University Press

$$\left(\overset{\infty}{\underset{i=1}{\tau}} F_i\right)(x) \geqq \left(\overset{\infty}{\underset{i=1}{\tau}} F_i\right)\left(\frac{x}{2} + \sum_{i=M+1}^{\infty} a_i\right)$$

$$\geqq T\left(\overset{M}{\underset{i=1}{T}} F_i\left(\frac{x}{2M}\right), \overset{\infty}{\underset{i=M+1}{T}} F_i(a_i)\right) > 0,$$

completing the proof.

We close with a somewhat surprising result about infinite τ_T products. As mentioned previously, a crucial question is whether the weak limit of the pointwise non-increasing sequence $\{\tau_T(F_1, \cdots, F_n)\}$ is not identically zero, that is, not equal to ε_∞. For a given sequence $\{F_i\}$, it would appear that the answer to this question should depend strongly on the particular t-norm T being used. But, at least for Product and $T_m(a, b) = \max\{a + b - 1, 0\}$, this is not so, for we have:

THEOREM 3.5. *Let* $\{F_i\}$ *be a sequence in* $\mathscr{D}^+$. *Then*

$$\tau_{\text{Prod}}(F_1, \cdots, F_n) \xrightarrow{w} \varepsilon_\infty \tag{3.4}$$

if and only if

$$\tau_{T_m}(F_1, \cdots, F_n) \xrightarrow{w} \varepsilon_\infty. \tag{3.5}$$

PROOF. Suppose (3.4) does not hold. Then there exists a sequence of positive numbers $\{a_i\}$ such that $\Sigma_{i=1}^\infty a_i < \infty$ and $\Pi_{i=1}^\infty F_i(a_i) > 0$. But then, by a well-known result on infinite products, we have that $\Sigma_{i=1}^\infty (1 - F_i(a_i)) < \infty$. In particular, for some $N > 0$, we have $\Sigma_{i=N+1}^\infty (1 - F_i(a_i)) < 1/2$. Now, since we are in $\mathscr{D}^+$, for each integer i with $1 \leqq i \leqq N$, we can find a number $b_i > 0$ so that $F_i(b_i) > 1 - (2N)^{-1}$. Letting $b_i = a_i$ for $i > N$, we then have that $\Sigma_{i=1}^\infty b_i < \infty$ and

$$\overset{\infty}{\underset{i=1}{T_m}} F_i(b_i) = \max\left\{\lim_{n\to\infty}\left[\left(\sum_{i=1}^{n} F_i(b_i)\right) - (n-1)\right], 0\right\}$$

$$= \max\left\{1 - \sum_{i=1}^{\infty}(1 - F_i(b_i)), 0\right\} > 0,$$

since $\Sigma_{i=1}^\infty (1 - F_i(b_i)) < 1$. But then, by Theorem 1.1, (3.5) does not hold.

The converse is easily established by the fact that Product is stronger than T_m, that is, $a \cdot b \geqq T_m(a, b)$ for all $a, b \in [0, 1]$. Thus

$$\tau_{\text{Prod}}(F_1, \cdots, F_n) \geqq \tau_{T_m}(F_1, \cdots, F_n)$$

for all n, whence (3.4) implies (3.5), completing the proof.

Theorem 3.5 does not hold in Δ^+, but is easily shown to generalize as follows:

Corollary 3.1. *Let* $\{F_i\}$ *be a sequence in* Δ^+. *If* $\Sigma_{i=1}^{\infty}(1-\sup_x F_i(x)) \geqq 1$ *then* (3.5) *holds. If* $\Sigma_{i=1}^{\infty}(1-\sup_x F_i(x)) < 1$, *then* (3.5) *holds if and only if* (3.4) *holds.*

Remark. Product and T_m are the two standard non-isomorphic examples of Archimedean t-norms. Thus one might conjecture whether convergence to ε_∞ of an infinite τ_T product of a given sequence $\{F_i\}$ in $\mathcal{D}^+$ is a class property of Archimedean t-norms. But this conjecture is false, as is seen by the following:

Example. For each integer $i > 0$, define $F_i \in \mathcal{D}^+$ by

$$F_i(x) = \begin{cases} 0, & x \leqq 0, \\ 1 - \dfrac{1}{i^2}, & 0 < x \leqq i, \\ 1, & i < x. \end{cases}$$

then it is easily checked using Theorem 1.1 that (3.4) does not hold. However, if we let T be the Archimedean t-norm which is multiplicatively generated using (1.4) by $h(x) = 1 - \sqrt{1-x}$, then

$$hF_i(x) = \begin{cases} 1 - \dfrac{1}{i}, & 0 < x \leqq i, \\ 1, & 1 < x. \end{cases}$$

Hence, for any sequence of positive numbers $\{a_i\}$ satisfying $\Sigma_{i=1}^{\infty} a_i < \infty$, it is clear that

$$\mathop{T}_{i=1}^{\infty} F_i(a_i) = h^{[-1]}\left(\prod_{i=1}^{\infty} hF_i(a_i)\right) = h^{[-1]}(0) = 0,$$

whence, by Theorem 1.1, $\tau_T(F_1, \cdots, F_n) \xrightarrow{w} \varepsilon_\infty$.

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