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COMPLETE DECOMPOSABILITY IN THE EXTERIOR ALGEBRA OF A FREE MODULE

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Recall the classical criterion for the complete decomposability of exterior vectors: the completely decomposable vectors in $\Lambda^p R^n$, R a field, are precisely the "Plücker vectors," i.e. those whose coordinates (relative to the standard bases for $\Lambda^p R^n$) satisfy the Plücker equations. For R an arbitrary commutative ring, completely decomposable exterior vectors are still Plücker vectors, but the converse is not generally true. Rings for which the converse is true (for all $1 \leqq p \leqq n$) are called *Towber rings*. Noetherian Towber rings are regular and, in fact, are characterized by the property that every Plücker vector in $\Lambda^2 R^4$ is completely decomposable. (See [**10**] for these two results as well as for the above mentioned facts.) The present note develops a new characterization of Towber rings, combining it with results of Kleiner [**9**] and Estes-Matijevic [**5**] in (1) below.

Notation. In the sequel R is always a noetherian ring, all modules are finitely generated and all projective modules are of constant rank. Recall that

$$SK_0(R) = \operatorname{Ker}(\tilde{K}_0(R) \rightarrow \operatorname{Pic} R) \quad \text{induced by } [P] \rightarrow [\Lambda^{\operatorname{rk} P} P].$$

A projective module P is *oriented* provided that $\Lambda^{\operatorname{rk} P} P \simeq R$. The condition "every Plücker vector in $\Lambda^p R^n$ is completely decomposable" is abbreviated T_p^n. Given a module M, $v(M)$ is defined by: M is generated by $v(M)$ elements, but not by fewer.

1. Implications of T_2^3.

(1) THEOREM. *For regular R, the following are equivalent:*

(a) *R is a Towber ring.*

(b) *For all n, every vector in $\Lambda^n R^{n+1}$ is completely decomposable.*

(c) *Every vector in $\Lambda^2 R^3$ is completely decomposable.*

(d) *Every maximal ideal of R is generated by two elements, and stably-free projective R-modules are free.*

(e) $\dim(R) \leqq 2$, *every projective R-module is stably isomorphic to the direct sum of a free module and an invertible ideal* (*i.e.* $SK_0(R) = 0$) *and stably-free projective R-modules are free.*

(f) $\dim(R) \leqq 2$ *and all oriented projective R-modules are free.*

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Remarks. (a) $\Leftrightarrow$ (b) was conjectured in [**11**] and proved in [**9**]; thus (a) $\Leftrightarrow$ (c) affirms an even stronger result, obtained independently in [**5**]. We'll see how the Estes-Matijevic result gives a proof of (c) $\Rightarrow$ (a) independent of the Kleiner result; but, we'll also see—(8) below—how to prove (c) $\Rightarrow$ (a) by using the Kleiner result and [**10**].

The proof of (1) is contained in (2)–(7) as follows: Since every exterior vector in $\Lambda^n R^{n+1}$ is a Plücker vector (see e.g. [**10**]) we have (a) $\Rightarrow$ (b) $\Rightarrow$ (c) immediately. (c) $\Rightarrow$ (d) is contained in (3) and (4); (d) $\Rightarrow$ (e) is contained in (5), an unpublished result of Murthy. (e) $\Rightarrow$ (a) is proved in [**5**] and (e) $\Leftrightarrow$ (f) is contained in (6) and (7).

(2) LEMMA. *$T_2^3 \Rightarrow v(mR_m) \leqq 2$ for all maximal ideals m of R. (Hence $T_2^3 \Rightarrow$ dim $R \leqq 2$ and R_m is regular if* ht$(m) = 2$.)

Proof. See proof of [**17**, 2.4].

Remark. From (2) it follows that if R has no maximal ideals of ht < 2—e.g. if R is a 2-dimensional affine domain over a field—then R is regular; hence, given (a) $\Leftrightarrow$ (c) of (1), it follows that (a) $\Leftrightarrow$ (c) holds for such an R. This was observed in [**7**] for the special case of R the polynomial ring in two variables over a field.

(3) LEMMA. *$T_2^3 \Rightarrow$ every stably-free projective R-module is free.*

Proof. Let P be a stably-free projective R-module. Then $P \oplus R^s \simeq R^{\text{rk } P+s}$ for some s. If rk $P = 1$ P must be free (take the $(s + 1)$st exterior power). By (2), dim $R \leqq 2$ and so by Bass' Cancellation Theorem if rk $P > 2$ then P is free. Hence to prove the lemma we need only consider P projective, rk $P = 2$ and $P \oplus R \simeq R^3$. Thus P is defined by the unimodular row $[\alpha\,\beta\,\gamma]$ which we must show can be completed to a 3×3 matrix, with entries in R, having determinant 1. Now $\alpha a + \beta b + \gamma c = 1$ for some $a, b, c \in R$, so consider the exterior vector

$$v = ae_2 \wedge e_3 - be_1 \wedge e_3 + ce_1 \wedge e_2 \in \Lambda^2 R^3$$

where e_1, e_2, e_3 are a basis for R^3. By property T_2^3, $v = v_1 \wedge v_2$ and the coordinates of v_1 and v_2 provide us with the two rows needed to complete $[\alpha\beta\gamma]$.

(4) PROPOSITION. *Assume either T_2^3 or that R is regular satisfying hypothesis (f) of* (1). *Then $v(m) \leqq 2$ for every maximal ideal m of R.*

Proof. In general, $v(m) \leqq v(mR_m) + 1$ and $v(m) = v(mR_m)$ if R_m is not regular ([**2**], Theorem 1). Thus in view of (2), it suffices to consider the case where R_m is 2-dimensional and regular.

In this case mR_m is generated by a regular sequence of length 2, and $mR_p = R_p$ if $p \neq m$. Thus, pd$(m) = 1$. Furthermore, $\text{Ext}_R{}^1(m, R)$ is locally cyclic with 0-dimensional support (see [**15**]) and hence cyclic. By Serre's

Lemma [**15**, Proposition 1] then, there is a projective resolution:

(*) $0 \to R \to P \to m \to 0$

Now, since m is locally generated by a regular sequence, the Koszul complex

$$0 \to \Lambda^2 P \to P \to m \to 0$$

over the map $P \to m$ is locally exact and hence exact. (See [**13**], proof of Lemma 4.4 for more details.) Thus $\Lambda^2 P \simeq R$. Hence under hypothesis (f), $P \simeq R^2$ and so $\upsilon(m) = 2$.

We finish the proof by showing that also under the hypothesis $T_2{}^3$ we have $P \simeq R^2$.

Since $\upsilon(mR_m) = 2$ we have $\upsilon(m) \leqq 3$. Let x_1, x_2, x_3 be three generators for m and let e_1, e_2, e_3 be the standard bases for R^3. Let

$$v \in \Lambda^2 R^3, \quad v = x_1 e_2 \wedge e_3 + x_2 e_1 \wedge e_3 + x_3 e_1 \wedge e_2.$$

By $T_2{}^3$ we have $v = v_1 \wedge v_2$ where

$$v_i = a_i e_1 + b_i e_2 + c_i e_3, \quad i = 1, 2.$$

Consider the sequence

$$(**) \quad 0 \to R^2 \xrightarrow{\begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \\ c_1 & c_2 \end{pmatrix}} R^3 \xrightarrow{(x_1, \, -x_2, \, x_3)} m \to 0$$

By the usual facts about cofactors we see that this is a complex. The last map is onto by the choice of generators for m. The Buchsbaum-Eisenbud criterion [**1**] then applies, as depth $(m, R) = 2$, to assert that this complex is exact.

By Schanuel's Lemma applied to (*) and (**) we see that P is a stably free projective R-module. By (3) then, P is free.

Remark. Independently and by entirely different methods, this proposition was also proved by Estes and Matijevic in [**5**].

(5) Proposition (Murthy). *Let* dim $R \leqq 2$ *and suppose* $\upsilon(m) = 2$ *for all maximal ideals of height* 2. *Then* $SK_0(R) = 0$.

Proof (Murthy). Since dim $R \leqq 2$ it will be enough to show that if P is projective, rk $P = 2$, then $P \oplus R \simeq R^2 \oplus K$.

By a theorem of Bass [**16**, 3.3], since rk $P = 2$, there is a surjective homomorphism $f : P^* \to q$ where q is an ideal of R and ht $q \geqq 2$. If $q = R$ then P^* (and hence P) has a free summand, so we may assume ht $q = 2$. Since $\upsilon(m) = 2$ for any maximal ideal of height 2 we have q is contained only in maximal ideals m for which R_m is 2-dimensional and regular. Thus, pd $q = 1$.

Let $m \supset q$ be maximal. Since $v(m) = 2$, m is generated by a regular sequence and so the Koszul complex based on these generators gives a free resolution of m. Hence $[m] = [R]$, in $G_0(R)$.

Since R/q has finite length as an R-module we have that R/q has a finite filtration by modules of the form R/m where m is maximal and $m \supseteq q$. Thus, we also obtain $[q] = [R]$ in $G_0(R)$.

Now, we have the exact sequence

$$0 \to L \to P^* \xrightarrow{f} q \to 0,$$

where $L = \operatorname{Ker} f$. Since pd $q = 1$, L is projective. Hence

$$[P^*] = [L] + [q] = [L] + [R] \quad \text{in } K_0(R),$$

i.e. P^* and $L \oplus R$ are stably isomorphic. Since $\dim R = 2$, $P^* \oplus R \simeq (L \oplus R) \oplus R$ and so $P \oplus R \simeq R^2 \oplus K$, where $K = (L)^*$, as was to be shown.

(6) Lemma. *If $SK_0(R) = 0$ then the hypotheses imposed on the projective modules of R by* (1d) *and* (1f) *are equivalent.*

Proof. Observe that if $SK_0(R) = 0$ then any projective module P is stably isomorphic to $\Lambda^{\mathrm{rk}\,P}P \oplus R^{\mathrm{rk}\,P-1}$.

If $\Lambda^{\mathrm{rk}\,P}P \simeq R$ then P is stably-free and hence free if the (1d) hypothesis holds. Conversely, if P is stably free then so is $\Lambda^{\mathrm{rk}\,P}P$ and hence $\Lambda^{\mathrm{rk}\,P}P \simeq R$. Since the (1f) hypothesis holds, P is then free.

(7) Corollary. *Assume either T_2^3 or that R is regular satisfying hypothesis* (f) *of* (1). *Then $SK_0(R) = 0$, all oriented projective R-modules are free, and stably free projective R-modules are free.*

Proof. $SK_0(R) = 0$ by (4) and (5). The rest follows from (3) and (6).

Remarks. 1) (7) should be compared with ([**10**], 6.4) where the conclusion is deduced from the stronger hypothesis T_2^5.

2) Note that Theorem (1) is now proved. See Remarks after Theorem (1).

(8) Corollary. *If R is normal then $T_2^3 \Rightarrow R$ is a Towber ring.*

Proof. Given a maximal ideal m of R, R_m is regular; by (2), if $\operatorname{ht}(m) = 2$, or by the hypothesis "normal" if ht $m < 2$. (The corollary now follows from (c) $\Rightarrow$ (a) of (1). We continue with an alternate argument for the reason explained in the Remarks following the statement of Theorem (1).) Thus R is a direct sum of regular domains of dimension at most 2. Since the properties T_p^n get along with direct sums, we may assume R is a domain. By (7) and Bass cancellation, every projective R-module of rank > 2 is of the form "free $\oplus$ ideal." Hence by ([**10**], 3.4) we have T_p^n for all $2 < p \leqq n$. In particular then, we have T_n^{n+1} for all n. Then R is a Towber ring by ([**9**], Theorem 1).

2. Towber rings and certain theorems and conjectures of Eisenbud-Evans. Consider the following statements concerning a d-dimensional noetherian ring R.

i) Every ideal of R is generated, up to radical, by d elements.

ii) For a finitely generated R-module M,

$$v(M) \leqq \delta(M) = \max\{v(M_p) + \dim(R/p) | p \in \operatorname{Spec} R, \dim R/p < d\}.$$

iii) If P is a projective R-module of rank d, then P has a free summand.

All these statements are false in general. However, if $R = S[X]$, then

i) is true [**4**].

ii) is conjectured in ([**3**], Conjecture 3) and proved if S is a domain in [**14**] and [**12**].

iii) is conjectured in ([**3**], Conjecture 1) and is proved there if S is local. One can then use Qullen's Localization Theorem to show that iii) is true if S is regular.

In this section we prove i) and ii) for Towber rings. These results lend further support to the notion expressed in [**10**] that Towber rings of dimension two behave, in certain respects, as if they were one dimensional. (Perhaps one should say: as if they were of the form $S[X]$, where S is one dimensional.)

The validity of iii) for Towber rings would prove that every projective module over a Towber ring is of the form "free $\oplus$ rank 1" and that Towber rings enjoy the cancellation property for all projective modules (not just the stably-free ones). If one really expects a Towber domain of dimension two to behave as if it were $S[X]$, S a Dedekind domain, then one should conjecture the "free $\oplus$ rank 1" result for arbitrary Towber domains. By [**13**], iii) is valid for a Towber ring which is a finitely generated affine algebra over an algebraically closed field. We do not know if iii) is valid in general, however, for a two dimensional Towber domain.

We now proceed to a proof of i) and ii) for Towber domains.

(9) Proposition. *Let R be a Towber ring. Then*

a) $v(M) \leqq \delta(M)$ *for any finitely generated R-module M.*

b) *Every radical ideal of R is generated by 2 elements.*

Remark. For the case $R = S[X]$, S a Dedekind domain, (9) a) and (9) b) are unpublished results of M. P. Murthy and R. Gilmer respectively. (9) b) is observed, independently, in [**5**].

Proof. With no loss of generality, we assume R is a domain, since a Towber ring, being regular, is a direct sum of Towber domains.

We first show that b) follows from a).

Let $0 \neq I = \operatorname{rad}(I)$; we will show that $\delta(I) \leqq 2$. Note that $\nu(IR_p) + \dim(R/p) \leqq 2$ for any prime p of height one, since R_p is a discrete valuation ring. Now let m be a height two maximal ideal such that $m \supset I$. If m is minimal over I then $IR_m = mR_m$ and so $v(IR_m) = 2$. If m is not minimal over I then IR_m is pure height one and so $v(IR_m) = 1$ since R_m is factorial. Thus $\delta(I) \leqq 2$ and b) follows from a).

We now turn to the proof of a). Since R is a domain it suffices, by an observation of Sathaye [**14**], to prove a) only for ideals of R. Furthermore, if $\delta(I) > 2$ then $\delta(I)$ is the "Forster bound" and $v(I) \leqq \delta(I)$ by ([**6**], Satz 1). Since $\delta(I) < 2$ only in the trivial cases of $I = (0)$ or $\dim R < 2$, it remains to prove that $v(I) \leqq 2$ in case R is a 2-dimensional Towber domain and $v(IR_M) \leqq 2$ for all maximal ideals m of R. Hence a) follows from the following proposition, which proves that a) holds for a (possibly) larger class than Towber rings.

(10) Proposition. *Let R be a 2-dimensional locally factorial domain such that all oriented projective R-modules are free. Let I be an ideal of R such that $v(IR_m) \leqq 2$ for all maximal ideals m of R. Then $v(I) \leqq 2$.*

Proof. We may assume $I \neq 0$. By "locally factorial" we have for any prime p, if $\operatorname{ht} p < 2$, $I_p \simeq R_p$ and if $\operatorname{ht} p = 2$, either $I_p \simeq R_p$ or I_p is isomorphic to an R_p-ideal generated by a regular sequence of length 2. Thus, $\operatorname{Ext}_R{}^1(I, R)$ is locally cyclic with 0-dimensional support, and so cyclic. Moreover $\operatorname{pd}(I) \leqq 1$. By Serre's lemma again, there is a projective resolution

(*) $\quad 0 \rightarrow R \rightarrow P \rightarrow I \rightarrow 0.$

Claim. $\Lambda^2 P \simeq J = (I^{-1})^{-1}$.

Conclusion of proof assuming the claim. Since J is an invertible ideal, IJ is locally isomorphic to I and so Serre's lemma, applied to IJ, gives a projective resolution

$$0 \rightarrow R \rightarrow Q \rightarrow IJ \rightarrow 0.$$

By the claim we have $\Lambda^2 Q \simeq J^2$. Identify Q with a sub-module of K^2 (K = quotient field of R). Tensoring this exact sequence with J^{-1} (observe that given our identification, this amounts to multiplication by J^{-1}) we obtain $QJ^{-1} \rightarrow I \rightarrow 0$ exact. We see that $\Lambda^2(QJ^{-1}) = (\Lambda^2 Q)J^{-2}$. Since $(\Lambda^2 Q)J^{-2} \simeq R$ and, by hypothesis, $QJ^{-1} \simeq R^2$, we obtain $v(I) \leqq 2$.

Proof of claim. We consider three cases:

(i) I is unmixed of ht 1. In this case the hypothesis "locally factorial" implies I is invertible and hence $P \simeq R \oplus I$. Then $\Lambda^2 P \simeq I = (I^{-1})^{-1}$.

(ii) I is unmixed of ht 2. In this case I is not contained in an associated prime of a principal ideal, since the hypothesis implies that R is Cohen-Macaulay. Thus $I^{-1} = R$, and so $J = R$. Now, since I is locally generated by a regular sequence, (*) is a Koszul complex (see proof of (4)) and hence $\Lambda^2 P \simeq R$. Notice also, then, that in this case $P \simeq R^2$.

(iii) I is of mixed height. In this case $I = JH$ where $H = II^{-1}$. (*Proof.* By "locally factorial," $I_m = J_m H_m$ for every maximal ideal m of R.)

Observe that for any prime p, $I_p \simeq H_p$ since J is invertible whence J_p is principle. Moreover, $H_p = R_p$ if ht $p = 1$. So, H is a locally 2-generated ideal to which case (ii) applies. Thus, by the remark at the end of (ii) we have an exact sequence: $0 \to R \to R^2 \to H \to 0$. Tensoring with J gives an exact sequence:

(**) $\quad 0 \to J \to J \oplus J \to I \to 0.$

Schanuel's lemma, applied to (*) and (**) gives:

$$J \oplus J \oplus R \simeq P \oplus J.$$

Applying Λ^3 to this "equation" gives $J^2 \simeq (\Lambda^2 P)J$. So $\Lambda^2 P \simeq J$ as was to be shown.

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