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A NOTE ON SOME PRIME HAUSDORFF METHODS OF SUMMABILITY

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Given a matrix $A=(a_{nk})$ $(n, k=0, 1, 2, \ldots)$, let (A) denote the set of all sequences $x=\{x_k\}$ such that $\{A_n(x)\} \in c$ where $A_n(x)=\sum_{k=0}^{\infty} a_{nk}x_k$ $(n\geq 0)$ and c denotes the set of all convergent sequences. It is well known (see e.g. Zeller [6] or Zeller and Beekmann [7], p. 48) that given an unbounded sequence x, there exists a regular (=permanent) matrix A with $a_{nk}=0$ for $k>n$ (and indeed with $a_{nn}\neq 0$) such that $(A)=c\oplus x$, the linear space spanned by c and x. We call A an Einfolgenverfahren. (See [7].) In [4] Rhoades considered, inconclusively, the question whether there exists a Hausdorff matrix H such that $(H)=c\oplus x$ (for arbitrary unbounded sequence x). The present author showed in [3] that there are many sequences x for which there exist no Hausdorff methods H with $(H)=c\oplus x$ and suggested the possibility that there exist pairs $[H, x]$, x unbounded and H Hausdorff, such that $(H)=c\oplus x$. Rhoades [5] settled this question by proving the following result.

THEOREM. *Let H_μ be the Hausdorff method defined by the moment sequence* $\{\mu_n\}$ where $\mu_n=(n-a/n+1)(\mathscr{Re}\ a>0)$ *and let $x=\{x_n\}$ be the sequence*

$$x_n = \frac{\Gamma(n+1)}{\Gamma(n-a+1)},$$

where we set $x_n=0$ if $n-a+1$ is 0 or a negative integer. Then $(H_\mu)=c\oplus x$. {Note that if $\nu_n=-\mu_n/a$, then H_ν is a regular Hausdorff matrix with $(H_\nu)=(H_\mu)=c\oplus x$.} By well known elementary properties of Hausdorff methods, if $\lambda=\{\lambda_n\}$ is the moment sequence where $\lambda_n=(n-a/n+b)$, $\mathscr{Re}\ a>0$, $\mathscr{Re}\ b>0$, then $(H_\lambda)=(H_\mu)$ and hence $(H_\lambda)=c\oplus x$ by the Theorem. This shows that all the known primes in the Banach algebra of all multiplicative Hausdorff methods are in fact Einfolgenverfahren. {For the definitions, terminology and classic results, see Hardy [1] *or Zeller and Beekmann* [7].}

Rhoades' proof of the Theorem depends on deep results on Hausdorff methods as well as on Zeller's technique for constructing Einfolgenverfahren. He uses the former to prove that $(H_\mu)\supseteq c\oplus x$ and the latter to obtain a regular A with $(A)=c\oplus x$; he then shows that $(A)\supseteq(H_\mu)$. In the present note we give a short alternative proof which is both simple and direct.

Proof of the Theorem. We have

$$\mu_n = \frac{n-a}{n+1} = 1-\frac{a+1}{n+1}.$$

Hence H_μ is in fact the matrix method $H_\mu = I-(a+1)C_1$ where I is the identity and C_1 is the Cesàro matrix of order 1. Now, as explicitly stated in [2], it is easy to see from Hardy's proof (see [1], Theorem 52) of Mercer's theorem that $u=\{u_n\} \in (H_\mu)$ implies $u_n = Kx_n + s_n$ where K is a constant and $\{s_n\} \in c$; i.e. $u \in c \oplus x$. Thus, $(H_\mu) \subseteq c \oplus x$. We prove the reverse inclusion relation by direct calculation as follows. Let $t=\{t_n\}=H_\mu x = [I-(a+1)C_1](x)$. Then

$$t_n = \frac{\Gamma(n+1)}{\Gamma(n+1-a)} - \frac{a+1}{n+1}\sum_{\nu=0}^{n} \frac{\Gamma(\nu+1)}{\Gamma(\nu+1-a)}. \tag{1}$$

Now, if a is not a positive integer, then

$$\begin{aligned}\frac{\Gamma(\nu+1)}{\Gamma(\nu+1-a)} &= \frac{(\nu+1)\Gamma(\nu+1)-\nu\Gamma(\nu+1)}{\Gamma(\nu+1-a)}\\ &= \frac{\Gamma(\nu+2)}{\Gamma(\nu+1-a)} - \frac{(\nu-a+a)\Gamma(\nu+1)}{\Gamma(\nu+1-a)}\\ &= \frac{\Gamma(\nu+2)}{\Gamma(\nu+1-a)} - \frac{\Gamma(\nu+1)}{\Gamma(\nu-a)} - \frac{a\Gamma(\nu+1)}{\Gamma(\nu+1-a)}. \qquad (2)\end{aligned}$$

Hence

$$t_n = \frac{\Gamma(n+1)}{\Gamma(n+1-a)} - \frac{a+1}{n+1}\sum_{\nu=0}^{n}\frac{1}{a+1}\left\{\frac{\Gamma(\nu+2)}{\Gamma(\nu+1-a)} - \frac{\Gamma(\nu+1)}{\Gamma(\nu-a)}\right\} \tag{3}$$

$$= \frac{\Gamma(n+1)}{\Gamma(n+1-a)} - \frac{1}{n+1}\left\{\frac{\Gamma(n+2)}{\Gamma(n+1-a)} - \frac{\Gamma(1)}{\Gamma(-a)}\right\} \tag{4}$$

$$= 0 + o(1) = o(1).$$

Thus, $(H_\mu) \supseteq c\oplus x$, if a is not a positive integer. If a is a positive integer then $x_\nu = 0$ for $\nu < a$ and $x_a = a+1$; hence $t_n = 0$ for $n \le a$. So for $n > a$ we write $\sum_0^n x_\nu$ as $x_a + \sum_{a+1}^n x_\nu$. Then the symbol $\sum_{\nu=0}^n$ in (3) will be replaced by $x_a + \sum_{\nu=a+1}^n$ and the symbol $\Gamma(1)/\Gamma(-a)$ in (4) by 0. We see thus that if a is a positive integer, then $t_n = 0$ for all n. Thus whatever be a with $\mathscr{Re}\, a > 0$, we have $(H_\mu) \supseteq c \oplus x$. This completes the proof of the theorem.

References

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