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Shrinking parallelepiped targets for $\boldsymbol{\beta}$-dynamical systems

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Abstract. For $\beta > 1$, let T_β be the β-transformation on $[0, 1)$. Let $\beta_1, \ldots, \beta_d > 1$ and let $\mathcal{P} = \{P_n\}_{n\geq 1}$ be a sequence of parallelepipeds in $[0, 1)^d$. Define

$$W(\mathcal{P}) = \{\mathbf{x} \in [0, 1)^d : (T_{\beta_1} \times \cdots \times T_{\beta_d})^n(\mathbf{x}) \in P_n \text{ infinitely often}\}.$$

When each P_n is a hyperrectangle with sides parallel to the axes, the 'rectangle to rectangle' mass transference principle by Wang and Wu [Mass transference principle from rectangles to rectangles in Diophantine approximation. *Math. Ann.* **381** (2021) 243–317] is usually employed to derive the lower bound for $\dim_H W(\mathcal{P})$, where $\dim_H$ denotes the Hausdorff dimension. However, in the case where P_n is still a hyperrectangle but with rotation, this principle, while still applicable, often fails to yield the desired lower bound. In this paper, we determine the optimal cover of parallelepipeds, thereby obtaining $\dim_H W(\mathcal{P})$. We also provide several examples to illustrate how the rotations of hyperrectangles affect $\dim_H W(\mathcal{P})$.

Key words: β-expansion, Hausdorff dimension, parallelepiped, shrinking target problem
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1. *Introduction*

The classical theory of Diophantine approximation is concerned with finding good approximations of irrationals. For any irrational $x \in [0, 1]$, if one can find infinitely many rationals p/q such that $|x - p/q| < q^{-\tau}$ with $\tau > 2$, then x is said to be τ-well approximable. In [10], Hill and Velani introduced a dynamical analogue of the classical theory of τ-well approximable numbers. The study of these sets is known as the so-called *shrinking target problem*. More precisely, consider a transformation T on a metric space (X, d). Let $\{B_n\}_{n\geq 1}$ be a sequence of balls with radius $r(B_n) \to 0$ as $n \to \infty$. The shrinking target problem concerns the size, especially the Hausdorff dimension, of the set

$$W(T, \{B_n\}_{n\geq 1}) := \{x \in X : T^n x \in B_n \text{ i.o.}\},$$

where 'i.o.' stands for *infinitely often*. Since its initial introduction, $W(T, \{B_n\}_{n\geq 1})$ has been studied intensively in many dynamical systems. See, for example, [1, 2, 4, 8, 10–16, 20, 22] and reference therein.

The set $W(T, \{B_n\}_{n\geq 1})$ can be thought of as trajectories which hit shrinking targets $\{B_n\}_{n\geq 1}$ infinitely often. Naturally, one would like to consider different targets, such as hyperrectangles, rather than just balls. To this end, motivated by the weighted theory of Diophantine approximation, the following set had also been introduced in β-dynamical system. For $d \geq 1$, let $\beta_1, \ldots, \beta_d > 1$ and let $\mathcal{P} = \{P_n\}_{n\geq 1}$ be a sequence of parallelepipeds in $[0, 1)^d$. Define

$$W(\mathcal{P}) = \{\mathbf{x} \in [0, 1)^d : (T_{\beta_1} \times \cdots \times T_{\beta_d})^n(\mathbf{x}) \in P_n \text{ i.o.}\},$$

where $T_{\beta_i} : [0, 1) \to [0, 1)$ is given by

$$T_{\beta_i} x = \beta_i x - \lfloor \beta_i x \rfloor.$$

Here, $\lfloor \cdot \rfloor$ denotes the integer part of a real number. Under the assumption that each P_n is a hyperrectangle with sides parallel to the axes, the Hausdorff dimension of $W(\mathcal{P})$, denoted by $\dim_{\mathrm{H}} W(\mathcal{P})$, was calculated by Li *et al* [15, Theorem 12]. It should be pointed out that their result crucially relies on this assumption. To see this, observe that $W(\mathcal{P})$ can be written as

$$\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} (T_{\beta_1} \times \cdots \times T_{\beta_d})^{-n} P_n.$$

In the presence of such an assumption, $(T_{\beta_1} \times \cdots \times T_{\beta_d})^{-n} P_n$ will be the union of hyperrectangles whose sides are also parallel to the axes. Thus, the 'rectangle to rectangle' mass transference principle by Wang and Wu [21] can be employed to obtain the desired lower bound of $\dim_{\mathrm{H}} W(\mathcal{P})$. However, if this assumption is removed, then $(T_{\beta_1} \times \cdots \times T_{\beta_d})^{-n} P_n$ is in general the union of parallelepipeds, and the mass transference principle, while still applicable, does not work well for this case. The main purpose of this paper is to determine $\dim_{\mathrm{H}} W(\mathcal{P})$ without assuming each P_n is a hyperrectangle. We further show that $W(\mathcal{P})$ has large intersection properties introduced by Falconer [6], which means that the set $W(\mathcal{P})$ belongs, for some $0 \leq s \leq d$, to the class $\mathscr{G}^s([0, 1]^d)$ of G_δ-sets, with the property that any countable intersection of bi-Lipschitz images of sets in $\mathscr{G}^s([0, 1]^d)$ has Hausdorff dimension at least s. In particular, the Hausdorff dimension of $W(\mathcal{P})$ is at least s.

Let

$$f = \mathrm{diag}(\beta_1^{-1}, \ldots, \beta_d^{-1}).$$

In slightly less rigorous words, the set $(T_{\beta_1} \times \cdots \times T_{\beta_d})^{-n} P_n$ consists of parallelepipeds with the same shape as $f^n P_n$. Note that up to a translation, each P_n can be uniquely determined by d column vectors $\alpha_j^{(n)}$. In Lemma 3.3, we establish the existence of a rearrangement $f^n\alpha_{i_1}^{(n)}, \ldots, f^n\alpha_{i_d}^{(n)}$ of $f^n\alpha_1^{(n)}, \ldots, f^n\alpha_d^{(n)}$, which ensures that upon the Gram–Schmidt process, the resulting pairwise orthogonal vectors, denoted by $\gamma_1^{(n)}, \ldots, \gamma_d^{(n)}$, satisfy the inequality

$$|\gamma_1^{(n)}| \geq \cdots \geq |\gamma_d^{(n)}| > 0. \tag{1.1}$$

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Most importantly, this yields that up to a multiplicative constant, the optimal cover of $f^n P_n$ is the same as that of the hyperrectangle with sidelengths $|\gamma_1^{(n)}| \geq \cdots \geq |\gamma_d^{(n)}| > 0$. To describe the optimal cover of $(T_{\beta_1} \times \cdots \times T_{\beta_d})^{-n} P_n$, let

$$\mathcal{A}_n = \{\beta_1^{-n}, \ldots, \beta_d^{-n}, |\gamma_1^{(n)}|, \ldots, |\gamma_d^{(n)}|\},$$

and define

$$s_n := \min_{\tau \in \mathcal{A}_n} \left\{ \sum_{i \in \mathcal{K}_{n,1}(\tau)} 1 + \sum_{i \notin \mathcal{K}_{n,1}(\tau)} -\frac{n \log \beta_i}{\log \tau} + \sum_{i \in \mathcal{K}_{n,2}(\tau)} \left(1 - \frac{\log |\gamma_i^{(n)}|}{\log \tau}\right) \right\},$$

where the sets $\mathcal{K}_{n,1}(\tau)$ and $\mathcal{K}_{n,2}(\tau)$ are defined as

$$\mathcal{K}_{n,1}(\tau) := \{1 \leq i \leq d : \beta_i^{-n} \leq \tau\} \quad \text{and} \quad \mathcal{K}_{n,2}(\tau) = \{1 \leq i \leq d : |\gamma_i^{(n)}| \geq \tau\}. \tag{1.2}$$

THEOREM 1.1. *Let $\mathcal{P} = \{P_n\}_{n \geq 1}$ be a sequence of parallelepipeds. For any $n \in \mathbb{N}$, let $\gamma_1^{(n)}, \ldots, \gamma_d^{(n)}$ be the vectors described in equation (1.1). Then,*

$$\dim_{\mathrm{H}} W(\mathcal{P}) = \limsup_{n \to \infty} s_n =: s^*.$$

Further, we have $W(\mathcal{P}) \in \mathscr{G}^{s^*}([0,1]^d)$.

Remark 1.2. In fact, orthogonalizing the vectors $f^n\alpha_1^{(n)}, \ldots, f^n\alpha_d^{(n)}$ in different orders will result in different pairwise orthogonal vectors. However, not all of them can be well used to illustrate the optimal cover of $f^n P_n$, only those satisfying equation (1.1) do. For example, let P be a parallelogram which is determined by two column vectors $\alpha_1 = (1, 0)^\top$ and $\alpha_2 = (m, m)^\top$, $m > 1$. Orthogonalizing in the order of α_1 and α_2 (respectively α_2 and α_1), we get the orthogonal vectors $\gamma_1 = \alpha_1 = (1, 0)^\top$ and $\gamma_2 = (0, m)^\top$ (respectively $\eta_1 = \alpha_2 = (m, m)^\top$ and $\eta_2 = (1/2, -1/2)^\top$). Denote the rectangles determined by γ_1 and γ_2 (respectively η_1 and η_2) as R (respectively $\tilde{R}$). As one can easily see from Figure 1. P is contained in the rectangle obtained by scaling $\tilde{R}$ by a factor of 2, whereas for R, a factor of m is required. Note that $|\gamma_1| < |\gamma_2|$, while $|\eta_1| > |\eta_2|$. This simple example partially inspires us to choose a suitable order to orthogonalize $f^n\alpha_1^{(n)}, \ldots, f^n\alpha_d^{(n)}$ so that the resulting vectors satisfy equation (1.1), which turns out to be crucial (see Lemma 3.3 and equations (3.2) and (3.3)).

Remark 1.3. Li *et al* [**15**, Theorem 12] studied an analogous problem, where P_n is restricted to be the following form:

$$P_n = \prod_{i=1}^{d} [-\psi_i(n), \psi_i(n)],$$

and where ψ_i is a positive function defined on natural numbers for $1 \leq i \leq d$. They further posed an additional condition that $\limsup_{n\to\infty} -\log \psi_i(n)/n < \infty$ $(1 \leq i \leq d)$, as their proof of lower bound for $\dim_{\mathrm{H}} W(\mathcal{P})$ relies on the 'rectangle to rectangle' mass transference principle [**21**, Theorem 3.3], which demands a similar condition. Their strategy is to investigate the accumulation points of the sequence $\{(-\log \psi_1(n)/n, \ldots, -\log \psi_d(n)/n)\}_{n \geq 1}$, and subsequently selecting a suitable

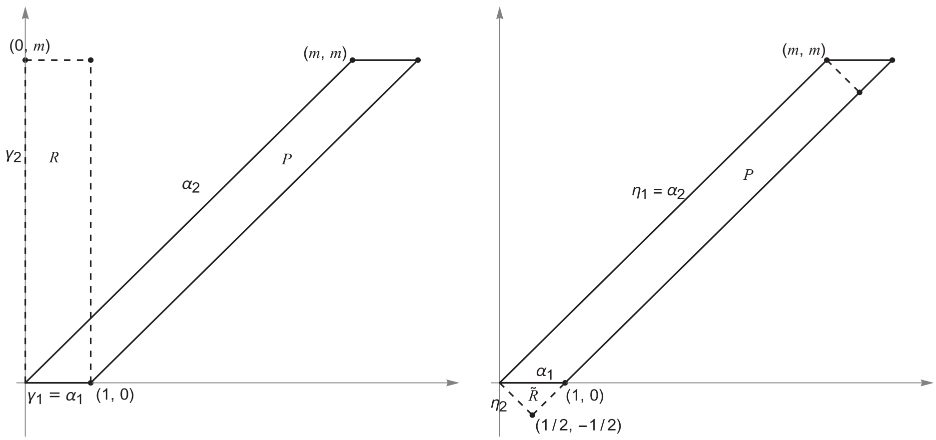

FIGURE 1. Orthogonal in the order of α_1 and α_2 (left), α_2 and α_1 (right).

accumulation point to construct a Cantor subset of $W(\mathcal{P})$, thereby obtaining the lower bound for $\dim_{\mathrm{H}} W(\mathcal{P})$. However, if $\limsup_{n\to\infty} -\log \psi_i(n)/n = \infty$ for some i, they illustrated by an example [15, §5.3] that this strategy may not achieve the desired lower bound. This problem has been addressed in their recent paper [14]. We stress that Theorem 1.1 does not pose any similar condition on P_n, and our approach differs from [14].

To gain insight into Theorem 1.1, we present two examples to illustrate how the rotations of rectangles affect the Hausdorff dimension of $W(\mathcal{P})$.

Example 1.4. Let $\beta_1 = 2$ and $\beta_2 = 4$. Let $\{H_n\}_{n\geq 1}$ be a sequence of rectangles with $H_n = [0, 2^{-n}] \times [0, 4^{-n}]$. For a sequence $\{\theta_n\}_{n\geq 1}$ with $\theta_n \in [0, \pi/2]$, let

$$P_n = R_{\theta_n} H_n + (1/2, 1/2), \tag{1.3}$$

where R_θ denotes the counterclockwise rotation by an angle θ. The translation $(1/2, 1/2)$ here is only used to ensure $P_n \subset [0, 1)^d$. Suppose that $\theta_n \equiv \theta$ for all $n \geq 1$. For any $n \geq 1$, we have

$$|\gamma_1^{(n)}| = \sqrt{2^{-4n}\cos^2\theta + 2^{-6n}\sin^2\theta} \quad \text{and} \quad |\gamma_2^{(n)}| = 2^{-6n}/|\gamma_1^{(n)}|.$$

By Theorem 1.1, we get

$$\dim_{\mathrm{H}} W(\mathcal{P}) = \begin{cases} 5/4 & \text{if } \theta \in [0, \pi/2), \\ 1 & \text{if } \theta = \pi/2. \end{cases}$$

Example 1.5. Let P_n be as in equation (1.3) but with $\theta_n = \arccos 2^{-an}$ for some $a > 0$. Then,

$$|\gamma_1^{(n)}| = \sqrt{2^{-n(4+2a)} + 2^{-6n}(1 - 2^{-na})^2} \quad \text{and} \quad |\gamma_2^{(n)}| = 2^{-6n}/|\gamma_1^{(n)}|.$$

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By Theorem 1.1, we get

$$\dim_{\mathrm{H}} W(\mathcal{P}) = \begin{cases} 1 + \dfrac{1-a}{4-a} & \text{if } a \le 1, \\ 1 & \text{if } a > 1. \end{cases}$$

The structure of the paper is as follows. In §2, we recall several notions and elementary properties of β-transformation. In §3, we estimate the optimal cover of parallelepipeds in terms of Falconer's singular value function. In §4, we prove Theorem 1.1.

2. *β-transformation*

We start with a brief discussion that sums up various fundamental properties of β-transformation.

For $\beta > 1$, let T_β be the β-transformation on $[0, 1)$. For any $n \ge 1$ and $x \in [0, 1)$, define $\epsilon_n(x, \beta) = \lfloor \beta T_\beta^{n-1} x \rfloor$. Then, we can write

$$x = \frac{\epsilon_1(x, \beta)}{\beta} + \frac{\epsilon_2(x, \beta)}{\beta^2} + \cdots + \frac{\epsilon_n(x, \beta)}{\beta^n} + \cdots,$$

and we call the sequence

$$\epsilon(x, \beta) := (\epsilon_1(x, \beta), \epsilon_2(x, \beta), \ldots)$$

the β-expansion of x. From the definition of T_β, it is clear that, for $n \ge 1$, $\epsilon_n(x, \beta)$ belongs to the alphabet $\{0, 1, \ldots, \lceil \beta - 1 \rceil\}$, where $\lceil x \rceil$ denotes the smallest integer greater than or equal to x. When β is not an integer, then not all sequences of $\{0, 1, \ldots, \lceil \beta - 1 \rceil\}^{\mathbb{N}}$ are the β-expansions of some $x \in [0, 1)$. This leads to the notion of β-admissible sequence.

Definition 2.1. A finite or an infinite sequence $(\epsilon_1, \epsilon_2, \ldots) \in \{0, 1, \ldots, \lceil \beta - 1 \rceil\}^{\mathbb{N}}$ is said to be β-admissible if there exists an $x \in [0, 1)$ such that the β-expansion of x begins with $(\epsilon_1, \epsilon_2, \ldots)$.

Denote by Σ_β^n the collection of all admissible sequences of length n. The following result of Rényi [**19**] implies that the cardinality of Σ_β^n is comparable to β^n.

LEMMA 2.2. [**19**, equation (4.9)] *Let $\beta > 1$. For any $n \ge 1$,*

$$\beta^n \le \#\Sigma_\beta^n \le \frac{\beta^{n+1}}{\beta - 1},$$

where # denotes the cardinality of a finite set.

Definition 2.3. For any $\boldsymbol{\epsilon}_n := (\epsilon_1, \ldots, \epsilon_n) \in \Sigma_\beta^n$, we call

$$I_{n,\beta}(\boldsymbol{\epsilon}_n) := \{x \in [0, 1) : \epsilon_j(x, \beta) = \epsilon_j, 1 \le j \le n\}$$

an nth level cylinder.

From the definition, it follows that $T_\beta^n|_{I_{n,\beta}(\boldsymbol{\epsilon}_n)}$ is linear with slope β^n, and it maps the cylinder $I_{n,\beta}(\boldsymbol{\epsilon}_n)$ into $[0, 1)$. If β is not an integer, then the dynamical system $(T_\beta, [0, 1))$ is not a full shift, and so $T_\beta^n|_{I_{n,\beta}(\boldsymbol{\epsilon}_n)}$ is not necessary onto. In other words, the length of

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$I_{n,\beta}(\epsilon_n)$ may be strictly less than β^{-n}, which makes describing the dynamical properties of T_β more challenging. To get around this barrier, we need the following notion.

Definition 2.4. A cylinder $I_{n,\beta}(\epsilon_n)$ or a sequence $\epsilon_n \in \Sigma_\beta^n$ is called β-full if it has maximal length, that is, if

$$|I_{n,\beta}(\epsilon_n)| = \frac{1}{\beta^n},$$

where $|I|$ denotes the diameter of I.

When there is no risk of ambiguity, we will write full instead of β-full. The importance of full sequences is based on the fact that the concatenation of any two full sequences is still full.

PROPOSITION 2.5. [7, Lemma 3.2] *An nth level cylinder $I_{n,\beta}(\epsilon_n)$ is full if and only if, for any β-admissible sequence $\epsilon'_m \in \Sigma_\beta^m$ with $m \geq 1$, the concatenation $\epsilon_n\epsilon'_m$ is still β-admissible. Moreover,*

$$|I_{n+m,\beta}(\epsilon_n\epsilon'_m)| = |I_{n,\beta}(\epsilon_n)| \cdot |I_{m,\beta}(\epsilon'_m)|.$$

So, for any two full cylinders $I_{n,\beta}(\epsilon_n)$, $I_{m,\beta}(\epsilon'_m)$, the cylinder $I_{n+m,\beta}(\epsilon_n\epsilon'_m)$ is also full.

For an interval $I \subset [0, 1)$, let $\Lambda_\beta^n(I)$ denote the set of full sequences ϵ_n of length n with $I_{n,\beta}(\epsilon_n) \subset I$. In particular, if $I = [0, 1)$, then we simply write Λ_β^n instead of $\Lambda_\beta^n([0, 1))$. For this case, the cardinality of Λ_β^n can be estimated as follows.

LEMMA 2.6. [17, Lemma 1.1.46] *Let $\beta > 1$ and $n \in \mathbb{N}$.*

(1) *If $\beta \in \mathbb{N}$, then*

$$\#\Lambda_\beta^n = \beta^n.$$

(2) *If $\beta > 2$, then*

$$\#\Lambda_\beta^n > \frac{\beta - 2}{\beta - 1}\beta^n.$$

(3) *If $1 < \beta < 2$, then*

$$\#\Lambda_\beta^n > \bigg(\prod_{i=1}^{\infty}(1 - \beta^{-i})\bigg)\beta^n.$$

The general case $I \neq [0, 1)$ requires the following technical lemma due to Bugeaud and Wang [4].

LEMMA 2.7. [4, Proposition 4.2] *Let $\delta > 0$. Let $n_0 \geq 3$ be an integer such that $(\beta n_0)^{1+\delta} < \beta^{n_0\delta}$. For any interval $I \subset [0, 1)$ with $0 < |I| < n_0\beta^{-n_0}$, there exists a full cylinder $I_{m,\beta}(\epsilon_m) \subset I$ such that $|I|^{1+\delta} < |I_{m,\beta}(\epsilon_m)| < |I|$.*

Now, we are ready to tackle with the general case.

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LEMMA 2.8. *Let $\delta > 0$. Let $n_0 \geq 3$ be an integer such that $(\beta n_0)^{1+\delta} < \beta^{n_0\delta}$. Then, for any interval I with $0 < |I| < n_0\beta^{-n_0}$, there exists a constant $c_\beta > 0$ depending on β such that for any $n \geq -(1+\delta)\log_\beta |I|$,*

$$\#\Lambda_\beta^n(I) \geq c_\beta |I|^{1+\delta}\beta^n.$$

Proof. Since $|I| < n_0\beta^{-n_0}$, by Lemma 2.7, there exists a full cylinder $I_{m,\beta}(\boldsymbol{\epsilon}_m)$ satisfying

$$I_{m,\beta}(\boldsymbol{\epsilon}_m) \subset I \quad \text{and} \quad |I|^{1+\delta} < |I_{m,\beta}(\boldsymbol{\epsilon}_m)| = \beta^{-m} < |I|.$$

For such m, we have $n \geq m$ whenever $n \geq -(1+\delta)\log_\beta |I|$. By Proposition 2.5, the concatenation of two full sequences $\boldsymbol{\epsilon}_{n-m} \in \Lambda_\beta^{n-m}$ and $\boldsymbol{\epsilon}_m$ is still full. Thus,

$$\#\Lambda_\beta^n(I) \geq \#\Lambda_\beta^{n-m} \geq c_\beta \beta^{n-m} \geq c_\beta |I|^{1+\delta}\beta^n,$$

where the constant $c_\beta > 0$ depending on β is given in Lemma 2.6. □

3. *Optimal cover of parallelepipeds*

The proof of Theorem 1.1 relies on finding efficient covering by balls of the lim sup set $W(\mathcal{P})$. With this in mind, we need to study the optimal cover of parallelepipeds, which is closely related to its Hausdorff content.

In what follows, for geometric reasons, it will be convenient to equip $\mathbb{R}^d$ with the maximal norm, and thus balls correspond to hypercubes. For any set $E \subset \mathbb{R}^d$, its s-dimensional *Hausdorff content* is given by

$$\mathcal{H}_\infty^s(E) = \inf\left\{\sum_{i=1}^\infty |B_i|^s : E \subset \bigcup_{i=1}^\infty B_i \text{ where } B_i \text{ are open balls}\right\}.$$

In other words, the optimal cover of a Borel set can be characterized by its Hausdorff content, which is generally estimated by putting measures or mass distributions on it, following the mass distribution principle described below.

PROPOSITION 3.1. (Mass distribution principle [3, Lemma 1.2.8]) *Let E be a subset of $\mathbb{R}^d$. If E supports a strictly positive Borel measure μ that satisfies*

$$\mu(B(\mathbf{x}, r)) \leq cr^s$$

for some constant $0 < c < \infty$ and for every ball $B(\mathbf{x}, r)$, then $\mathcal{H}_\infty^s(E) \geq \mu(E)/c$.

Following Falconer [5], when E is taken as a hyperrectangle R, its Hausdorff content can be expressed as the so-called *singular value function*. For a hyperrectangle $R \subset \mathbb{R}^d$ with sidelengths $a_1 \geq a_2 \geq \cdots \geq a_d > 0$ and a parameter $s \in [0, d]$, the singular value function φ^s is defined by

$$\varphi^s(R) = a_1 \cdots a_m a_{m+1}^{s-m}, \tag{3.1}$$

where $m = \lfloor s \rfloor$.

The next lemma allows us to estimate the Hausdorff content of a Borel set inside a hyperrectangle. Denote the d-dimensional Lebesgue measure by $\mathcal{L}^d$.

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LEMMA 3.2. *Let $E \subset \mathbb{R}^d$ be a bounded Borel set. Assume that there exists a hyperrectangle R with sidelengths $a_1 \geq a_2 \geq \cdots \geq a_d > 0$ such that $E \subset R$ and $\mathcal{L}^d(E) \geq c\mathcal{L}^d(R)$ for some $c > 0$, then for any $0 < s \leq d$,*

$$c2^{-d}\varphi^s(R) \leq \mathcal{H}^s_\infty(E) \leq \varphi^s(R).$$

Proof. The second inequality simply follows from $E \subset R$ and equation (3.1). So, we only need to prove the first one. Let ν be the normalized Lebesgue measure supported on E, that is,

$$\nu = \frac{\mathcal{L}^d|_E}{\mathcal{L}^d(E)}.$$

For any $0 < s \leq d$, let $m = \lfloor s \rfloor$ be the integer part of s. Now we estimate the ν-measure of arbitrary ball $B(\mathbf{x}, r)$ with $r > 0$ and $\mathbf{x} \in E$. The proof is split into two cases.

Case 1: $0 < r < a_d$. Then,

$$\begin{aligned}\nu(B(\mathbf{x}, r)) &= \frac{\mathcal{L}^d(E \cap B(x, r))}{\mathcal{L}^d(E)} \leq \frac{\mathcal{L}^d(R \cap B(\mathbf{x}, r))}{c\mathcal{L}^d(R)} \leq \frac{(2r)^d}{ca_1 \cdots a_d} \\ &= \frac{2^d r^s \cdot r^{d-s}}{ca_1 \cdots a_m a_{m+1}^{s-m} a_{m+1}^{m+1-s} a_{m+2} \cdots a_d} \leq \frac{2^d r^s}{ca_1 \cdots a_m a_{m+1}^{s-m}}.\end{aligned}$$

Case 2: $a_{i+1} \leq r < a_i$ for $1 \leq i \leq d-1$. It follows that

$$\nu(B(x, r)) \leq \frac{\mathcal{L}^d(R \cap B(x, r))}{c\mathcal{L}^d(R)} \leq \frac{(2r)^i \cdot a_{i+1} \cdots a_d}{ca_1 \cdots a_d} = \frac{2^i r^i}{ca_1 \cdots a_i}.$$

If $i > m = \lfloor s \rfloor$, then the right-hand side can be estimated in a way similar to Case 1,

$$\frac{2^i r^i}{ca_1 \ldots a_i} = \frac{2^i r^s \cdot r^{i-s}}{ca_1 \ldots a_m a_{m+1}^{s-m} a_{m+1}^{m+1-s} a_{m+2} \cdots a_i} \leq \frac{2^i r^s}{ca_1 \ldots a_m a_{m+1}^{s-m}}.$$

If $i \leq m = \lfloor s \rfloor$, then $i - s \leq 0$, and so

$$\frac{2^i r^i}{ca_1 \ldots a_i} = \frac{2^i r^s \cdot r^{i-s}}{ca_1 \ldots a_i} \leq \frac{2^i r^s \cdot a_{i+1}^{i-s}}{ca_1 \ldots a_i} = \frac{2^i r^s}{ca_1 \ldots a_i a_{i+1}^{s-i}} \leq \frac{2^i r^s}{ca_1 \ldots a_m a_{m+1}^{s-m}},$$

where the last inequality follows from the fact that $a_{i+1} \leq \cdots \leq a_{m+1}$.

With the estimation given above, by the mass distribution principle, we have

$$\mathcal{H}^s_\infty(E) \geq c2^{-d} a_1 \ldots a_m a_{m+1}^{s-m} = c2^{-d}\varphi^s(R),$$

as desired. □

By the above lemma, to obtain the optimal cover of a parallelepiped P, it suffices to find a suitable hyperrectangle containing it. Since the optimal cover of P does not depend on its location, we assume that one of its vertices lies in the origin. With this assumption, P is uniquely determined by d column vectors, say $\alpha_1, \ldots, \alpha_d$. Moreover, we have

$$P = \{x_1\alpha_1 + \cdots + x_d\alpha_d : (x_1, \ldots, x_d) \in [0, 1]^d\}.$$

LEMMA 3.3. *Let P be a parallelepiped given above. There exists a hyperrectangle R such that*

$$P \subset R \quad \text{and} \quad \mathcal{L}^d(P) = 2^{-d(d+1)}\mathcal{L}^d(R).$$

Proof. We will employ the Gram–Schmidt process to $\alpha_1, \ldots, \alpha_d$ in a proper way to obtain d pairwise orthogonal vectors that yield the desired hyperrectangle.

First, let $\gamma_1 = \alpha_{i_1}$ with $\alpha_{i_1} = \max_{1\le l\le d} |\alpha_l|$. For $1 < k \le d$, let γ_k be defined inductively as

$$\gamma_k = \alpha_{i_k} - \sum_{j=1}^{k-1} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)}\gamma_j, \tag{3.2}$$

where α_{i_k} is chosen so that

$$\left|\alpha_{i_k} - \sum_{j=1}^{k-1} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)}\gamma_j\right| = \max_{l\ne i_1,\ldots,i_{k-1}} \left|\alpha_l - \sum_{j=1}^{k-1} \frac{(\alpha_l, \gamma_j)}{(\gamma_j, \gamma_j)}\gamma_j\right|. \tag{3.3}$$

This is the standard Gram–Schmidt process and so $\gamma_1, \ldots, \gamma_d$ are pairwise orthogonal. In addition,

$$(\alpha_{i_1}, \ldots, \alpha_{i_d}) = (\gamma_1, \ldots, \gamma_d) \begin{pmatrix} 1 & -\dfrac{(\alpha_{i_2}, \gamma_1)}{(\gamma_1, \gamma_1)} & \cdots & -\dfrac{(\alpha_{i_d}, \gamma_1)}{(\gamma_1, \gamma_1)} \\ 0 & 1 & \cdots & -\dfrac{(\alpha_{i_d}, \gamma_2)}{(\gamma_2, \gamma_2)} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{3.4}$$

Denote the rightmost upper triangular matrix by U. For any $\mathbf{x} = x_{i_1}\alpha_{i_1} + \cdots + x_{i_d}\alpha_{i_d} \in P$ with $(x_{i_1}, \ldots, x_{i_d}) \in [0, 1]^d$, we have

$$\mathbf{x} = (\alpha_{i_1}, \ldots, \alpha_{i_d}) \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_d} \end{pmatrix} = (\gamma_1, \ldots, \gamma_d)U \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_d} \end{pmatrix}.$$

The proof of Lemma 3.3 will be completed with the help of the following lemma.

LEMMA 3.4. *The absolute value of each entry of U is not greater than* 2.

Proof. For any $1 < k \le d$, by the orthogonality of $\gamma_1, \ldots, \gamma_{k-1}$,

$$\begin{aligned} |\gamma_k|^2 = (\gamma_k, \gamma_k) &= \left(\alpha_{i_k} - \sum_{j=1}^{k-1} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)}\gamma_j, \alpha_{i_k} - \sum_{j=1}^{k-1} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)}\gamma_j\right) \\ &= \Bigg(\alpha_{i_k} - \sum_{j=1}^{k-2} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)}\gamma_j - \frac{(\alpha_{i_k}, \gamma_{k-1})}{(\gamma_{k-1}, \gamma_{k-1})}\gamma_{k-1}, \alpha_{i_k} \\ &\quad - \sum_{j=1}^{k-2} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)}\gamma_j - \frac{(\alpha_{i_k}, \gamma_{k-1})}{(\gamma_{k-1}, \gamma_{k-1})}\gamma_{k-1}\Bigg) \end{aligned}$$

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$$= \left(\alpha_{i_k} - \sum_{j=1}^{k-2} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)} \gamma_j, \alpha_{i_k} - \sum_{j=1}^{k-2} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)} \gamma_j \right) - \frac{(\alpha_{i_k}, \gamma_{k-1})^2}{(\gamma_{k-1}, \gamma_{k-1})} \leq |\gamma_{k-1}|^2,$$

where the last inequality follows from the definition of γ_{k-1} (see equation (3.3)). This gives

$$|\gamma_1| \geq |\gamma_2| \geq \cdots \geq |\gamma_d| > 0. \tag{3.5}$$

By the above inequality and equation (3.3), for any $1 \leq l \leq k$, it follows that

$$\begin{aligned} |\gamma_{l-1}| \geq |\gamma_l| \geq \left| \alpha_{i_k} - \sum_{j=1}^{l-1} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)} \gamma_j \right| &\geq \left| \frac{(\alpha_{i_k}, \gamma_{l-1})}{(\gamma_{l-1}, \gamma_{l-1})} \gamma_{l-1} \right| - \left| \alpha_{i_k} - \sum_{j=1}^{l-2} \frac{(\alpha_{i_k}, \gamma_j)}{(\gamma_j, \gamma_j)} \gamma_j \right| \\ &\geq \left(\left| \frac{(\alpha_{i_k}, \gamma_{l-1})}{(\gamma_{l-1}, \gamma_{l-1})} \right| - 1 \right) |\gamma_{l-1}|, \end{aligned}$$

which implies that

$$\left| \frac{(\alpha_{i_k}, \gamma_{l-1})}{(\gamma_{l-1}, \gamma_{l-1})} \right| \leq 2. \qquad \square$$

Now we proceed to prove Lemma 3.3.

Let $(U_{i1}, \ldots, U_{id})$ be the ith row of U. Since $0 \leq x_{i_k} \leq 1$, by Lemma 3.4, we have

$$\left| (U_{i1}, \ldots, U_{id}) \begin{pmatrix} x_{i_1} \\ \vdots \\ x_{i_d} \end{pmatrix} \right| = \left| \sum_{k=1}^{d} U_{ik} x_{i_k} \right| \leq 2^d,$$

and so

$$\mathbf{x} \in R := \{x_1\gamma_1 + \cdots + x_d\gamma_d : (x_1, \ldots, x_d) \in [-2^d, 2^d]^d\}. \tag{3.6}$$

Therefore, $P \subset R$ which finishes the proof of the first point.

However, by an elementary result of linear algebra,

$$\begin{aligned} \mathcal{L}^d(P) &= \text{the absolute value of the determinant } |(\alpha_{i_1}, \ldots, \alpha_{i_d})| \\ &= \text{the absolute value of the determinant } |(\gamma_1, \ldots, \gamma_d)U| \\ &= |\gamma_1| \cdots |\gamma_d| = 2^{-d(d+1)} \mathcal{L}^d(R), \end{aligned} \tag{3.7}$$

where the third equality follows from the fact that $\gamma_1, \ldots, \gamma_d$ are pairwise orthogonal and U is upper triangular with all diagonal entries equal to 1, and the last equality follows from equation (3.6). $\square$

4. *Proof of Theorem 1.1*

Throughout, we write $a \asymp b$ if $c^{-1} \leq a/b \leq c$, and $a \lesssim b$ if $a \leq cb$ for some unspecified constant $c \geq 1$.

4.1. *Upper bound of* $\dim_{\mathrm{H}} W(\mathcal{P})$.

Obtaining upper estimates for the Hausdorff dimension of a lim sup set is usually straightforward, as it involves a natural covering argument.

For $1 \le i \le d$ and any $\boldsymbol{\epsilon}_n^i = (\epsilon_1^i, \ldots, \epsilon_n^i) \in \Sigma_{\beta_i}^n$, we always take

$$z_i^* = \frac{\epsilon_1^i}{\beta_i} + \frac{\epsilon_2^i}{\beta_i^2} + \cdots + \frac{\epsilon_n^i}{\beta_i^n} \tag{4.1}$$

to be the left endpoint of $I_{n,\beta_i}(\boldsymbol{\epsilon}_n^i)$. Write $\mathbf{z}^* = (z_1^*, \ldots, z_d^*)$. Then, $W(\mathcal{P})$ is contained in the following set:

$$\bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{\boldsymbol{\epsilon}_n^1 \in \Sigma_{\beta_1}^n} \cdots \bigcup_{\boldsymbol{\epsilon}_n^d \in \Sigma_{\beta_d}^n} (f^n P_n + \mathbf{z}^*) =: \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} E_n. \tag{4.2}$$

For any $n \ge 1$, let $f^n \alpha_1^{(n)}, \ldots, f^n \alpha_d^{(n)}$ be the vectors that determine $f^n P_n$. By Lemma 3.3 and equation (3.2), there is a hyperrectangle R_n with sidelengths $2^{d+1}|\gamma_1^{(n)}| \ge \cdots \ge 2^{d+1}|\gamma_d^{(n)}| > 0$ such that $f^n P_n \subset R_n$.

Recall that $\mathcal{A}_n = \{\beta_1^{-n}, \ldots, \beta_d^{-n}, |\gamma_1^{(n)}|, \ldots, |\gamma_d^{(n)}|\}$, and for any $\tau \in \mathcal{A}_n$,

$$\mathcal{K}_{n,1}(\tau) := \{1 \le i \le d : \beta_i^{-n} \le \tau\} \quad \text{and} \quad \mathcal{K}_{n,2}(\tau) = \{1 \le i \le d : |\gamma_i^{(n)}| \ge \tau\}.$$

Let $\tau \in \mathcal{A}_n$. We now estimate the number of balls of diameter τ needed to cover the set E_n. We start by covering a fixed parallelepiped $P := f^n P_n + \mathbf{z}^*$. In what follows, one can regard P as a hyperrectangle, since $P = f^n P_n + \mathbf{z}^* \subset R_n + \mathbf{z}^*$. It is easily verified that we can find a collection $\mathcal{B}_n(P)$ of balls of diameter τ that covers P with

$$\#\mathcal{B}_n(P) \lesssim \prod_{i \in \mathcal{K}_{n,2}(\tau)} \frac{|\gamma_i^{(n)}|}{\tau}.$$

Observe that the collection $\mathcal{B}_n(P)$ will also cover other parallelepipeds contained in E_n along the direction of the ith axis with $i \in \mathcal{K}_{n,1}(\tau)$. Namely, the collection of balls $\mathcal{B}_n(P)$ simultaneously covers

$$\asymp \prod_{i \in \mathcal{K}_{n,1}(\tau)} \frac{\tau}{\beta_i^{-n}}$$

parallelepipeds. Since the number of parallelepipeds contained in E_n is $\lesssim \beta_1^n \cdots \beta_d^n$, one needs at most

$$\begin{aligned} &\lesssim \bigg(\prod_{i \in \mathcal{K}_{n,2}(\tau)} \frac{|\gamma_i^{(n)}|}{\tau} \bigg) \cdot (\beta_1^n \cdots \beta_d^n) \Big/ \prod_{i \in \mathcal{K}_{n,1}(\tau)} \frac{\tau}{\beta_i^{-n}} \\ &= \prod_{i \in \mathcal{K}_{n,1}(\tau)} \tau^{-1} \prod_{i \notin \mathcal{K}_{n,1}(\tau)} \beta_i^n \prod_{i \in \mathcal{K}_{n,2}(\tau)} \frac{|\gamma_i^{(n)}|}{\tau} \end{aligned} \tag{4.3}$$

balls of diameter τ to cover E_n.

Now suppose that $s > s^* = \limsup s_n$, where

$$s_n := \min_{\tau \in \mathcal{A}_n} \bigg\{ \sum_{i \in \mathcal{K}_{n,1}(\tau)} 1 + \sum_{i \notin \mathcal{K}_{n,1}(\tau)} \frac{n \log \beta_i}{-\log \tau} + \sum_{i \in \mathcal{K}_{n,2}(\tau)} \bigg(1 - \frac{\log |\gamma_i^{(n)}|}{\log \tau} \bigg) \bigg\}.$$

Let $\varepsilon < s - s^*$. For any large n, we have $s > s_n + \varepsilon$. Let $\tau_0 \in \mathcal{A}_n$ be such that the minimum in the definition of s_n is attained. In particular, equation (4.3) holds for τ_0. The s-volume of the cover of E_n is majorized by

$$\begin{aligned}
&\lesssim \bigg(\prod_{i \in \mathcal{K}_{n,1}(\tau_0)} \tau_0^{-1} \prod_{i \notin \mathcal{K}_{n,1}(\tau_0)} \beta_i^n \prod_{i \in \mathcal{K}_{n,2}(\tau_0)} \frac{|\gamma_i^{(n)}|}{\tau_0} \bigg) \cdot \tau_0^s \\
&= \exp \bigg(\sum_{i \in \mathcal{K}_{n,1}(\tau_0)} - \log \tau_0 + \sum_{i \notin \mathcal{K}_{n,1}(\tau_0)} n \log \beta_i + \sum_{i \in \mathcal{K}_{n,2}(\tau_0)} (\log |\gamma_i^{(n)}| - \log \tau_0) + s \log \tau_0 \bigg) \\
&= \exp \bigg(- \log \tau_0 \bigg(\sum_{i \in \mathcal{K}_{n,1}(\tau_0)} 1 + \sum_{i \notin \mathcal{K}_{n,1}(\tau_0)} \frac{n \log \beta_i}{- \log \tau_0} + \sum_{i \in \mathcal{K}_{n,2}(\tau_0)} \bigg(1 - \frac{\log |\gamma_i^{(n)}|}{\log \tau_0} \bigg) - s \bigg) \bigg) \\
&= \exp(- \log \tau_0 (s_n - s)) \le \exp(\varepsilon \log \tau_0).
\end{aligned}$$

Since the elements of $\mathcal{A}_n$ decay exponentially, the last equation is less than $e^{-n\delta\varepsilon}$ for some $\delta > 0$ independent of n and ε. It follows from the definition of s-dimensional Hausdorff measure that for any s, $\delta > 0$ and ε given above,

$$\mathcal{H}^s(W(\mathcal{P})) \le \liminf_{N \to \infty} \sum_{n=N}^{\infty} e^{-n\delta\varepsilon} = 0.$$

Therefore, $\dim_{\mathrm{H}} W(\mathcal{P}) \le s$. Since this is true for all $s > s^*$, we have

$$\dim_{\mathrm{H}} W(\mathcal{P}) \le s^* = \limsup_{n \to \infty} s_n.$$

4.2. *Lower bound of* $\dim_{\mathrm{H}} W(\mathcal{P})$. The proof crucially relies on the following lemma.

LEMMA 4.1. [9, Corollary 2.6] *Let* $\{F_n\}_{n \ge 1}$ *be a sequence of open sets in* $[0, 1]^d$ *and* $F = \limsup F_n$. *Let* $s > 0$. *If for any* $0 < t < s$, *there exists a constant* c_t *such that*

$$\limsup_{n \to \infty} \mathcal{H}^t_\infty(F_n \cap D) \ge c_t |D|^d \tag{4.4}$$

holds for all hypercubes $D \subset [0, 1]^d$, *then* $F \in \mathscr{G}^s([0, 1]^d)$. *In particular,* $\dim_{\mathrm{H}} F \ge s$.

Remark 4.2. A weaker version by Persson and Reeve [18, Lemma 2.1] also applies to the current proof, but is not adopted here because it results in a more complex proof.

Let

$$E_n = \bigcup_{\boldsymbol{\epsilon}_n^1 \in \Sigma_{\beta_1}^n} \cdots \bigcup_{\boldsymbol{\epsilon}_n^d \in \Sigma_{\beta_d}^n} (I_{n,\beta_1}(\boldsymbol{\epsilon}_n^1) \times \cdots \times I_{n,\beta_d}(\boldsymbol{\epsilon}_n^d)) \cap (f^n P_n + \mathbf{z}^*),$$

where $\mathbf{z}^* = (z_1^*, \ldots, z_d^*)$ is defined as in equation (4.1).

LEMMA 4.3. *For any* $0 < t < s^* = \limsup s_n$,

$$\limsup_{n \to \infty} \mathcal{H}^t_\infty(E_n \cap D) \gtrsim |D|^d$$

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holds for all hypercubes $D \subset [0,1]^d$, where the unspecified constant depends on d only. Therefore, $W(\mathcal{P}) \in \mathscr{G}^{s^*}([0,1]^d)$ and, in particular,

$$\dim_{\mathrm{H}} W(\mathcal{P}) \geq s^*.$$

Proof. Fix $0 < t < s^*$. Write $\varepsilon = s^* - t$. By definition, there exist infinitely many n such that

$$s_n > t. \tag{4.5}$$

In view of Lemma 2.8, let $D \subset [0,1]^d$ be a hypercube with $|D| \leq n_0 \beta_d^{-n_0}$, where n_0 is an integer such that $(\beta_i n_0)^{1+\varepsilon/d} < \beta_i^{n_0\varepsilon/d}$ for $1 \leq i \leq d$. Let n be an integer such that equation (4.5) holds and for any $1 \leq i \leq d$,

$$n \geq -(1+\varepsilon/d)\log_{\beta_i}|D| \quad \text{and} \quad \beta_i^{-n}/2 \leq |D|^{d+\varepsilon}. \tag{4.6}$$

Obviously, there are still infinitely many n that satisfy these conditions. Write $D = I_1 \times \cdots \times I_d$ with $|I_1| = \cdots = |I_d|$. The first inequality in equation (4.6) ensures that Lemma 2.8 is applicable to bound $\#\Lambda^n_{\beta_i}(I_i)$ from below for $1 \leq i \leq d$.

Recall from Lemma 3.3 and equation (3.2) that for any $n \geq 1$, $f^n P_n + z^*$ is contained in some hyperrectangle with sidelengths $2^{d+1}|\gamma_1^{(n)}| \geq \cdots \geq 2^{d+1}|\gamma_d^{(n)}| > 0$. For any $n \in \mathbb{N}$ satisfying equations (4.5) and (4.6), define a probability measure μ_n supported on $E_n \cap D$ by

$$\mu_n = \sum_{\boldsymbol{\epsilon}_n^1 \in \Lambda^n_{\beta_1}(I_1)} \cdots \sum_{\boldsymbol{\epsilon}_n^d \in \Lambda^n_{\beta_d}(I_d)} \frac{\nu_{\mathbf{z}^*}}{\#\Lambda^n_{\beta_1}(I_1)\cdots\#\Lambda^n_{\beta_d}(I_d)}, \tag{4.7}$$

where $\nu_{\mathbf{z}^*}$ is defined by

$$\nu_{\mathbf{z}^*} := \frac{\mathcal{L}^d|_{f^n P_n + \mathbf{z}^*}}{\mathcal{L}^d(f^n P_n + \mathbf{z}^*)} = \frac{\mathcal{L}^d|_{f^n P_n + \mathbf{z}^*}}{|\gamma_1^{(n)}|\cdots|\gamma_d^{(n)}|}. \tag{4.8}$$

The equality $\mathcal{L}^d(f^n P_n + \mathbf{z}^*) = |\gamma_1^{(n)}|\cdots|\gamma_d^{(n)}|$ can be deduced from equation (3.7).

Let $\mathbf{x} \in E_n \cap D$ and $r > 0$. Suppose that $\mathbf{x} \in f^n P_n + \mathbf{y}^* \subset E_n \cap D$. Now, we estimate $\mu_n(B(\mathbf{x}, r))$, and the proof is divided into four distinct cases.

Case 1: $r \geq |D|$. Clearly, since $t < s \leq d$,

$$\mu_n(B(\mathbf{x}, r)) \leq 1 = \frac{|D|^d}{|D|^d} \leq \frac{r^d}{|D|^d} \leq \frac{r^t}{|D|^d}.$$

Case 2: $r \leq |\gamma_d^{(n)}|$. Note that in the definition of μ_n, all the cylinders under consideration are full. We see that the ball $B(\mathbf{x}, r)$ intersects at most 2^d parallelepipeds with the form $f^n P_n + \mathbf{z}^*$. For any such parallelepiped, by the definition of $\nu_{\mathbf{z}^*}$ (see equation (4.8)) and Lemma 2.8, we have

$$\begin{aligned}\frac{\nu_{\mathbf{z}^*}(B(\mathbf{x}, r))}{\#\Lambda^n_{\beta_1}(I_1)\cdots\#\Lambda^n_{\beta_d}(I_d)} &\lesssim \frac{1}{\beta_1^n\cdots\beta_d^n|D|^{d+\varepsilon}} \cdot \frac{r^d}{|\gamma_1^{(n)}|\cdots|\gamma_d^{(n)}|} \\ &= \frac{r^{d - \sum_{i=1}^d (\log\beta_i^n + \log|\gamma_i^{(n)}|)/\log r}}{|D|^{d+\varepsilon}}. \end{aligned} \tag{4.9}$$

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Since $f^n P_n + \mathbf{z}^*$ is contained in some $I_{n,\beta_1}(\boldsymbol{\epsilon}_n^1) \times \cdots \times I_{n,\beta_d}(\boldsymbol{\epsilon}_n^d)$, by a volume argument, we have $\sum_{i=1}^d (\log \beta_i^n + \log |\gamma_i^{(n)}|) < 0$. This combined with $r \le |\gamma_d^{(n)}| < 1$ gives

$$\begin{aligned} d - \sum_{i=1}^d (\log \beta_i^n + \log |\gamma_i^{(n)}|)/\log r &\ge d - \sum_{i=1}^d (\log \beta_i^n + \log |\gamma_i^{(n)}|)/\log |\gamma_d^{(n)}| \\ &= \sum_{i=1}^d \frac{n \log \beta_i}{-\log |\gamma_d^{(n)}|} + \sum_{i=1}^d 1 - \frac{\log |\gamma_i^{(n)}|}{\log |\gamma_d^{(n)}|}. \end{aligned} \tag{4.10}$$

One can see that the right-hand side of equation (4.10) is just the one in equation (1.2) defined by choosing $\tau = |\gamma_d^{(n)}|$, since

$$\mathcal{K}_{n,1}(|\gamma_d^{(n)}|) = \emptyset \quad \text{and} \quad \mathcal{K}_{n,2}(|\gamma_d^{(n)}|) = \{1, \ldots, d\}.$$

This means that the quantity in equation (4.10) is greater than or equal to s_n, and so by equation (4.9), one has

$$\mu_n(B(\mathbf{x}, r)) \lesssim 2^d \cdot \frac{r^{s_n}}{|D|^{d+\varepsilon}} \lesssim \frac{r^{s_n-\varepsilon}}{|D|^d} \le \frac{r^t}{|D|^d}.$$

Case 3: $\beta_1^{-n} < r \le |D|$. In this case, the ball $B(\mathbf{x}, r)$ is sufficiently large so that for any hyperrectangle $R := I_{n,\beta_1}(\boldsymbol{\epsilon}_n^1) \times \cdots \times I_{n,\beta_d}(\boldsymbol{\epsilon}_n^d)$,

$$B(\mathbf{x}, r) \cap R \ne \emptyset \Longrightarrow R \subset B(\mathbf{x}, 3r).$$

A simple calculation shows that $B(\mathbf{x}, r)$ intersects at most $\lesssim r^d \beta_1^n \cdots \beta_d^n$ hyperrectangles with the form $I_{n,\beta_1}(\boldsymbol{\epsilon}_n^1) \times \cdots \times I_{n,\beta_d}(\boldsymbol{\epsilon}_n^d)$. By the definition of μ_n, one has

$$\begin{aligned} \mu_n(B(\mathbf{x}, r)) &\lesssim \frac{1}{\#\Lambda_{\beta_1}^n(I_1) \cdots \#\Lambda_{\beta_d}^n(I_d)} \cdot r^d \beta_1^n \cdots \beta_d^n \\ &\lesssim \frac{r^d}{|D|^{d+\varepsilon}} \le \frac{r^{d-\varepsilon}}{|D|^d} \le \frac{r^t}{|D|^d}. \end{aligned}$$

Case 4: Arrange the elements in $\mathcal{A}_n$ in non-descending order. Suppose that $\tau_{k+1} \le r < \tau_k$ with τ_k and τ_{k+1} being two consecutive terms in $\mathcal{A}_n$. Let

$$\mathcal{K}_{n,1}(\tau_{k+1}) := \{1 \le i \le d : \beta_i^{-n} \le \tau_{k+1}\} \quad \text{and} \quad \mathcal{K}_{n,2}(\tau_k) = \{1 \le i \le d : |\gamma_i^{(n)}| \ge \tau_k\}$$

be defined in the same way as in equation (1.2). It is easy to see that $B(\mathbf{x}, r)$ can intersects at most

$$\lesssim \prod_{i \in \mathcal{K}_{n,1}(\tau_{k+1})} r\beta_i^n$$

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parallelepipeds with positive μ_n-measure. Moreover, the μ_n-measure of the intersection of each parallelepiped with $B(\mathbf{x}, r)$ is majorized by

$$\begin{aligned}
&\lesssim \frac{1}{|D|^{d+\varepsilon}\beta_1^n \cdots \beta_d^n} \cdot \frac{1}{|\gamma_1^{(n)}| \cdots |\gamma_d^{(n)}|} \cdot \prod_{i \in \mathcal{K}_{n,2}(\tau_k)} r \cdot \prod_{i \notin \mathcal{K}_{n,2}(\tau_k)} |\gamma_i^{(n)}| \\
&= \frac{1}{|D|^{d+\varepsilon}\beta_1^n \cdots \beta_d^n} \cdot \prod_{i \in \mathcal{K}_{n,2}(\tau_k)} \frac{r}{|\gamma_i^{(n)}|}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\mu_n(B(\mathbf{x}, r)) &\lesssim \bigg(\prod_{i \in \mathcal{K}_{n,1}(\tau_{k+1})} r\beta_i^n \bigg) \cdot \bigg(\frac{1}{|D|^{d+\varepsilon}\beta_1^n \cdots \beta_d^n} \cdot \prod_{i \in \mathcal{K}_{n,2}(\tau_k)} \frac{r}{|\gamma_i^{(n)}|} \bigg) \\
&= \frac{1}{|D|^{d+\varepsilon}} \cdot \prod_{i \in \mathcal{K}_{n,1}(\tau_{k+1})} r \cdot \prod_{i \notin \mathcal{K}_{n,1}(\tau_{k+1})} \beta_i^{-n} \cdot \prod_{i \in \mathcal{K}_{n,2}(\tau_k)} \frac{r}{|\gamma_i^{(n)}|} \\
&= \frac{r^{s(r)}}{|D|^{d+\varepsilon}},
\end{aligned}$$

where

$$s(r) = \sum_{i \in \mathcal{K}_{n,1}(\tau_{k+1})} 1 + \sum_{i \notin \mathcal{K}_{n,1}(\tau_{k+1})} \frac{-n \log \beta_i}{\log r} + \sum_{i \in \mathcal{K}_{n,2}(\tau_k)} \bigg(1 - \frac{\log |\gamma_i^{(n)}|}{\log r} \bigg).$$

Clearly, as a function of r, $s(r)$ is monotonic on the interval $[\tau_{k+1}, \tau_k]$. So the minimal value is attained when $r = \tau_{k+1}$ or τ_k. First, suppose that the minimum is attained at $r = \tau_k$. If $\mathcal{K}_{n,1}(\tau_k) = \mathcal{K}_{n,1}(\tau_{k+1})$, then there is nothing to be proved. So we may assume that $\mathcal{K}_{n,1}(\tau_k) \neq \mathcal{K}_{n,1}(\tau_{k+1})$. Since $\mathcal{K}_{n,1}(\tau_{k+1}) \subsetneq \mathcal{K}_{n,1}(\tau_k)$, one can see that $\tau_k = \beta_j^{-n}$ for some j and

$$\mathcal{K}_{n,1}(\tau_k) = \mathcal{K}_{n,1}(\tau_{k+1}) \cup \{j\}.$$

It follows that

$$\sum_{i \in \mathcal{K}_{n,1}(\tau_{k+1})} 1 + \sum_{i \notin \mathcal{K}_{n,1}(\tau_{k+1})} \frac{-n \log \beta_i}{\log \tau_k} = \sum_{i \in \mathcal{K}_{n,1}(\tau_k)} 1 + \sum_{i \notin \mathcal{K}_{n,1}(\tau_k)} \frac{-n \log \beta_i}{\log \tau_k},$$

which implies that

$$s(r) \geq s_n.$$

By a similar argument, one still has $s(r) \geq s_n$ if the minimum is attained at $r = \tau_{k+1}$. Therefore,

$$\mu_n(B(\mathbf{x}, r)) \lesssim \frac{r^{s_n}}{|D|^{d+\varepsilon}} \leq \frac{r^{s_n - \varepsilon}}{|D|^d} \leq \frac{r^t}{|D|^d}.$$

Summarizing the estimates of the μ_n-measures of arbitrarily balls presented in Cases 1–4, we get

$$\mu_n(B(\mathbf{x}, r)) \lesssim \frac{r^t}{|D|^d} \quad \text{for all } r > 0,$$

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where the unspecified constant does not depend on D. Finally, by the mass distribution principle,

$$\mathcal{H}^t_\infty(E_n \cap D) \gtrsim |D|^d.$$

This is true for infinitely many n, and the proof is completed. $\square$

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