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A result on tournaments with an application to counting

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A generating function is derived for the number of transitions to the first passage through a particular vertex of a random walk on a doubly regular tournament. As an application of this result, we obtain a generating function for the number of solutions $(q_1, \ldots, q_k)$ of $\sum_{i=1}^{k} q_i \equiv r \pmod{p}$, where p is a prime of the form $4l+3$ and the q_i are quadratic residues modulo p.

1. Introduction

By a tournament T_n on n vertices we mean a directed graph on n vertices (which we assume are labelled with the numbers $0, \ldots, n-1$) for which every pair of distinct vertices form the endpoints of exactly one (directed) edge. In this note we shall employ the usual terminology for tournaments, so that any undefined terms are to be found in Moon [2].

If each vertex has the same score, the tournament is called regular. If, in addition, for any pair of distinct vertices there are exactly m other vertices dominated by both members of the pair, then the tournament is called doubly regular (of order m). If T_n is doubly regular, it follows easily that each vertex has score $2m+1$ and $n = 4m+3$.

Our first step will be to obtain a generating function for $q_k(i)$, the number of paths starting at vertex i of the doubly regular tournament T_n and passing through vertex 0 for the first time at the kth

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transition. For convenience, we define $q_0(i) = 0$ for $i \neq 0$ and $q_0(0) = 1$. The generating function is defined by $Q(i, t) = \sum_{k=0}^{\infty} q_k(i)t^k$ and (as with all subsequent generating functions) may be regarded as a formal power series. When the result for $Q(i, t)$ is used for a particular class of tournaments defined for prime numbers of the form $4l + 3$, we obtain an expression for $W(p, r, t) = \sum_{k=0}^{\infty} w(p, r, k)t^k$, the generating function for $w(p, r, k)$, the number of solutions $(q_1, \ldots, q_k)$ of $\sum_{i=1}^{k} q_i \equiv r \pmod p$, where $q_i \in Q$, the set of quadratic residues modulo p.

2. Counting results

We proceed straight to the result about paths in T_n (doubly regular).

THEOREM. *If* P *denotes the vertices in* T_n *which are dominated by* 0*, and* P' *the vertices which dominate* 0*, then*

$$Q(i, t) = t[2m+(m-1)t]\left[mt^2+2mt-1\right]^{-1} \quad (i \in P),$$

$$= t[(2m-1)+mt]\left[mt^2+2mt-1\right]^{-1} \quad (i \in P').$$

Proof. Let us denote by $p_k(i)$ the number of paths of length k which start at vertex i and do not pass through vertex 0, so that $q_k(i) = p_{k-1}(i) - p_k(i)$, $k \geq 1$, where $p_0(i) = 1$. Therefore, if $P(i, t) = \sum_{k=0}^{\infty} p_k(i)t^k$, then $Q(i, t) = 1 + (t-1)P(i, t)$. We shall thus find $Q(i, t)$ by establishing the recurrence relation

(1) $$p_{k+1}(i) = 2mp_k(i) + mp_{k-1}(i),$$

independently of i, from which $P(i, t)$ may then be calculated.

Clearly $p_k(i)$ is the number of paths of length k starting at vertex i in the 'deleted tournament' obtained by removing all edges

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incident with 0 . Then in the deleted tournament, vertices in P' have their scores reduced to $2m$, while vertices in P have their scores unchanged at $2m + 1$. Thus every path of length k produces $2m$ or $2m + 1$ paths of length $k + 1$ according as it ends in P' or P respectively; whence

$$(2) \qquad p_{k+1}(i) = 2m \sum_{j \in P'} p_k(i, j) + (2m+1) \sum_{j \in P} p_k(i, j)$$
$$= 2mp_k(i) + \sum_{j \in P} p_k(i, j) ,$$

where $p_k(i, j)$ denotes the number of paths of length k from i to j in the deleted tournament. Now each path of length k from i to j arises from one of length $k - 1$ to some vertex v which dominates j , so that writing $I(j)$ for the set of vertices, excluding 0 , which dominate j , we conclude that

$$\sum_{j \in P} p_k(i, j) = \sum_{j \in P} \sum_{v \in I(j)} p_{k-1}(i, v) .$$

But since T_n is doubly regular, this implies that

$$\sum_{j \in P} p_k(i, j) = m \sum_{v=1}^{p-1} p_{k-1}(i, v)$$
$$= mp_k(i) ,$$

and we have established (1).

In order to determine the necessary boundary conditions for $p_k(i)$, we first notice that $p_1(i)$ is simply the score of i in the deleted tournament, namely $2m + 1$ if $i \in P$ and $2m$ if $i \in P'$. It is also true that $\sum_{j \in P} p_1(i, j) = m$ (independently of i) since the left hand side is the number of vertices in T_n dominated by both 0 and i . From (2) it now follows that $p_2(i) = m(4m+3)$ if $i \in P$ and $p_2(i) = m(4m+1)$ if $i \in P'$.

Using (1) and the boundary conditions we conclude that

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$$P(i,\ t) = (1+t)\left(1-2mt-mt^2\right)^{-1} \quad (i \in P)\ ,$$
$$= \left(1-2mt-mt^2\right)^{-1} \quad (i \in P')\ .$$

Substituting for $P(i,\ t)$ in the formula for $Q(i,\ t)$ then proves the theorem.

REMARK. Notice that proving the result for vertex 0 is simply for convenience, as the theorem holds for the first passage through any vertex of the tournament.

We now apply the theorem to a particular class of tournaments called p-tournaments $(p = 4l + 3$, a prime) which are defined by the requirement that the edge directed from i to j is part of the tournament if and only if $j - i$ is a quadratic residue modulo p. These tournaments are indeed doubly regular, since the $2m + 1$ elements of Q form a difference set ([3, pp. 131-134]). The number of paths of length k from i to j in a p-tournament equals $w(p,\ j-i,\ k)$.

To establish a result about $W(p,\ r,\ t)$ we require the straightforward relationship

$$W(p,\ r,\ t) = W(p,\ 0,\ t)Q(p-r,\ t)\ . \tag{3}$$

We can show that

$$W(p,\ 0,\ t) = \left[1-2mt-mt^2\right]\left[\left(1-(2m+1)t\right)\left(1+\tfrac{1}{2}t\left(ip^{\frac{1}{2}}+1\right)\right)\left(1-\tfrac{1}{2}t\left(ip^{\frac{1}{2}}-1\right)\right)\right]^{-1}\ . \tag{4}$$

Combining the theorem with (3) and (4) we then find that

$$W(p,\ r,\ t) = \left[-t\left(2m+(m-1)t\right)\right]\left[\left(1-(2m+1)t\right)\left(1+\tfrac{1}{2}t\left(ip^{\frac{1}{2}}+1\right)\right)\left(1-\tfrac{1}{2}t\left(ip^{\frac{1}{2}}-1\right)\right)\right]^{-1} \quad (r \in -Q)\ ,$$
$$= \left[-t\left((2m-1)+mt\right)\right]\left[\left(1-(2m+1)t\right)\left(1+\tfrac{1}{2}t\left(ip^{\frac{1}{2}}+1\right)\right)\left(1-\tfrac{1}{2}t\left(ip^{\frac{1}{2}}-1\right)\right)\right]^{-1} \quad (r \in Q)\ .$$

The problem of obtaining an expression for $w(p,\ 0,\ k)$ is well-known and indeed arises as a particular case of a result attributed to Libri and Lebesgue (see, for instance, Williams [4]). It seems worth outlining a different approach which is short and leads us directly to the result we require for $W(p,\ 0,\ t)$. This approach is also in keeping with the random walk ideas we have used so far and the argument may be extended to more general situations.

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If A_p is the adjacency matrix of the p-tournament, then $pw(p, 0, k) = \operatorname{trace}\left[(A_p)^k\right]$, so that we can obtain $w(p, 0, k)$ from the eigenvalues λ_v of A_p. But A_p is a circulant and so its eigenvalues are just

$$\lambda_{v+1} = \sum_{j=1}^{p-1} \psi(j)\exp[(2\pi ijv)/p] \ , \quad v = 0, \ldots, p-1 \ ,$$

where

$$\begin{aligned} \psi(j) &= 1 \quad \text{if} \quad j \in Q \ , \\ &= 0 \quad \text{if} \quad j \in -Q \ . \end{aligned}$$

Then $\lambda_{v+1} = \frac{1}{2}\left(-1+S(v, p)\right)$, where $S(v, p)$ is the gaussian sum whose value may be shown to be $(v/p)ip^{\frac{1}{2}}$ $\left((v/p)\right.$ is the Legendre symbol – for a proof of this result see Chowla [1]$\left.\right)$. Notice that of the p eigenvalues, $2m + 1$ have the value $\frac{1}{2}\left(ip^{\frac{1}{2}}-1\right)$, $2m + 1$ have the value $-\frac{1}{2}\left(ip^{\frac{1}{2}}+1\right)$, and 1 has the value $2m + 1$, so that

$$pw(p, 0, k) = (2m+1)^k + (2m+1)\left[\left(\tfrac{1}{2}\left(ip^{\frac{1}{2}}-1\right)\right)^k+\left(-\tfrac{1}{2}\left(ip^{\frac{1}{2}}+1\right)\right)^k\right] \ ,$$

from which (4) follows after some simplification.

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