Appendix A Two-component spinors

The Dirac equation simplifies dramatically in the case where the fermion mass is zero. The equation

$$\not{D}\psi = 0 \tag{A1}$$

has the feature that if ψ is a solution then so is $\gamma_5\psi$:

$$\not{D}(\gamma_5\psi) = 0. \tag{A2}$$

The matrices

$$P_\pm = \frac{1}{2}(1 \pm \gamma_5) \tag{A3}$$

are projectors:

$$P_\pm^2 = P_\pm, \quad P_+P_- = P_-P_+ = 0. \tag{A4}$$

To understand the physical significance of these projectors it is convenient to use a particular basis for the Dirac matrices γ^μ, often called the chiral or Weyl basis:

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix}, \tag{A5}$$

where

$$\sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma}^\mu = (1, -\vec{\sigma}). \tag{A6}$$

In this basis,

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, \tag{A7}$$

so that

$$P_+ = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad P_- = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \tag{A8}$$

We will adopt certain notation that follows the text of Wess and Bagger:

$$\psi = \begin{pmatrix} \chi_\alpha \\ \phi^{*\dot{\alpha}} \end{pmatrix}. \tag{A9}$$

Correspondingly, we label the indices on the matrices σ^μ and $\bar{\sigma}^\mu$ as

$$\sigma^\mu = \sigma^\mu_{\alpha\dot{\alpha}}, \quad \bar{\sigma}^\mu = \bar{\sigma}^{\mu\dot{\beta}\beta}. \tag{A10}$$

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This allows us to match the "upstairs" and "downstairs" indices and will prove quite useful. The Dirac equation now becomes

$$i\sigma^{\mu}_{\alpha\dot{\alpha}}\partial_\mu \phi^{*\dot{\alpha}} = 0, \quad i\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu \chi_\alpha = 0. \tag{A11}$$

Note that χ and ϕ^* are equivalent representations of the Lorentz group; χ and ϕ obey identical equations. We may proceed by complex-conjugating the second of Eqs. (A11) and noting that $\sigma_2\sigma^{\mu *}\sigma_2 = \bar{\sigma}^\mu$.

Before discussing this identification in terms of representations of the Lorentz group, it is helpful to introduce some further notation. First, we define the action of complex conjugation as that of changing dotted to undotted indices. So, for example,

$$\phi^{*\dot{\alpha}} = (\phi^\alpha)^*. \tag{A12}$$

Then we define the antisymmetric matrices $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ by

$$\epsilon^{12} = 1 = -\epsilon^{21}, \quad \epsilon_{\alpha\beta} = -\epsilon^{\alpha\beta}. \tag{A13}$$

The matrices with dotted indices are defined identically. Note that, for the upstairs indices, $\epsilon = i\sigma_2$ and $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = \delta_\alpha^\gamma$. We can use these matrices to raise and lower indices on spinors. Define $\phi_\alpha = \epsilon_{\alpha\beta}\phi^\beta$, and similarly for the dotted indices. So

$$\phi_\alpha = \epsilon_{\alpha\beta}(\phi^{*\dot{\beta}})^*. \tag{A14}$$

Finally, we will define the complex conjugation of a product of spinors as inverting the order of factors; so, for example, $(\chi_\alpha\phi_\beta)^* = \phi^*_{\dot{\beta}}\chi^*_{\dot{\alpha}}$.

With this in hand, the reader should check that the action for our original four-component spinor is:

$$\begin{aligned} S = \int d^4x\,\mathcal{L} &= \int d^4x \left(i\chi^*_{\dot{\alpha}}\bar{\sigma}^{\mu\dot{\alpha}\alpha}\partial_\mu\chi_\alpha + i\phi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi^{*\dot{\alpha}}\right) \\ &= \int d^4x \left(i\chi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\chi^{*\dot{\alpha}} + i\phi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\phi^{*\dot{\alpha}}\right). \end{aligned} \tag{A15}$$

At the level of Lorentz-invariant Lagrangians or equations of motion, there is *only one* irreducible representation of the Lorentz algebra for massless fermions.

Two-component fermions have definite helicity. For a single-particle state with momentum $\vec{p} = p\hat{z}$, the Dirac equation reads

$$p(1 \pm \sigma_z)\phi = 0. \tag{A16}$$

Similarly, the reader should check that the antiparticle has the opposite helicity.

It is instructive to describe quantum electrodynamics with a massive electron in two-component language. Write

$$\psi = \begin{pmatrix} e \\ \bar{e}^* \end{pmatrix}. \tag{A17}$$

In the Lagrangian we need to replace ∂_μ with the covariant derivative, D_μ. Note that e contains annihilation operators for the left-handed electron and creation operators for the corresponding antiparticle. Note also that $\bar{e}$ contains annihilation operators for

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particles with the opposite helicity and charge to e and $\bar{e}^*$ and creation operators for the corresponding antiparticle.

The mass term $m\bar{\psi}\psi$ becomes:

$$m\bar{\psi}\psi = me^{\alpha}\bar{e}_{\alpha} + me^*_{\dot{\alpha}}\bar{e}^{*\dot{\alpha}}. \tag{A18}$$

Again, note that both terms preserve electric charge. Note also that the equations of motion now couple e and $\bar{e}$.

It is helpful to introduce one last piece of notation. Set

$$\psi\chi = \psi^{\alpha}\chi_{\alpha} = -\psi_{\alpha}\chi^{\alpha} = \chi^{\alpha}\psi_{\alpha} = \chi\psi. \tag{A19}$$

Similarly,

$$\psi^*\chi^* = \psi^*_{\dot{\alpha}}\chi^{*\dot{\alpha}} = -\psi^{*\dot{\alpha}}\chi^*_{\dot{\alpha}} = \chi^*_{\dot{\alpha}}\psi^{*\dot{\alpha}} = \chi^*\psi^*. \tag{A20}$$

Finally, note that, with these definitions,

$$(\chi\psi)^* = \chi^*\psi^*. \tag{A21}$$

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