MAXIMAL SUBGROUPS OF INFINITE SYMMETRIC GROUPS

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The purpose of this paper is to extend results of Ball [1] concerning maximal subgroups of the group S(X) of all permutations of the infinite set X. The basic idea is to consider S(X) as a group of operators on objects more complicated than X. The objects we consider here are subspaces of the Stone-Čech compactification of the discrete space X and the Boolean algebra of "big setoids" of X.

In [1] Ball exhibited several classes of maximal subgroups, to wit:

I. All permutations which fix (setwise) a finite subset of X.

II. All permutations which "almost" fix an infinite subset A of smaller cardinality than X.

III. All permutations which either almost fix or almost interchange two complementary subsets of X of the same cardinality.

In this paper we shall extend classes I and III.

Before starting we fix some notation and terminology. If A and B are sets A + B means the symmetric difference of A and B, $|A|$ is the cardinality of A. The symbol $< >$ is used to denote the group generated by whatever is placed within. A permutation σ <u>almost fixes</u> an infinite set A if $|A+\sigma A|<|A|$. <u>Almost interchange</u> and <u>almost permute</u> then have the obvious meanings. A group of operators G is <u>transitive on sets of type</u> T if any 1-1 correspondence between two sets of type T is realizable by (the restriction of) an element of G. Small Greek letters will designate operators (e.g. permutations), small Roman letters the elements they act upon.

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1. *A change of scene.* To extend class I we change the domain of $S(X)$ from X to suitable subspaces of the Stone-Čech compactification of the discrete space X. The relevant theorem concerns maximal subgroups of groups acting on sets.

THEOREM 1. *Let* G *be a group acting on an infinite set* S. *Suppose* G *is transitive on finite subsets of* S. *If* J *is a nonempty finite subset of* S, *set* $H = \{\pi \in G \mid \pi J = J\}$. *Then* H *is a maximal subgroup of* G.

Proof. Let $\pi \in G \setminus H$. We first show the existence of a $\sigma \in \langle H, \pi \rangle$ such that $J \cap \sigma J = \phi$. Choose $\sigma \in \langle H, \pi \rangle$ minimizing $|J \cap \sigma J|$. If σ is not as desired then there exist $x, y \in J$ such that $\sigma x \notin J$, $\sigma y \in J$. Choose $\tau \in H$ such that $\tau(\sigma y) = x$ and $\tau(\sigma z) \notin \sigma^{-1} J$ for all $z \in J$ such that $\sigma z \notin J$. Then $\sigma \tau \sigma \in \langle H, \pi \rangle$ contrary to the minimality of σ.

We now show that $G = \langle H, \sigma \rangle$. Let $\rho \in G$. Choose $\lambda \in H$ such that $(\lambda \sigma J) \cap (\rho^{-1} J) = \phi$ and $\mu \in H$ such that $\mu \rho \lambda \sigma J = \sigma^{-1} J$. Then $\sigma \mu \rho \lambda \sigma \in H$ and so $\rho \in \langle H, \sigma \rangle$.

Observe that distinct finite sets J give rise to distinct maximal subgroups H.

If X is an infinite set let βX be the set of all ultrafilters on X, i.e. the Stone-Čech compactification of the discrete space X. We note that $S(X)$ acts on βX in a natural way. Two points of βX are said to be *equivalent* if there is an element of $S(X)$ taking one to the other.

THEOREM 2. *If* S *is an equivalence class of ultrafilters then* S *is infinite and* $S(X)$ *is transitive on finite subsets of* S.

Proof. Let $\mathfrak{F} \in S$, $A \in \mathfrak{F}$. If $\mathfrak{F}$ is fixed (i.e. is an element of X) the theorem is trivial. Otherwise $A = B \cup C$, $B \cap C = \phi$, $|B| = |C| = |A|$. Then, say, $B \in \mathfrak{F}$, $C \notin \mathfrak{F}$. Partition C into an infinite number of copies B_i of B. Clearly there exist $\pi_i \in S(X)$ such that $\pi_i B = B_i$ and so $\pi_i \mathfrak{F}$ are all distinct and in S; hence S is infinite.

Now suppose $\mathfrak{F}_1, \ldots, \mathfrak{F}_n$ are distinct elements of S and similarly $\mathcal{G}_1, \ldots, \mathcal{G}_n$. Choose $A_j \in \mathfrak{F}_j$ such that $A_i \cap A_j = \phi$

for $i \neq j$. Let $\pi_j \in S(X)$ be such that $\pi_j \mathcal{F}_j = \mathcal{S}_j$. We may arrange to have $\pi_j A_j \cap \pi_i A_i = \phi$ for $i \neq j$ and $|X \setminus \bigcup A_i| = |X \setminus \bigcup \pi_i A_i|$. Choose $\pi \in S(X)$ such that $\pi | A_j = \pi_j | A_j$, $j = 1, \ldots, n$. Then $\pi \mathcal{F}_j = \mathcal{S}_j$, $j = 1, \ldots, n$.

COROLLARY. *If J is a finite set of equivalent ultrafilters on X then the set of all $\pi \in S(X)$ such that $\pi J = J$ is a maximal subgroup of $S(X)$.*

COROLLARY. *There are $2^{2^{|X|}}$ maximal subgroups of $S(X)$.*

2. *Another view.* In this section we take a look at our extended class from another angle and observe some of the limitations to further extension. If $\pi \in S(X)$ we denote by *$fs\pi$* the *fixed set of* π , i.e. the elements $x \in X$ such that $\pi x = x$. We use the notation *$spt\,\pi$* to stand for the *support*, $X \setminus fs\pi$, of π .

PROPOSITION 1. *Let $\mathcal{F}$ be an ultrafilter on X . Then* $\pi \mathcal{F} = \mathcal{F} \iff fs\pi \in \mathcal{F}$.

Proof. $\Leftarrow$ If $F \in \mathcal{F}$ so is $F \cap fs\pi$. But $F \cap fs\pi \subseteq \pi F$. Thus $\pi F \in \mathcal{F}$.

$\Rightarrow$ If $fs\pi \notin \mathcal{F}$ then $spt\pi \in \mathcal{F}$. By examining the cycles of π it is clear that we can break up $spt\,\pi$ into a disjoint union $A \cup B \cup C$ where $\pi A = B$, $\pi B \subseteq C \cup A$ and $\pi C \subseteq A$. Since exactly one of A , B and C is in $\mathcal{F}$ this contradicts $\pi \mathcal{F} = \mathcal{F}$.

By an *n-partition* of X we mean a partition of X into sets of cardinality n . If $\{X_i\}$, $i \in I$, is an n-partition of X and $\mathcal{F}$ is an ultrafilter on I we can define a subgroup H of $S(X)$ as $\{\pi \in S(X) \mid \exists F \in \mathcal{F} , \pi X_i = X_i \text{ for all } i \in F\}$.

PROPOSITION 2. H *is a subgroup associated with an n-partition of* $X \iff H = \{\pi \in S(X) | \pi J = J\}$ *where* J *is a set of* n *distinct equivalent points of* βX .

Proof. $\Rightarrow$ Label the points in X_i, $x_{i1}, \ldots, x_{in}$. Let $\tau x_{ij} = x_{i(j+1)}$ (mod n in the second index). Let $\mathcal{F}_1$ be the ultrafilter on X generated by the ultrafilter induced on $\{x_{i1}\}$ by the ultrafilter $\mathcal{F}$ on I . Set $\mathcal{F}_j = \tau^{j-1} \mathcal{F}_1$ for $j = 1, \ldots, n$

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and let $J = \{ \mathcal{F}_j \}$. We shall show that $\pi J = J$ implies $\pi \in H$ and by the maximality of $\{\pi \mid \pi J = J\}$ and the properness of H we are done.

If $\pi \notin H$ then $\{i \mid \pi X_i \neq X_i\} \in \mathcal{F}$ and therefore there is a j such that $\{i \mid \pi x_{ij} \notin X_i\} \in \mathcal{F}$ and therefore is a j_o such that $\{i \mid \pi x_{ij} = x_{i_o j_o}$ for some $i_o \neq i\} = F \in \mathcal{F}$. Clearly $\pi \mathcal{F}_j = \mathcal{F}_{j_o}$. Consider the function $g: F \to I$, $g(i) = i_o$. Let $A \subseteq F$ be maximal with respect to $g(A) \cap A = \phi$. Then $F = A \cup (g(A) \cap F) \cup (g^{-1}(A) \cap F)$. Now $A \notin \mathcal{F}$ lest also $g(A) \in \mathcal{F}$. On the other hand $g(A) \notin \mathcal{F}$ because $F \setminus A$ and therefore $g(F \setminus A)$ is in $\mathcal{F}$ (note that g is 1-1) . Finally $g^{-1}(A) \notin \mathcal{F}$ lest $A \in \mathcal{F}$, completing the contradiction.

$\Leftarrow$ Let $J = \{\pi_1 \mathcal{F}, \ldots, \pi_n \mathcal{F}\}$, $\pi_j \in S(X)$, $\mathcal{F}$ an ultrafilter on X . Choose $F \in \mathcal{F}$ such that the $\pi_j F$'s are mutually disjoint and their union has infinite complement. For $x_i \in F$ set $X_i = \{\pi_1 x_i, \ldots, \pi_n x_i\}$ and enlarge to an n-partition of X . $\mathcal{F}$ induces an ultrafilter on this n-partition and we get an associated proper subgroup H . Again, it suffices to show that $\pi J = J$ implies $\pi \in H$ and the proof is the same as above.

Observe that we gain no generality by considering partitions of X into sets of cardinality $\leq m$ for if $\mathcal{F}$ is an ultrafilter on such a partition we may always find an n and an $F \in \mathcal{F}$ such that every set of the partition indexed by F is of cardinality n . On the other hand we dare not allow unbounded partitions in view of:

PROPOSITION 3. Let $\mathcal{F}$ be an ultrafilter on the index set of a partition of X such that if $F \in \mathcal{F}$ then the sets X_i , $i \in F$, are not of bounded finite cardinality. Then the subgroup $H = \{\pi \in S(X) \mid$ for some $F \in \mathcal{F}$, $\pi X_i = X_i$ for all $i \in F\}$ is not maximal.

Proof. Let $\tilde{H} = \{\pi \in S(X) \mid$ for some $F \in \mathcal{F}$ and n, $|\pi X_i + X_i| \leq n$ for all $i \in F\}$. It is readily verified that $\tilde{H}$ is a subgroup and $H \subsetneq \tilde{H} \subsetneq S(X)$.

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3. Operators on Boolean algebras. In order to extend the third class of maximal subgroups we view $S(X)$ as acting on the Boolean algebra of subsets of X modulo the ideal of subsets of cardinality less than X. This puts the notions of "almost fixing" and "almost interchanging" in their proper setting. The relevant theorem concerns maximal subgroups of groups acting on Boolean algebras. Recall that a Boolean algebra B is atomless if whenever $b \in B$ and $b \neq 0$ there exists an $a \in B$ such that $0 \lneq a \lneq b$.

THEOREM 3. *Let B be an atomless Boolean algebra and G a group of operators on B which is transitive on finite sets of nonzero disjoint elements whose supremum is 1. Let $a_1, \ldots, a_n$ be such a set, $n \geq 2$. Let $H = \{\pi \in G \mid$ for all i there exists a j, $\pi a_i = a_j\}$. Then H is a maximal subgroup of G.*

Proof. We first treat the case when $n = 2$. Let $\pi \in G \setminus H$. We wish to show that $G = \langle H, \pi \rangle$. Partition $G \setminus H$ into five classes of elements: 1. $\pi a_1 < a_1$; 2. $\pi a_1 < a_2$; 3. $\pi a_1 > a_1$; 4. $\pi a_1 > a_2$; 5. all others. If π is in a class C it is clear that $C \subseteq \langle H, \pi \rangle$. Therefore we need only show that if π is in a class C and D is any class then there is a $\tau \in D$ such that $\tau \in \langle H, \pi \rangle$. We employ a semi-circular proof.

$C = 1$, $D = 2$: Let $\tau = \sigma\pi$ where $\sigma a_1 = a_2$.

$C = 2$, $D = 3$: Let $\tau = (\sigma\pi)^{-1}$, σ as above.

$C = 3$, $D = 4$: Let $\tau = \sigma\pi$, σ as above.

$C = 4$, $D = 1$: Let $\tau = (\sigma\pi)^{-1}$, σ as above.

$C = 1$, $D = 5$: Let $\pi a_1 = x \vee y$ where $x, y \neq 0$ and $x \wedge y = 0$. Let $a_2 \wedge \pi^{-1} a_1 = z_1 \vee x_1$ where $x_1, z_1 \neq 0$ and $x_1 \wedge z_1 = 0$. Let $a_2 \setminus \pi^{-1} a_1 = z_2 \vee y_1$ where $y_1, z_2 \neq 0$ and $y_1 \wedge z_2 = 0$ (these steps are justified by the fact that B is atomless). Choose σ such that $\sigma x = x_1$, $\sigma y = y_1$, $\sigma(a_1 \setminus \pi a_1) = z_1 \vee z_2$ and $\sigma a_2 = a_1$. Set $\tau = \pi\sigma\pi$.

$C = 5$, $D = 1$: Let $x_i, y_i, z_i, u_i \neq 0$ be such that $u_1 \wedge u_2 = 0$, $y_2 \wedge z_2 = 0$, $\pi a_1 = x_1 \vee y_1$, $x_1 < a_1$, $y_1 < a_2$, $\pi^{-1} a_1 = x_2 \vee y_2 \vee z_2$, $x_2 < a_1$, $(y_2 \vee z_2) < a_2$, $a_2 \setminus y_1 = u_1 \vee u_2$. Choose σ such that $\sigma x_1 = x_2$, $\sigma y_1 = y_2$, $\sigma u_1 = z_2$, $\sigma u_2 = a_2 \setminus (y_2 \vee z_2)$, $\sigma(a_1 \setminus x_1) = a_1 \setminus x_2$.

Note that $\sigma a_1 = a_1$. Set $\tau = \pi\sigma\pi$.

For $n > 2$ the proof breaks into two parts:

(i) If $\pi \notin H$ then there is a $\tau \in \langle H, \pi \rangle$ such that $\tau \notin H$ and $\tau a_i = a_j$ for some i and j (and we may just as well assume $i = j = 1$).

(ii) If τ is as above then, by induction, $\langle H, \tau \rangle$ contains all σ such that for some i and j, $\sigma a_i = a_j$. Under this assumption we show that any π is an $\langle H, \tau \rangle$.

(i) Proof. $\pi^{-1} a_j$ has nontrivial meet with two a_i's for some j lest $\pi \in H$. Without loss of generality we may assume that πa_2 and πa_3 have nontrivial meets with a_1. Let $a_1 \wedge \pi a_2 = x \vee y$, $a_1 \wedge \pi a_3 = z \vee w$, $x, y, z, w, \neq 0$, $x \wedge y = z \wedge w = 0$. Let σ fix all elements $a_i \wedge \pi a_j$, $(i, j) \neq (1, 2), (1, 3)$, interchange x and z and fix y and w. Set $\tau = \pi^{-1}\sigma\pi$. Then $\sigma \in H$ and $\tau a_1 = a_1$ but τa_2 meets a_2 and a_3 and hence $\tau \notin H$.

(ii) Proof. Let $x_{ij} = \pi^{-1} a_j \wedge a_i$, $y_{ij} = a_j \wedge \pi a_i = \pi x_{ij}$. Observe that $a_i = \bigvee_j x_{ij}$ and $\pi^{-1} a_j = \bigvee_i x_{ij}$.

(1) We may assume that $\pi a_1 \nleq a_1$ and $\pi a_1 \ngeq a_1$. For consider πa_j vis à vis a_1. We cannot have $\pi a_j \leq a_1$ for all j. If $\pi a_j \geq a_1$ simply choose some other j. We may assume that $j = 1$ since it will suffice to prove $\pi\sigma$ is in $\langle H, \tau \rangle$ where $\sigma a_1 = a_j$.

(2) We may assume that $x_{22} \neq 0$. For if $\pi a_j \leq a_1$ for all $j \neq 1$, reverse the roles of 1 and 2. Otherwise use suitable interchanges fixing a_1.

(3) Choose λ such that

$$\lambda x_{ij} = y_{ij} \qquad \text{for } i, j \neq 1 \quad (i, j) \neq (2, 2)$$

$$\lambda y_{i1} = y_{i1}$$

$$\bigvee_{i \neq 1} \lambda x_{i1} = y_{22}$$

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$$\lambda x_{22} = \bigvee_{j \neq 1} y_{1j}$$

Note that $\lambda a_1 = a_1$.

(4) Consider $\sigma = \lambda^{-1} \pi$.

$$\sigma x_{ij} = x_{ij} \qquad \text{for } i, j \neq 1 \qquad (1, j) \neq (2, 2)$$

$$\sigma x_{i1} = y_{i1}$$

$$\sigma x_{22} = \bigvee_{i \neq 1} x_{i1}$$

$$\sigma x_{1j} \leq x_{22} \qquad j \neq 1 .$$

We can conclude

$$\sigma a_1 \leq a_1 \vee a_2$$

$$\sigma a_i \leq a_1 \vee a_i \qquad i \neq 1, 2 .$$

If $\sigma a_1 \neq a_1 \vee a_2$ we can find ρ fixing $a_3, \ldots, a_n$ such that $\rho \sigma a_1 = a_1$. If $\sigma a_1 = a_1 \vee a_2$ then $\sigma a_3 \neq a_1 \vee a_3$ and so we can find ρ fixing $a_2, a_4, \ldots, a_n$ such that $\rho \sigma a_3 = a_3$. In any event $\rho \lambda^{-1} \pi$ fixes an a_j as do ρ and λ .

COROLLARY. _If X is partitioned into a finite number of sets of equal cardinality then the subgroup of_ S(X) _which almost permutes these sets is maximal._

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