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OVERCONVERGENT DE RHAM EICHLER–SHIMURA MORPHISMS

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Contents

1. Introduction

This article represents our attempt to improve the previous results on defining and understanding overconvergent Eichler–Shimura maps in [5] and [6].

We fix a prime integer $p > 2$. We recall that an (overconvergent) Eichler–Shimura morphism is a comparison map describing weight k overconvergent modular symbols, seen as pro-Kummer étale cohomology classes of a sheaf of weight k distributions $\mathbb{D}_k$ (where $k\colon \mathbb{Z}_p^* \to B^*$ is a B-valued weight as in Definition 3.6) tensored some period ring, in terms of overconvergent modular forms of weight $k+2$, tensored with the same period ring. In [5] and in [6], we defined and studied Hodge-Tate Eichler-Shimura maps while in this article we'll have Hodge–Tate, de Rham and crystalline variants.

To really explain what the main issues are that we deal with in this article, let us observe that there has been remarkable recent progress in p-adic Hodge theory and especially in integral p-adic Hodge theory, and let us just mention [8], [9], [11], [14]. The articles quoted here deal with various cohomology theories on formal schemes or adic spaces with constant coefficients. On the other hand, it has been clear for some time that for applications to p-adic automorphic forms one needs to work with cohomology with very large coefficients. In this article, we try to understand p-adic Hodge theory (comparison morphisms really) with large coefficients, and therefore, unfortunately, we cannot use the recent results quoted above.

More precisely, let $\mathcal{X} := \mathcal{X}_0(p^m,N)$ be the log adic space defined by the modular curve over $\mathbb{Q}_p$ associated to the congruence subgroup $\Gamma_0(p^m)\cap\Gamma_1(N)$. For any $r \geq 0$, we have open subspaces $\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big)$, where Ha is a (any) local lift of the Hasse invariant. We fix k,B as above and let $h \in \mathbb{N}$. We denote by ω_E the sheaf of invariant differentials of the universal semiabelian scheme E over $\mathcal{X}$ and by $\mathcal{X}_{\mathrm{pke}}$, the log adic space $\mathcal{X}$ equipped with the pro-Kummer étale topology, see section §2.2. We let $\mathbb{D}_k$ denote the pro-Kummer étale sheaf of weight k-modular symbols on this $\mathcal{X}_{\mathrm{pke}}$, and we recall from [6] that both the pro-Kummer étale cohomology and the sheaf cohomology groups $\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k(1)\big)$ and $\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big),\omega_E^{k+2}\big)$ have finite slope decompositions for the action of the compact operator U_p. If $h \in \mathbb{N}$ is a slope, we denote, for a Hecke module M by $M^{(h)}$ the submodule of slope $\leq h$ submodule of M.

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Then, Theorem 5.1 states that there is an r depending on k, and therefore a neighbourhood $\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big)$ of a component of the ordinary locus in $\mathcal{X}$, and a canonical $\mathbb{C}_p$-linear, Galois and Hecke equivariant map:

$$\Psi_{\mathrm{HT,h}}\colon \mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k(1)\big)^{(h)}\widehat{\otimes}\mathbb{C}_p \longrightarrow \mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big),\omega_E^{k+2}\big)^{(h)}\widehat{\otimes}\mathbb{C}_p.$$

In a slightly different way and only for analytic weights, the map $\Psi_{\mathrm{HT,h}}$ was constructed in [5]. There we also proved that the map is generically surjective. The new result in this paper is:

Theorem 1.1. *If* $\prod_{i=0}^{h-1}(u_k-i)\in\big(B[1/p]\big)^*$, *then* $\Psi_{\mathrm{HT,h}}$ *is surjective, for all* $h\geq 1$, *and it is surjective if* $h=0$.

Our next result in this article is a de Rham overconvergent Eichler–Shimura map. We fix now, as in section §6, $\mathcal{X}:=\mathcal{X}(N)$ the modular curve with full level N structure for the remainder of this introduction. We construct modular sheaves with connections and filtrations $\mathbf{W}_{k,\mathrm{dR}}$ on $\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big)$, for an $r\geq 0$ depending on k, which interpolate p-adically the family of sheaves $\{\mathrm{Sym}^v\mathrm{H}^1_{\mathrm{dR}}\big(E/\mathcal{X}\big)\}_{v\in\mathbb{N}}$, with their filtrations and connections, and we denote by $\mathbf{W}_{k,\mathrm{dR},\bullet}\colon \mathbf{W}_{k,\mathrm{dR}}\xrightarrow{\nabla}\mathbf{W}_{k+2,\mathrm{dR}}$ the de Rham complex of $(\mathbf{W}_{k,\mathrm{dR}},\nabla)$. Here, of course, we use the Kodaira–Spencer isomorphism in order to see the connection as a morphism of abelian sheaves with values in $\mathbf{W}_{k+2,\mathrm{dR}}$. Assuming the hypothesis and notations above we prove:

Theorem 1.2. a) *There is a natural, Galois and Hecke equivariant* $B\widehat{\otimes}B^+_{\mathrm{dR}}$*-semilinear map*

$$(*)\qquad \rho_k\colon \mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}_k\big)^{(h)}\widehat{\otimes}B^+_{\mathrm{dR}}\longrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big),\mathbf{W}_{k,\mathrm{dR},\bullet}\big)^{(h)}\widehat{\otimes}\mathrm{Fil}^{-1}B_{\mathrm{dR}},$$

b) *If* $\prod_{i=0}^{h-1}(u_k-i)\in\big(B[1/p]\big)^*$ *then the display* $(*)$ *above becomes:*

$$\rho_k\colon \mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}_k\big)^{(h)}\widehat{\otimes}B^+_{\mathrm{dR}}\longrightarrow \mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big),\omega_E^{k+2}\big)^{(h)}\widehat{\otimes}\mathrm{Fil}^{-1}B_{\mathrm{dR}}$$

and it is surjective.

In order to make it clear what improvements we were able to produce in this article, we now list the new ideas.

1) *Neighbourhoods of the ordinary loci in modular curves.*

Both in [5] and [6], we worked on the (log) adic modular curves $\mathcal{X}_1(N)$ and $\mathcal{X}_0(p,N)$; these are the (log) adic spaces associated to the modular curves over $\mathbb{Q}_p$ of level $\Gamma_1(N)$ and, respectively $\Gamma_1(N)\cap\Gamma_0(p)$, which have a connected, respectively two connected, components of ordinary loci. We worked with strict neighbourhoods of these ordinary loci of depth $n\in\mathbb{N}$ defined as the points x with the property $v_x(\mathrm{Ha})\leq 1/n$. These neighbourhoods are defined over $\mathrm{Spa}(L,\mathcal{O}_L)$ over which the point x is defined, where L is some complete extension of $\mathbb{Q}_p$ and these neighbourhoods are also used in this very article for the de Rham Eichler–Shimura maps.

For the Hodge–Tate comparison maps in this article, we use a better technology, inspired by the work of [12]. Namely, let $\mathcal{X}(p^\infty,N)$ be the perfectoid adic space over $\mathrm{Spa}(\mathbb{C}_p,\mathcal{O}_{\mathbb{C}_p})$

associated to the projective limit of adic modular curves $\lim\limits_{\leftarrow,m}\mathcal{X}(p^m,N)$ and the Hodge–Tate period map

$$\pi_{\mathrm{HT}}\colon \mathcal{X}(p^\infty,N)\longrightarrow \mathbb{P}^1_{\mathbb{Q}_p}.$$

We define interesting opens $U^{(n)}_{\#}\subset\mathbb{P}^1_{\mathbb{Q}_p}$, for $n\geq 1$ and the symbol $\#\in\{0,\infty\}$, which are invariant under the action of the m-th Iwahori subgroup $\mathrm{Iw}_m\subset\mathbf{GL}_2(\mathbb{Z}_p)$ as follows: If $\#=\infty$, then $m\geq n$ if $\#=0$, then $m\geq 1$ and on which we understand the dynamic of the U_p-operator. Then by the properties of π_{HT}, for every $n\geq 1$, there are: an $m\geq 1$ as above and neighbourhoods of the ordinary loci in $\mathcal{X}_0(p^m,N)$ denoted $\mathcal{Z}^{(n)}_0,\mathcal{Z}^{(n)}_\infty$ such that if $\pi_m\colon\mathcal{X}(p^\infty,N)\to\mathcal{X}_0(p^m,N)$ is the natural projection, then $\pi^{-1}_{\mathrm{HT}}(U^{(n)}_{\#})=\pi_m^{-1}(\mathcal{Z}^{(n)}_{\#})$, where $\#\in\{0,\infty\}$ such that we understand well the dynamic of the U_p-operator on sections of modular sheaves on $\mathcal{X}_0(p^m,N)$. We remark that $\mathcal{X}_0(p^m,N)$ has many connected components of the ordinary locus if m is large and a complicated semistable integral model; therefore, it would have been difficult to apply the previous method, that is, defining neighbourhoods of the ordinary loci using Ha, in $\mathcal{X}_0(p^m,N)$ for $m>1$.

2) *Payman Kassaei's method for the cohomology of pro-Kummer étale sheaves.*

Let us now explain our new take on the overconvergent Hodge–Tate Eichler–Shimura morphism. We fix a slope $h\in\mathbb{N}$ and a weight $k\colon\mathbb{Z}_p^*\longrightarrow B^*$ as in Definition 3.6. This weight is N-analytic, for some $N\in\mathbb{N}$, that is, there is $u_k\in B[1/p]$ such that $k(t)=\exp\big(u_k\log(t)\big)$ for all $t\in 1+p^N\mathbb{Z}_p$. These data determine integers n,u,m such that on $\mathcal{X}:=\mathcal{X}_0(p^m,N)$ we have our neighbourhoods $\mathcal{Z}^{(u)}_\infty$ for $u\leq m$ and $\mathcal{Z}^{(n)}_0,\mathcal{Z}^{(n+1)}_0$. We base-change $\mathcal{X},\mathcal{Z}^{(u)}_\infty,\mathcal{Z}^{(n)}_0,\mathcal{Z}^{(n+1)}_0$ over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ to $\mathrm{Spa}(B[1/p],B)$ and still denote them $\mathcal{X},\mathcal{Z}^{(u)}_\infty,\mathcal{Z}^{(n)}_0,\mathcal{Z}^{(n+1)}_0$. We let $\mathbb{D}^o_k$ be the integral sheaf of weight k-distributions, seen as a pro-Kummer étale sheaf on $\mathcal{X}$, and denote by $\mathbb{D}_k:=\mathbb{D}^o_k\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

First, let us recall that the map $\Psi_{\mathrm{HT,h}}$ appears, after passing to the the open subspaces defined in (1), as the following composition:

$$\begin{aligned}\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k(1)\big)^{(h)}\widehat{\otimes}\mathbb{C}_p &\cong \Big(\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathfrak{D}_k(1)\big)[1/p]\Big)^{(h)}\\ &\xrightarrow{\mathcal{R}} \Big(\mathrm{H}^1\big((\mathcal{Z}^{(u)}_\infty)_{\mathrm{pke}},\mathfrak{D}_k(1)\big)[1/p]\Big)^{(h)} \xrightarrow{\Phi}\\ &\xrightarrow{\Phi} \mathrm{H}^0\big(\mathcal{Z}^{(u)}_\infty,\omega_E^{k+2}\big)^{(h)}\widehat{\otimes}\mathbb{C}_p,\end{aligned}$$

where $\mathfrak{D}_k:=\mathbb{D}_k\widehat{\otimes}\mathcal{O}_{\mathcal{Z}_{\mathrm{pke}}}$, $\mathcal{R}$ is the restriction map and Φ was defined in [6] and in 4.15, and it was proved in loc. cit. that it is an isomorphism. Therefore, in order to prove Theorem 5.1, we need to show that restriction from $\mathcal{Z}_{\mathrm{pke}}$ to $(\mathcal{Z}^{(u)}_\infty)_{\mathrm{pke}}$ induces a surjective map on the H^1's if $\prod_{i=0}^{h-1}(u_k-i)$ is a unit in $B[1/p]$.

To do this, we use Payman Kassaei's idea of proving classicity of overconvergent modular forms of integral weight and small slope. More precisely, given $x\in\mathrm{H}^1\big((\mathcal{Z}^{(u)}_\infty)_{\mathrm{pke}},\mathfrak{D}_k\big)^{(h)}$, we may see it as an element of $\mathrm{H}^1\big((\mathcal{Z}^{(u)}_\infty)_{\mathrm{pke}},\mathfrak{D}^o_k\ \big)$ which is annihilated by $Q(U_p)$, where $Q(T)\in(B\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})[T]$ is a polynomial all of whose roots have valuations $\leq h$. We write

$Q(T) = P(T) - \alpha$, with $P(0) = 0$, and denote by a the valuation of α. Then by applying to x the operator $\big(P(U_p)\big)^{n+u+1}$ we can see it as a class $\tilde{x}$ in $\mathrm{H}^1\big((\mathcal{X}\backslash\mathcal{Z}_0^{(n+1)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big)$. On the other hand, following Kassaei, we can define a new operator $\big(P(U_p)^n\big)^{\mathrm{good}}$ by choosing all the isogenies defining the correspondence U_p^n which map $\mathcal{Z}_0^{(n)}$ to $\mathcal{Z}_\infty^{(1)}$. Let $\mathcal{P}(x) := \big(P(U_p)^n\big)^{\mathrm{good}}\big(P(U_p)^{u+1}(x)\big) \in \mathrm{H}^1\big(\mathcal{Z}_0^{(n)}, \mathfrak{D}_k^o\big)$. As the family $\{\mathcal{X}\backslash\mathcal{Z}_0^{(n+1)}, \mathcal{Z}_0^{(n)}\}$ is an open covering of $\mathcal{X}$, one can use a Mayer–Vietoris sequence in order to glue $p^s\tilde{x}, p^s\mathcal{P}(x)$ for a certain fixed power of p, s modulo p^r, where r was chosen in the beginning large enough so that $r \geq 2(s+d+1+(u+n+1)a)$ for a certain constant d (see Section §5). We obtain a class $z \in \mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}}, \mathfrak{D}_k^0\big)$ annihilated by $Q(U_p)$ and such that its restriction to $\mathcal{Z}_\infty^{(n)}$ is congruent to $p^{s+d+1}\alpha^{u+n+1}x$ modulo p^r, that is, there is $x_1 \in \mathrm{H}^1\big((\mathcal{Z}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big)$ annihilated by $Q(U_p)$ such that $\mathcal{R}(z) = p^{s+d+1}\alpha^{n+u+1}(x - p^{r/2}x_1)$. Now, we iterate the process for x_1 and in the end obtain an element $y \in \mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}}, \mathfrak{D}_k\big)^{(h)}$ such that $\mathcal{R}(y) = x$.

3) *On the de Rham comparison.*

It is interesting to note, about the de Rham Eichler–Shimura map ρ_k in theorem 1.2, the 'decalage' between the filtrations on B_{dR} that appear. This decalage is explained as follows: On the pro-Kummer étale site of $\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big)$, we have the sheaves with filtrations and connections: $\nabla' : \mathbf{W}_{k,\mathrm{dR}}\widehat{\otimes}\mathcal{O}\mathbb{B}_{\mathrm{dR}} \longrightarrow \mathbf{W}_{k+2,\mathrm{dR}}\widehat{\otimes}\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ (see Section §6.4 for the details), where $\nabla' = \nabla_k\widehat{\otimes}1 + 1\widehat{\otimes}\nabla_{\mathrm{dR}}$ and $\mathbf{W}_{k,\mathrm{dR}}$ has an increasing, infinite filtration, while $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ has a decreasing, infinite filtration. Both ∇_k and ∇_{dR} satisfy the Griffith transversality property with respect to the respective filtrations, but on the tensor product, we don't have a natural filtration. We have, however, the following fact:

$$\nabla' : \mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR}}\widehat{\otimes}\mathrm{Fil}^0\mathcal{O}\mathbb{B}_{\mathrm{dR}} \longrightarrow \mathrm{Fil}_{m+1}\mathbf{W}_{k+2,\mathrm{dR}}\widehat{\otimes}\mathrm{Fil}^{-1}\mathcal{O}\mathbb{B}_{\mathrm{dR}}.$$

This explains the decalage.

As an immediate consequence of the above theorem, we have a 'big exponential map'. More precisely, let K be the finite extension of $\mathbb{Q}_p$ over which $\mathcal{X}$ and $\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big)$ are both defined, and let G denote the absolute Galois group of K for a fixed algebraic closure $\overline{K}$ of K.

Then we have a Hecke equivariant, B-linear map

$$\mathrm{Exp}_k^* : \mathrm{H}^1\left(G, \mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}_k(1)\big)^{(h)}\right) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big), \mathbf{W}_{k,\mathrm{dR},\bullet}\big)^{(h)},$$

which has the property that, for every classical weight k_0-specialization, it is compatible with the classical dual exponential map, as follows:

a) If $k_0 > h-1$, that is, k_0 is a noncritical weight for the slope h, then we have the following commutative diagram with horizontal isomorphisms. Here, we denoted by $\exp_{k_0}^*$ the Kato dual exponential map associated to weight k_0 modular forms.

$$\begin{array}{ccc}
\left(\mathrm{H}^1\left(G, \mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}_k(1)\big)^{(h)}\right)\right)_{k_0} & \overset{\left(\mathrm{Exp}_k^*\right)_{k_0}}{\longrightarrow} & \left(\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^r}\big), \mathbf{W}_{k,\mathrm{dR},\bullet}\big)^{(h)}\right)_{k_0} \\
\downarrow\cong & & \downarrow\cong \\
\mathrm{H}^1\left(G, \mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathrm{Sym}^{k_0}(T_p(E)^\vee)(1)\big)^{(h)}\right) & \overset{\exp_{k_0}^*}{\longrightarrow} & \mathrm{Fil}^0\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}, \mathrm{Sym}^{k_0}(\mathrm{H}_E)\big)^{(h)}.
\end{array}$$

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b) If $0 \le k_0 \le h+1$, that is, k_0 is critical with respect to b, we only have a commutative diagram of the form

$$\begin{array}{ccc} \left(\mathrm{H}^1\left(G,\mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}_k(1)\right)^{(h)}\right)\right)_{k_0} & \xrightarrow{\left(\mathrm{Exp}_k^*\right)_{k_0}} & \left(\mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}\left(p/\mathrm{Ha}^{p^r}\right),\mathbf{W}_{k,\mathrm{dR},\bullet}\right)^{(h)}\right)_{k_0} \\ \downarrow & & \uparrow \\ \mathrm{H}^1\left(G,\mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathrm{Sym}^{k_0}(T_p(E)^\vee)(1)\right)^{(h)}\right) & \xrightarrow{\exp_{k_0}^*} & \mathrm{Fil}^0\mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X},\mathrm{Sym}^{k_0}(\mathrm{H}_E)\right)^{(h)}, \end{array}$$

where the right vertical arrow is induced by restriction.

2. Preliminaries

We will denote by X, Y, Z, ... log schemes and by caligraphic letters $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{Z}$, ... log adic spaces. We refer to [15] for generalities on those.

2.1. Pro-Kummer étale site

Given a finite saturated (for short 'fs') locally noetherian log scheme X (resp. an fs locally noetherian log adic space $\mathcal{X}$) we denote by X_{ket}, X_{fket} (resp. $\mathcal{X}_{\mathrm{ket}}$, $\mathcal{X}_{\mathrm{fket}}$) the Kummer étale site, respectively the finite Kummer étale site (see [18, Def. 2.1], [15, Def. 4.1.2]). Following Scholze [22], we denote by X_{pke}, resp. X_{profket} (resp. $\mathcal{X}_{\mathrm{pke}}$, $\mathcal{X}_{\mathrm{profket}}$) the pro-Kummer étale site, resp. the pro-finite Kummer étale site (see [15, Def. 5.1.2 & 5.1.9]) of X, respectively $\mathcal{X}$.

As a category, it is the full subcategory of pro-X_{ket}, resp. pro-X_{fket} (resp. pro-$\mathcal{X}_{\mathrm{ket}}$, pro-$\mathcal{X}_{\mathrm{fket}}$) of pro-objects that are pro-Kummer étale over X, resp. pro-finite Kummer étale over X (resp. $\mathcal{X}$), that is, objects that are equivalent to cofiltered systems $\varprojlim Z_i$ such that $Z_i \to X$ is Kummer étale, resp. finite Kummer étale, for every i and there exists an index i_0 such that $Z_j \to Z_i$ is finite Kummer étale and surjective for i and $j \ge i_0$ (and similarly for $\mathcal{X}$). For the covering families we refer to loc. cit.

We have a natural projection $\nu\colon X_{\mathrm{pke}} \to X_{\mathrm{ket}}$ (resp. $\nu\colon \mathcal{X}_{\mathrm{pke}} \to \mathcal{X}_{\mathrm{ket}}$) sending $U \in X_{\mathrm{ket}}$ (resp. in $\mathcal{X}_{\mathrm{ket}}$) to the constant inverse system defined by U. Then, by [15, Prop. 5.1.6 & 5.1.7] for every sheaf of abelian groups $\mathcal{F}$ on X_{ket} (resp. in $\mathcal{X}_{\mathrm{ket}}$) and any quasi-compact and quasi-separated object $U = \varprojlim U_j$ in X_{pke} (resp. in $\mathcal{X}_{\mathrm{pke}}$), we have natural isomorphisms of δ-functors

$$\mathrm{H}^i\left(U_{\mathrm{pke}},\nu^{-1}(\mathcal{F})\right) \cong \varinjlim \mathrm{H}^i\left(U_{j,\mathrm{ket}},\mathcal{F}\right), \qquad \mathcal{F} \to \mathrm{R}\nu_*\nu^{-1}\left(\mathcal{F}\right).$$

2.2. Sheaves on the pro-Kummer étale site

We then have the following sheaves on $\mathcal{X}_{\mathrm{pke}}$ defined in [15, Def. 5.4.1] and in [16, Def. 2.2.3] following [22, Def. 6.1]:

i. The structure sheaf $\mathcal{O}_{\mathcal{X}_{\mathrm{pke}}} := \nu^{-1}\left(\mathcal{O}_{\mathcal{X}_{\mathrm{ket}}}\right)$ and its subsheaf of integral elements $\mathcal{O}^+_{\mathcal{X}_{\mathrm{pke}}} := \nu^{-1}\left(\mathcal{O}^+_{\mathcal{X}_{\mathrm{ket}}}\right)$. It comes endowed with a morphism of sheaves of multiplicative monoids $\alpha\colon \mathcal{M} \to \mathcal{O}_{\mathcal{X}_{\mathrm{pke}}}$ defined by taking ν^{-1} of the morphism of sheaves of multiplicative monoids defining the log structure on $\mathcal{X}$.

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ii. The completed sheaf $\widehat{\mathcal{O}}^+_{\mathcal{X}_{\mathrm{pke}}} := \lim_{\infty \leftarrow n} \mathcal{O}^+_{\mathcal{X}_{\mathrm{pke}}}/p^n\mathcal{O}^+_{\mathcal{X}_{\mathrm{pke}}}$ and the completed structure sheaf $\widehat{\mathcal{O}}_{\mathcal{X}_{\mathrm{pke}}} := \widehat{\mathcal{O}}^+_{\mathcal{X}_{\mathrm{pke}}}\left[\frac{1}{p}\right]$.

iii. Let K be a perfectoid field of characteristic 0 with an open and bounded subring K^+. Assume that $\mathcal{X}$ is defined over $\mathrm{Spa}(K,K^+)$. Then we have the tilted integral structure sheaf $\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat_{\mathrm{pke}}} := \varprojlim_{\varphi} \mathcal{O}^+_{\mathcal{X}_{\mathrm{pke}}}/p\mathcal{O}^+_{\mathcal{X}_{\mathrm{pke}}}$ and the tilted structure sheaf $\widehat{\mathcal{O}}_{\mathcal{X}^\flat_{\mathrm{pke}}} := \widehat{\mathcal{O}}^+_{\mathcal{X}^\flat_{\mathrm{pke}}} \otimes_{K^{\flat+}} K^\flat$. It comes endowed with a morphism of monoids $\alpha^\flat \colon \mathcal{M}^\flat \to \widehat{\mathcal{O}}_{\mathcal{X}^\flat_{\mathrm{pke}}}$, where $\mathcal{M}^\flat$ is the inverse limit $\varprojlim \mathcal{M}$ indexed by $\mathbb{N}$ with transition maps given by raising to the p-th power, $\widehat{\mathcal{O}}_{\mathcal{X}^\flat}$ is identified as a sheaf of mutiplicative monoids with the inverse limit $\varprojlim \widehat{\mathcal{O}}_{\mathcal{X}_{\mathrm{pke}}}$ indexed by $\mathbb{N}$ with transition maps given by rasing to the p-th power and the map $\alpha^\flat$ is the inverse limit of the maps α composed with the natural maps $\mathcal{O}_{\mathcal{X}_{\mathrm{pke}}} \to \widehat{\mathcal{O}}_{\mathcal{X}_{\mathrm{pke}}}$.

iv. The period sheaf $\mathbb{A}_{\mathrm{inf}} := \mathrm{W}\big(\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat_{\mathrm{pke}}}\big)$ and the period map $\vartheta \colon \mathbb{A}_{\mathrm{inf}} \to \widehat{\mathcal{O}}^+_{\mathcal{X}_{\mathrm{pke}}}$.

2.3. Log affinoid perfectoid opens

Consider a locally noetherian fs log adic space $\mathcal{X}$ over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$. Following [15, Def. 5.3.1 & Rmk. 5.3.2], an object $U = \lim_{i\in I} U_i$, with $U_i = \big(\mathrm{Spa}(R_i,R_i^+),\mathcal{M}_i\big)$ in $\mathcal{X}_{\mathrm{pke}}$ is called *log affinoid perfectoid* if:

a. There is an initial object $0 \in I$.

b. Each U_i admits a global sharp finite saturated chart P_i such that each transition map $U_j \to U_i$ is modeled on the Kummer chart $P_i \to P_j$;

c. $\big(\mathrm{Spa}(R_i,R_i^+)\big)_i$ is affinoid perfectoid, that is, the p-adic completion (R,R^+) of $\lim_i (R_i,R_i^+)$ is a perfectoid affinoid $\mathbb{Q}_p$-algebra;

d. The monoid $P = \lim_i P_i$ is n-divisible for all n.

Given a log affinoid perfectoid U as above, we denote by $\widehat{U} := \mathrm{Spa}\big(R,R^+\big)$ the associated perfectoid affinoid space. By [15, Lemma 5.3.6], it has the same underlying topological space as U (which is defined as the inverse limit of topological spaces $\varprojlim_i |U_i|$). Moreover, by [15, Thm. 5.4.3] and [22, Thm. 6.5], we have that

$$\widehat{\mathcal{O}}^+_{\mathcal{X}_{\mathrm{pke}}}(U) = R^+, \quad \widehat{\mathcal{O}}_{\mathcal{X}_{\mathrm{pke}}}(U) = R, \quad \widehat{\mathcal{O}}^+_{\mathcal{X}^\flat_{\mathrm{pke}}}(U) = R^{\flat+},$$
$$\widehat{\mathcal{O}}_{\mathcal{X}^\flat_{\mathrm{pke}}}(U) = R^\flat, \quad \mathbb{A}_{\mathrm{inf}}(U) = W\big(R^{\flat+}\big)$$

and the cohomology groups

$$\mathrm{H}^i\big(U,\widehat{\mathcal{O}}^+_{\mathcal{X}_{\mathrm{pke}}}\big) \sim 0, \quad \mathrm{H}^i\big(U,\widehat{\mathcal{O}}^+_{\mathcal{X}^\flat_{\mathrm{pke}}}\big) \sim 0, \quad \mathrm{H}^i\big(U,\mathbb{A}_{\mathrm{inf}}\big) \sim 0 \quad \forall i \geq 1$$

(where $\sim$ means almost 0).

Thanks to [15, Prop. 5.3.12 & Prop. 5.3.13], there exists a basis $\mathcal{B}$ for the site $\mathcal{X}_{\mathrm{pke}}$ given by log affinoid perfectoid objects such that for every locally constant p-torsion sheaf $\mathbb{L}$ on $\mathcal{X}_{\mathrm{ket}}$ and every $U \in \mathcal{B}$ we have $\mathrm{H}^i\big(\mathcal{X}_{\mathrm{pke}}|_U,\mathbb{L}\big) = 0$ for $i \geq 1$. In case X is a fs log scheme over

$\mathbb{Q}_p$, there is an analogous notion of log affinoid perfectoid opens of X_{pke}, and it follows from the arguments in loc. cit. that there exists a basis of X_{pke} with the same property.

We recall that K was defined in the previous section as a perfectoid field of characteristic 0 with an open and bounded subring K^+. Assume that $\mathcal{X}$ is defined over $\mathrm{Spa}(K,K^+)$. In this case, $\mathbb{A}_{\mathrm{inf}}$, resp. $\widehat{\mathcal{O}}^+_{\mathcal{X}}$, is a sheaf of algebras over the classical period ring $\mathrm{A}_{\mathrm{inf}} := \mathrm{W}\big(K^{\flat+}\big)$, resp. over K^+, and given a generator $\zeta \in \mathrm{A}_{\mathrm{inf}}$ for the kernel of the canonical ring homomorphism $\mathrm{A}_{\mathrm{inf}} \to K^+$, it follows from [22, Lemma 6.3] that we have an exact sequence

$$0 \longrightarrow \mathbb{A}_{\mathrm{inf}} \xrightarrow{\cdot\zeta} \mathbb{A}_{\mathrm{inf}} \xrightarrow{\vartheta} \widehat{\mathcal{O}}^+_{\mathcal{X}_{\mathrm{pke}}} \longrightarrow 0.$$

2.4. Comparison results

Assume that $\mathcal{X}$ is a finite saturated locally noetherian log adic space over a perfectoid field $\mathrm{Spa}(K,K^+)$ with K algebraically closed of characteristic 0. Firstly, the main result of [15], namely Theorem 6.2.1, states that if the underlying adic space to $\mathcal{X}$ is log smooth and proper and $\mathbb{L}$ is an $\mathbb{F}_p$-local system on $\mathcal{X}_{\mathrm{ket}}$, then the cohomology groups $\mathrm{H}^i\big(\mathcal{X}_{\mathrm{ket}},\mathbb{L}\big)$ are finite for all i, they vanish for $i \gg 0$ and the natural map

$$\mathrm{H}^i(\mathcal{X}_{\mathrm{ket}},\mathbb{L}) \otimes K^+/p \to \mathrm{H}^i(\mathcal{X}_{\mathrm{ket}},\mathbb{L} \otimes \mathcal{O}^+_{\mathcal{X}_{\mathrm{ket}}}/p)$$

is an almost isomorphism for every $i \geq 0$. As $\mathcal{F} \cong \mathrm{R}\nu_*\nu^{-1}\big(\mathcal{F}\big)$ for any sheaf of abelian goups, we obtain the same cohomology groups replacing $\mathcal{X}_{\mathrm{ket}}$ with $\mathcal{X}_{\mathrm{pke}}$ in the isomorphisms above. Here, we denoted $\nu\colon \mathcal{X}_{\mathrm{pke}} \longrightarrow \mathcal{X}_{\mathrm{ket}}$ the natural morphism of sites.

Second, in the case X is finite separated locally noetherian log scheme, proper and log smooth over K, we have a géométrie algébrique et géométrie analytique (GAGA) type comparison isomorphism. Let $\mathcal{X}$ be the associated log adic space over $\mathrm{Spa}(K,K^+)$. We have a natural morphism of sites $\gamma\colon \mathcal{X}_{\mathrm{ket}} \to X_{\mathrm{ket}}$. Let $\mathbb{L}$ be an $\mathbb{F}_p$-local system on X_{ket}. Then

Proposition 2.1. *For every $i \geq 0$ the natural morphism $\mathrm{H}^i\big(X_{\mathrm{ket}},\mathbb{L}\big) \longrightarrow \mathrm{H}^i\big(\mathcal{X}_{\mathrm{ket}},\gamma^*\big(\mathbb{L}\big)\big)$ is an isomorphism.*

Proof. Let X^o and $\mathcal{X}^o$ be the scheme, resp. the adic space defined by X and $\mathcal{X}$ forgetting the log structures. In this case, the morphism of sites $\gamma^o\colon \mathcal{X}^o_{\mathrm{et}} \to X^o_{\mathrm{et}}$ induces the map $\mathrm{H}^i\big(X^o_{\mathrm{et}},F\big) \longrightarrow \mathrm{H}^i\big(\mathcal{X}^o_{\mathrm{et}},\gamma^{o,*}\big(F\big)\big)$ for every sheaf of torsion abelian groups F. It is an isomorphism due to [17, Thm. 3.2.10]. Consider the commutative diagram of sites

$$\begin{array}{ccc} \mathcal{X}_{\mathrm{et}} & \xrightarrow{\gamma} & X_{\mathrm{et}} \\ \alpha\downarrow & & \downarrow\beta \\ \mathcal{X}^o_{\mathrm{et}} & \xrightarrow{\gamma^o} & X^o_{\mathrm{et}}. \end{array}$$

Using the compatibility of the Leray spectral sequences $\mathrm{H}^i\big(\mathcal{X}^o_{\mathrm{et}},\mathrm{R}^j\alpha_*\gamma^*(\mathbb{L})\big) \Longrightarrow \mathrm{H}^{i+j}\big(\mathcal{X}_{\mathrm{ket}},\gamma^*\big(\mathbb{L}\big)\big)$ and $\mathrm{H}^i\big(X^o_{\mathrm{et}},\mathrm{R}^j\beta_*(\mathbb{L})\big) \Longrightarrow \mathrm{H}^{i+j}\big(X_{\mathrm{ket}},\mathbb{L}\big)$ and the result of Huber, it suffices to prove that the natural morphism

$$\gamma^{o,*}\big(\mathrm{R}^j\beta_*(\mathbb{L})\big) \longrightarrow \mathrm{R}^j\alpha_*\big(\gamma^*\big(\mathbb{L}\big)$$

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is an isomorphism of sheaves for every j. It suffices to prove that we get an isomorphism after passing to stalks at geometric points $\zeta = \mathrm{Spa}\big(l,l^+\big) \to \mathcal{X}^o$ as those form a conservative family by [17, Prop. 2.5.5]. Recall that ζ might consist of more than one point but it has a unique closed point ζ_0. Taking the stalk at ζ is equivalent to take global sections over the associated strictly local adic space $\mathcal{X}^o(\zeta)$ (see [17, Lemma 2.5.12]). Let $X^o(\zeta_0)$ be the spectrum of the strict Henselization of X at ζ_0. Taking the stalk at ζ of $\gamma^{o,*}$ of a sheaf is equivalent to taking the sections of that sheaf over $X^o(\zeta_0)$. We have a natural map of sites $\mathcal{X}^o(\zeta)_{\mathrm{ket}} \to X^o(\zeta_0)_{\mathrm{ket}}$, considering on $X^o(\zeta)$ and on $\mathcal{X}^o(\zeta)$ the log structures coming from X and $\mathcal{X}$. We need to show that it is an equivalence. In both cases, the Kummer étale sites are the same as the finite Kummer étale sites; indeed by definition the Kummer étale topology is generated in both cases by finite Kummer étale covers and classical étale morphisms and a Kummer cover of $X^o(\zeta)$, resp. $\mathcal{X}^o(\zeta)$, is still strictly local and hence does not admit any nontrivial classical étale cover (see [17, Lemma 2.5.6] in the adic setting). Both in the schematic and in the adic setting, the finite Kummer étale sites are equivalent to the category of finite sets with continuous action of the group $\mathrm{Hom}(\overline{M}^{\mathrm{gp}},\widehat{\mathbb{Z}})$ with $\overline{M}$ the stalk of the log structure at ζ, modulo l^*. See [18, Ex. 4.7(a)] in the schematic case and [15, Prop. 4.4.7] in the adic case. As such quotient is the same in the schematic and adic cases, the conclusion follows. $\square$

3. VBMS and dual VBMS

3.1. VBMS, that is, vector bundles with marked sections

We recall the main constructions of [2] and [4]. Let $\mathcal{X}$ denote an adic analytic space over $\mathrm{Spa}(\mathbb{Q}_p,\mathbb{Z}_p)$ and let $(\mathcal{E},\mathcal{E}^+)$ denote a pair consisting of a locally free $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{E}$ of rank 2 and a subsheaf $\mathcal{E}^+$ of $\mathcal{E}$ which is a locally free $\mathcal{O}_{\mathcal{X}}^+$-module of rank 2 such that $\mathcal{E} = \mathcal{E}^+ \otimes_{\mathcal{O}_{\mathcal{X}}^+} \mathcal{O}_{\mathcal{X}}$. Let $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}^+$ be an invertible ideal such that $\mathcal{I}$ gives the topology on $\mathcal{O}_{\mathcal{X}}^+$, and let $r \geq 0$ be an integer such that $\mathcal{I} \subset p^r \mathcal{O}_{\mathcal{X}}^+$.

We suppose that there is a section $s \in \mathrm{H}^0(\mathcal{X},\mathcal{E}^+/\mathcal{I}\mathcal{E}^+)$ such that the submodule $\big(\mathcal{O}_{\mathcal{X}}^+/\mathcal{I}\big)s$ is a direct summand of $\mathcal{E}^+/\mathcal{I}\mathcal{E}^+$. We have the following.

Theorem 3.1 [4]. a) *The functor attaching to every adic space $\gamma\colon \mathcal{Z} \to \mathcal{X}$ such that $t^*(\mathcal{I})$ is an invertible ideal in $\mathcal{O}_{\mathcal{Z}}^+$, the set (group in fact):*

$$\mathbb{V}(\mathcal{E},\mathcal{E}^+)\big(\gamma\colon \mathcal{Z} \to \mathcal{X}\big) := \mathrm{Hom}_{\mathcal{O}_{\mathcal{Z}}^+}\big(\gamma^*(\mathcal{E}^+),\mathcal{O}_{\mathcal{Z}}^+\big) = \mathrm{H}^0\big(\mathcal{Z},\gamma^*(\mathcal{E}^+)^\vee\big),$$

is represented by the adic vector bundle $\mathbb{V}(\mathcal{E},\mathcal{E}^+) := \mathrm{Spa}_{\mathcal{X}}\big(\mathrm{Sym}(\mathcal{E}),\mathrm{Sym}(\mathcal{E}^+)\big) \to \mathcal{X}$.

b) *The subfunctor of $\mathbb{V}(\mathcal{E},\mathcal{E}^+)$ denoted $\mathbb{V}_0(\mathcal{E}^+,s)$ which associates to every adic space $\gamma\colon \mathcal{Z} \to \mathcal{X}$ as above, the set:*

$$\mathbb{V}_0(\mathcal{E}^+,s)\big(\gamma\colon \mathcal{Z} \to \mathcal{X}\big) := \Big\{h \in \mathbb{V}(\mathcal{E},\mathcal{E}^+)\big(\gamma\colon \mathcal{Z} \to \mathcal{X}\big) \mid h\big(\mathrm{mod}\ \gamma^*(\mathcal{I})\big)(\gamma^*(s)) = 1\Big\},$$

is represented by the open adic subspace of $\mathbb{V}(\mathcal{E},\mathcal{E}^+)$, also denoted $\mathbb{V}_0(\mathcal{E}^+,s)$, consisting of the points x such that $|\tilde{s}-1|_x \leq |\alpha|_x$, where $\tilde{s}$ is a (local) lift of s to $\mathcal{E}^+$ and α is a (local) generator of $\mathcal{I}$ at x.

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c) *Suppose that we have sections s and $t \in \mathrm{H}^0(\mathcal{X},\mathcal{E}^+/\mathcal{I}\mathcal{E}^+)$ which form an $\big(\mathcal{O}_{\mathcal{X}}^+/\mathcal{I}\big)$-basis of $\mathcal{E}^+/\mathcal{I}\mathcal{E}^+$. Then, the subfunctor $\mathbb{V}_0(\mathcal{E}^+,s,t)$ of $\mathbb{V}_0(\mathcal{E}^+,s)$ which associates to every adic space $\gamma\colon \mathcal{Z}\to\mathcal{X}$, the set:*

$$\mathbb{V}_0(\mathcal{E}^+,s,t)\big(\gamma\colon \mathcal{Z}\to\mathcal{X}\big) := \Big\{h\in \mathbb{V}_0(\mathcal{E}^+,s)\big(\gamma\colon \mathcal{Z}\to\mathcal{X}\big) \mid h\big(\mathrm{mod}\ \gamma^*(\mathcal{I})\big)(\gamma^*(t)) = 0\Big\},$$

is represented by the open adic subspace $\mathbb{V}_0(\mathcal{E}^+,s,t)$ of $\mathbb{V}_0(\mathcal{E}^+,s)$ consisting of the points x such that $|\tilde{t}|_x \le |\alpha|_x$ for a (any) lift $\tilde{t}$ of t to $\mathcal{E}^+$ and α a (local) generator of $\mathcal{I}$ at x.

Proof. The proof is local on $\mathcal{X}$. Assume that $U\subset\mathcal{X}$ is an affinoid open $U=\mathrm{Spa}(R,R^+)$ such that $\mathcal{I}|_U$ is principal generated by $\alpha\in R^+$ and $\mathcal{E}^+|_U$ is free with basis f_0, f_1 with $f_0(\mathrm{mod}\ \alpha) = s|_U$. Then $f_1(\mathrm{mod}\ \alpha)$ generates $\Big(\big(\mathcal{E}^+/\mathcal{I}\mathcal{E}^+\big)/s\big(\mathcal{O}_{\mathcal{X}}^+/\mathcal{I}\big)\Big)|_U$ and we assume in case (c) that $f_1(\mathrm{mod}\ \alpha) = t|_U$. Then by [2, §2] we have $\mathbb{V}(\mathcal{E},\mathcal{E}^+)|_U = \mathrm{Spa}\big(R\langle X,Y\rangle, R^+\langle X,Y\rangle\big)$ and

$$\mathbb{V}_0(\mathcal{E}^+,s)|_U = \mathrm{Spa}\big(R\langle X,Y\rangle\langle\frac{X-1}{\alpha}\rangle, R^+\langle X,Y\rangle\langle\frac{X-1}{\alpha}\rangle\big) = \mathrm{Spa}\big(R\langle Z,Y\rangle, R^+\langle Z,Y\rangle\big),$$

where $X = 1+\alpha Z$ giving also the map to $\mathbb{V}(\mathcal{E},\mathcal{E}^+)|_U$. Similarly,

$$\mathbb{V}_0(\mathcal{E}^+,s,t)|_U = \mathrm{Spa}\big(R\langle Z,W\rangle, R^+\langle Z,W\rangle\big)$$

with $Y=\alpha W$. We have the tautological map over $\mathbb{V}(\mathcal{E},\mathcal{E}^+)|_U$ given by

$$\mathcal{E}^+\otimes_{R^+} R^+\langle X,Y\rangle \to R^+\langle X,Y\rangle, \qquad f_0\mapsto X, f_1\mapsto Y$$

from which we deduce similarly the tautological maps over $\mathbb{V}_0(\mathcal{E}^+,s)|_U$ and $\mathbb{V}_0(\mathcal{E}^+,s,t)|_U$ providing the claimed representability and concluding the proof. □

3.2. Dual VBMS

In this article, we'll need a variant of the construction in Section §3.1 which we now present. Suppose that $\mathcal{X}$, $\mathcal{I}$, $(\mathcal{E},\mathcal{E}^+)$ are as in Section §3.1. Moreover, we assume that there is an exact sequence of locally free $\mathcal{O}_{\mathcal{X}}^+/\mathcal{I}$-modules

$$0\longrightarrow \mathcal{Q}\longrightarrow \mathcal{E}^+/\mathcal{I}\mathcal{E}^+\longrightarrow\mathcal{F}\longrightarrow 0$$

and a section $s\in\mathrm{H}^0\big(\mathcal{X},\mathcal{F}\big)$ such that $\big(\mathcal{O}_{\mathcal{X}}^+/\mathcal{I}\big)s=\mathcal{F}$. We have:

Theorem 3.2. *The subfunctor $\mathbb{V}_0^D(\mathcal{E}^+,\mathcal{Q},s)$ of $\mathbb{V}(\mathcal{E},\mathcal{E}^+)$, defined by associating to every adic space $t\colon\mathcal{Z}\to\mathcal{X}$ as in Section §3.1 the set*

$$\mathbb{V}_0^D(\mathcal{E}^+,\mathcal{Q},s)\big(\gamma\colon\mathcal{Z}\to\mathcal{X}\big) :=$$

$$:= \Big\{h\in\mathbb{V}(\mathcal{E},\mathcal{E}^+)\big(\gamma\colon\mathcal{Z}\to\mathcal{X}\big) \,|\, h\big(\mathrm{mod}\ \gamma^*(\mathcal{I})\big)\big(\gamma^*(\mathcal{Q})\big)=0 \text{ and } h\big(\mathrm{mod}\ \gamma^*(\mathcal{I})\big)\big(\gamma^*(s)\big)=1\Big\}$$

is represented by the the open adic subspace of $\mathbb{V}(\mathcal{E},\mathcal{E}^+)$ denoted $\mathbb{V}_0^D(\mathcal{E}^+,\mathcal{Q},s)$ and consisting of the points x such that $|q|_x\le|\alpha|_x$ and $|\tilde{s}-1|_x\le|\alpha|_x$, where q is a (local) lift to $\mathcal{E}^+$ of a local generator of $\mathcal{Q}$ at x, α is a (local) generator of $\mathcal{I}$ at x and $\tilde{s}$ is a (local) lift of s to $\mathcal{E}^+$.

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3.3. The sheaves $\mathbb{W}_k$ and $\mathbb{W}_k^D$

Let the hypothesis be as in Section §3.1 and §3.2. We assume that we have a morphism of adic spaces $\mathcal{X} \to \mathcal{W}$ where let us recall that $\mathcal{W}$ is the adic weight space for $\mathbf{GL}_{2,/\mathbb{Q}}$. For every adic space $\mathcal{Z}$, the morphisms $\mathcal{Z} \to \mathcal{W}$ classify continuous homomorphisms $\mathbb{Z}_p^* \to \Gamma\big(\mathcal{Z},\mathcal{O}_\mathcal{Z}\big)$. We denote by $k^{\mathrm{univ}}\colon \mathbb{Z}_p^* \to \Gamma\big(\mathcal{X},\mathcal{O}_\mathcal{X}\big)$ the continuous homomorphism defined by $\mathcal{X} \to \mathcal{W}$. We assume that k^{univ} satisfies the following analyticity assumption: There exists a section u_{univ} of $\mathcal{O}_\mathcal{X}$ such that $|u_{\mathrm{univ}}|_x < |p^{\frac{1}{p-1}-r}|_x$ for every $x \in \mathcal{X}$ and

$$k^{\mathrm{univ}}(t) = \exp u_{\mathrm{univ}} \log(t), \quad \forall t \in 1+p^r\mathbb{Z}_p.$$

We recall that the integer $r \geq 0$ is such that $\mathcal{I} \subset p^r\mathcal{O}_\mathcal{X}^+$. Let us denote by $\mathcal{T}$ the adic torus representing the functor which associates to an adic space $\gamma\colon \mathcal{Z} \to \mathcal{X}$ the group

$$\mathcal{T}(\gamma\colon \mathcal{Z} \to \mathcal{X}) := 1+\gamma^*(\mathcal{I}).$$

Then k^{univ} defines a character $k^{\mathrm{univ}}\colon \mathcal{T} \to \mathbb{G}_m$, that is, a morphism of adic spaces and group functors, using the fomula above.

We have natural actions of $\mathcal{T}$ on $\mathbb{V}_0(\mathcal{E}^+,s)$, $\mathbb{V}_0(\mathcal{E}^+,s,t)$ and $\mathbb{V}_0^D\big(\mathcal{E}^+,\mathcal{Q},s\big)$ defined on $\gamma\colon \mathcal{Z} \to \mathcal{X}$ points by: $u * h := uh$ and $u * h' := uh'$, where $u \in \mathcal{T}(\gamma\colon \mathcal{Z} \to \mathcal{X})$, $h \in \mathbb{V}_0(\mathcal{E}^+,s)(\gamma\colon \mathcal{Z} \to \mathcal{X})$, resp. $h \in \mathbb{V}_0(\mathcal{E}^+,s,t)(\gamma\colon \mathcal{Z} \to \mathcal{X})$, resp. $h' \in \mathbb{V}_0^D\big(\mathcal{E}^+,\mathcal{Q},s\big)(\gamma\colon \mathcal{Z} \to \mathcal{X})$. Let us denote by $f\colon \mathbb{V}_0(\mathcal{E}^+,s) \longrightarrow \mathcal{X}$, $g\colon \mathbb{V}_0(\mathcal{E}^+,s,t) \longrightarrow \mathcal{X}$ and by f^D : $\mathbb{V}_0^D\big(\mathcal{E}^+,\mathcal{Q},s\big) \longrightarrow \mathcal{X}$ the structural morphisms.

Definition 3.3. We denote

$$\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s) := f_*\big(\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E}^+,s)}\big)[k^{\mathrm{univ}}], \;\; \mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s,t) := g_*\big(\mathcal{O}^+_{\mathbb{V}_0(\mathcal{E}^+,s,t)}\big)[k^{\mathrm{univ}}]$$

and

$$\mathbb{W}^D_{k^{\mathrm{univ}}}\big(\mathcal{E}^+,\mathcal{Q},s\big) := f^D_*\Big(\mathcal{O}^+_{\mathbb{V}_0^D\big(\mathcal{E}^+,\mathcal{Q},s\big)}\Big)[k^{\mathrm{univ}}],$$

where if $\mathcal{G}$ is an $\mathcal{O}_\mathcal{X}^+$-module on $\mathcal{X}$ with an action by the torus $\mathcal{T} := 1+\mathcal{I}$, we denote $\mathcal{G}[k^{\mathrm{univ}}]$ the subsheaf of $\mathcal{G}$ of sections x such that $u * x = k^{\mathrm{univ}}(u)x$ for all corresponding sections u of $\mathcal{T}$.

3.3.1. Local descriptions of the sheaves $\mathbb{W}_k$ and $\mathbb{W}_k^D$. We assume all notations and assumptions of the proof of Theorem 3.1 and of Section §3.3. Let $U = \mathrm{Spa}(R,R^+) \subset \mathcal{X}$ be an affinoid open such that $\mathcal{E}^+|_U = f_0\mathcal{O}_U^+ + f_1\mathcal{O}_U^+$, $\mathcal{I}|_U = \alpha\mathcal{O}_U^+$ and such that $f_0 (\mathrm{mod}\ \alpha) = s|_U$ and $f_1 (\mathrm{mod}\ \alpha)$ generates $\Big(\big(\mathcal{E}^+/\mathcal{I}\mathcal{E}^+\big)/s\big(\mathcal{O}_\mathcal{X}^+/\mathcal{I}\big)\Big)|_U$ (respectively $f_1(\mathrm{mod}\ \alpha) = t|_U$).

In view of the next section, we also consider the dual situation, that is, recall that s is a global marked section of $\mathcal{E}^+$ modulo $\mathcal{I}$ and denote $\mathcal{F} := (\mathcal{E}^+/\mathcal{I}\mathcal{E}^+)/\big(s(\mathcal{O}_\mathcal{X}^+/\mathcal{I})\big)$. By dualizing, we obtain the exact sequence

$$0 \longrightarrow \mathcal{Q} \longrightarrow (\mathcal{E}^+)^\vee/\mathcal{I}(\mathcal{E}^+)^\vee \longrightarrow (\mathcal{O}_\mathcal{X}^+/\mathcal{I})s^\vee \longrightarrow 0,$$

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where $\mathcal{Q} := \mathcal{F}^\vee$, which defines data for $\mathbb{V}_0^D$. Then $(\mathcal{E}^+)^\vee|_U = f_0^\vee \mathcal{O}_U^+ + f_1^\vee \mathcal{O}_U^+$, $\mathcal{Q}|_U = f_1^\vee\big(\mathcal{O}_U^+/\alpha\mathcal{O}_U^+\big)$ and $f_0^\vee(\mathrm{mod}\ \alpha) = s^\vee|_U$. Therefore, as in [2, Lemma 2.4], one proves that

$$\mathbb{V}_0(\mathcal{E}^+,s)|_U = \mathrm{Spa}\big(R\langle X,Y\rangle\langle\frac{X-1}{\alpha}\rangle, R^+\langle X,Y\rangle\langle\frac{X-1}{\alpha}\rangle\big) = \mathrm{Spa}\big(R\langle Z,Y\rangle, R^+\langle Z,Y\rangle\big),$$

where $X = 1+\alpha Z$ and

$$\mathbb{V}_0(\mathcal{E}^+,s,t)|_U = \mathrm{Spa}\big(R\langle Z,W\rangle, R^+\langle Z,W\rangle\big)$$

with $Y = \alpha W$. Similarly, we have

$$\begin{aligned}\mathbb{V}_0^D\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big)|_U &= \mathrm{Spa}\Big(R\langle A,B\rangle\langle\frac{A-1}{\alpha},\frac{B}{\alpha}\rangle, R^+\langle A,B\rangle\langle\frac{A-1}{\alpha},\frac{B}{\alpha}\rangle\Big) = \\ &= \mathrm{Spa}\big(R\langle C,D\rangle, R^+\langle C,D\rangle\big),\end{aligned}$$

where $A = 1+\alpha C$ and $B = \alpha D$.

Therefore, we have

$$\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s)(U) = R^+\langle Z,Y\rangle[k^{\mathrm{univ}}] = R^+\langle\frac{Y}{1+\alpha Z}\rangle k^{\mathrm{univ}}(1+\alpha Z),$$

$$\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s,t)(U) = R^+\langle Z,W\rangle[k^{\mathrm{univ}}] = R^+\langle\frac{W}{1+\alpha Z}\rangle k^{\mathrm{univ}}(1+\alpha Z)$$

and

$$\mathbb{W}^D_{k^{\mathrm{univ}}}\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big)(U) = R^+\langle C,D\rangle[k^{\mathrm{univ}}] = R^+\langle\frac{D}{1+\alpha C}\rangle k^{\mathrm{univ}}(1+\alpha C).$$

3.4. The duality

Suppose that $\mathcal{X}$, $\mathcal{I}$, $(\mathcal{E},\mathcal{E}^+)$, s are as in Section §3.1, and let us denote by

$$\iota\colon \mathcal{F} := s\big(\mathcal{O}_{\mathcal{X}}^+/\mathcal{I}\big) \hookrightarrow \mathcal{E}^+/\mathcal{I}\mathcal{E}^+.$$

Let $\big(\mathcal{E}',(\mathcal{E}^+)^\vee\big)$ denote the pair where $(\mathcal{E}^+)^\vee$ is the $\mathcal{O}_{\mathcal{X}}^+$-dual of $\mathcal{E}^+$ and $\mathcal{E}' := (\mathcal{E}^+)^\vee \otimes_{\mathcal{O}_{\mathcal{X}}^+} \mathcal{O}_{\mathcal{X}}$. Moreover, let us consider $\mathcal{Q} := \mathrm{Ker}\big((\mathcal{E}^+/\mathcal{I}\mathcal{E}^+)^\vee \xrightarrow{\iota^\vee} \mathcal{F}^\vee\big)$, where $\mathcal{F}^\vee$ is the $\mathcal{O}_{\mathcal{X}}^+/\mathcal{I}$-dual of $\mathcal{F}$. As explained in Theorems 3.1 and 3.2, we have adic spaces $\mathbb{V}_0(\mathcal{E}^+,s)$ and $\mathbb{V}_0^D\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big)$ over $\mathcal{X}$. Consider the morphism of adic spaces

$$\langle\ ,\ \rangle\colon \mathbb{V}_0(\mathcal{E}^+,s)\times_{\mathcal{X}}\mathbb{V}_0^D\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big) \longrightarrow \mathbb{V}_0(\mathcal{O}_{\mathcal{X}}^+,1),$$

defined on points as follows. Fix a morphism of adic spaces $\gamma\colon \mathcal{Z}\to\mathcal{X}$. Consider $h\in\mathbb{V}_0(\mathcal{E}^+,s)\big(\gamma\colon\mathcal{Z}\to\mathcal{X}\big)$ and $h'\in\mathbb{V}_0^D\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big)\big(\gamma\colon\mathcal{Z}\to\mathcal{X}\big)$. By definition, this is equivalent to giving morphisms of $\mathcal{O}_{\mathcal{Z}}^+$-modules $h\colon \gamma^*(\mathcal{E}^+)\to\mathcal{O}_{\mathcal{Z}}^+$ with $h(\mathrm{mod}\ \mathcal{I})(\gamma^*(s)) = 1$ and $h'\colon \gamma^*(\mathcal{E}^+)^\vee\to\mathcal{O}_{\mathcal{Z}}^+$ with $h'(\mathrm{mod}\ \gamma^*(\mathcal{I}))(\mathcal{Q}) = 0$ and $h'(\mathrm{mod}\ \gamma^*(\mathcal{I}))(\gamma^*(s^\vee)) = 1$. Then $h'(h)\in\mathrm{H}^0(\mathcal{Z},\mathcal{O}_{\mathcal{Z}}^+)$ and $h'(h)(\mathrm{mod}\ \gamma^*(\mathcal{I})) = 1$, that is, $h'(h)\in\mathbb{V}_0(\mathcal{O}_{\mathcal{X}}^+,1)\big(\gamma\colon\mathcal{Z}\to\mathcal{X}\big)$. We define

$$\langle h,h'\rangle := h'(h)\in\mathbb{V}_0(\mathcal{O}_{\mathcal{X}}^+,1)\big(\gamma\colon\mathcal{Z}\to\mathcal{X}\big).$$

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Notice that $\mathbb{V}_0(\mathcal{O}_{\mathcal{X}}^+,1)$ is the affine one-dimensional space $\mathbb{A}^1_{\mathcal{X}}$ over $\mathcal{X}$, with standard coordinate T. The torus $\mathcal{T}\times\mathcal{T}$ acts componentwisely on $\mathbb{V}_0(\mathcal{E}^+,s)\times_{\mathcal{X}}\mathbb{V}_0^D\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big)$. Given sections (h,h') of the latter and (u,u') of $\mathcal{T}\times\mathcal{T}$, we have $\langle u*h,u'*h'\rangle=(u\cdot u')*\langle h,h'\rangle$.

Lemma 3.4. *There is a section $T^{k^{\mathrm{univ}}}$ of $\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{O}_{\mathcal{X}}^+,1)$ over $\mathcal{X}$ such that $\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{O}_{\mathcal{X}}^+,1)=T^{k^{\mathrm{univ}}}\cdot\mathcal{O}_{\mathcal{X}}^+$. Moreover, $\langle\ ,\ \rangle^*\big(T^{k^{\mathrm{univ}}}\big)\in\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s)\widehat{\otimes}\mathbb{W}^D_{k^{\mathrm{univ}}}\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big)$. Here, we see $\langle\ ,\ \rangle^*$ in a natural way, as a morphism of sheaves $\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{O}_{\mathcal{X}}^+,1)\longrightarrow\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s)\widehat{\otimes}\mathbb{W}^D_{k^{\mathrm{univ}}}\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big)$.*

Proof. The statement is local on $\mathcal{X}$. We use the explicit coordinates of §3.3.1. Then, $T=1+\alpha V$ and $T^{k^{\mathrm{univ}}}$ is the section $k^{\mathrm{univ}}(1+\alpha V)$. By loc. cit., we have $\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{O}_{\mathcal{X}}^+,1)(U)=T^{k^{\mathrm{univ}}}\cdot R^+$.

As $\langle\ ,\ \rangle^*(T)=X\otimes A+Y\otimes B=\big(1+\alpha Z)\big)\cdot\big(1+\alpha C\big)+\alpha Y\otimes D$, we conclude that

$$\langle\ ,\ \rangle^*\big(T^{k^{\mathrm{univ}}}\big)=k^{\mathrm{univ}}(1+\alpha Z)k^{\mathrm{univ}}(1+\alpha C)\cdot k^{\mathrm{univ}}\left(1+\alpha\frac{Y}{1+\alpha Z}\otimes\frac{D}{1+\alpha C}\right). \tag{1}$$

This concludes the proof. □

Definition 3.5. Write $\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s)^\vee$ for the $\mathcal{O}_{\mathcal{X}}^+$-dual of $\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s)$. Define the map of $\mathcal{O}_{\mathcal{X}}^+$-modules

$$\xi_{k^{\mathrm{univ}}}:\mathbb{W}_{k^{\mathrm{univ}}}(\mathcal{E}^+,s)^\vee\longrightarrow\mathbb{W}^D_{k^{\mathrm{univ}}}\big((\mathcal{E}^+)^\vee,\mathcal{Q},s^\vee\big),\quad\gamma\mapsto(\gamma\otimes1)\big(\langle\ ,\ \rangle^*\big(T^{k^{\mathrm{univ}}}\big)\big).$$

3.4.1. Local descriptions of the duality between $\mathbb{W}_k$ and $\mathbb{W}_k^D$. We put ourselves in the setting of §3.3.1 and compute explicitly the pairing $\xi_{k^{\mathrm{univ}}}$ on the affinoid U in terms of the local coordinates of loc. cit. As k^{univ} is supposed to be r-analytic on $\mathcal{X}$, we can write $k^{\mathrm{univ}}(t)=\exp u_{\mathrm{univ}}\log(t)$ for every $t\in1+p^r\mathbb{Z}_p$. We then claim that

$$\xi_{k^{\mathrm{univ}}}\left(k^{\mathrm{univ}}(1+\alpha Z)\big(\frac{Y}{1+\alpha Z}\big)^n\right)^\vee=\alpha^n\binom{u_{\mathrm{univ}}}{n}k^{\mathrm{univ}}(1+\alpha C)\big(\frac{D}{1+\alpha C}\big)^n,$$

where $\displaystyle\binom{u_{\mathrm{univ}}}{n}:=\frac{u_{\mathrm{univ}}(u_{\mathrm{univ}}-1)\cdots(u_{\mathrm{univ}}-n+1)}{n!}$ if $n\geq1$ and $\displaystyle\binom{u_{\mathrm{univ}}}{0}=1$.

First of all, one computes that, if we write $f(X):=\exp u_{\mathrm{univ}}\log(1+X)=\sum_{n=0}^\infty a_nX^n$ as a formal power series in X, then $a_n=\binom{u_{\mathrm{univ}}}{n}$. Using equation (1), we deduce that

$$\frac{\langle\ ,\ \rangle^*\big(T^{k^{\mathrm{univ}}}\big)}{k^{\mathrm{univ}}(1+\alpha Z)k^{\mathrm{univ}}(1+\alpha C)}=\sum_{n=0}^\infty\alpha^n\binom{u_{\mathrm{univ}}}{n}\left(\frac{Y}{1+\alpha Z}\right)^n\otimes\left(\frac{D}{1+\alpha C}\right)^n,$$

and the claim follows.

3.5. An example: locally analytic functions and distributions

We consider in this article weights defined as follows. Let $\mathcal{W}$ denote the weight space seen as an adic space.

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Definition 3.6. Let $U \subset \mathcal{W}$ be an open disk of an open affinoid of $\mathcal{W}$, and let Λ_U denote the $\mathbb{Z}_p$-subalgebra of sections in $\mathcal{O}_U^+(U)$ which are bounded; see Section §4 of [5]. We recall that Λ_U is a complete noetherian local $\mathbb{Z}_p$-algebra, and we call 'weak topology' the m_{Λ_U}-adic topology of Λ_U, where m_{Λ_U} is its maximal ideal. Let B denote either Λ_U for some open disk $U \subset \mathcal{W}$ or $\mathcal{O}_K$ for a finite extension K of $\mathbb{Q}_p$. In this article, we will work with B-valued weights $k\colon \mathbb{Z}_p^* \longrightarrow B^*$, which, if $B = \Lambda_U$, is the universal weight associated to U.

Let $k\colon \mathbb{Z}_p^* \longrightarrow B^*$ be a weight as in definition 3.6, and suppose it is an r-analytic character. Let $T = \mathbb{Z}_p \oplus \mathbb{Z}_p$; denote by $f_0 = (1,0)$ and $f_1 = (0,1) \in T$ the standard $\mathbb{Z}_p$-basis. Denote by $T^\vee$ the $\mathbb{Z}_p$-dual of T, and let $T_0^\vee \subset T^\vee$ be the subset of elements $\mathbb{Z}_p^* e_0 \times \mathbb{Z}_p e_1$ with $e_0 = f_0^\vee$ and $e_1 = f_1^\vee$, the $\mathbb{Z}_p$-basis of $T^\vee$ dual to (f_0, f_1). Then $T_0^\vee$ is a profinite set with an action of the Iwahori subgroup $\mathrm{Iw}_1 \subset \mathbf{GL}_2(\mathbb{Z}_p)$. Following [7, Def. 3.1], we set:

Definition 3.7. For every integer $n \geq r$, let $A_k^o(T_0^\vee)[n]$ denote the space of functions $f\colon T_0^\vee \longrightarrow B$ such that

(1) for every $a \in \mathbb{Z}_p^*$, $t \in T_0^\vee$, we have $f(at) = k(a) f(t)$

(2) the function $z \to f(e_0 + z e_1)$ extends to an n-analytic function, that is, for every $i \in \mathbb{Z}/p^n\mathbb{Z}$ the function $f\big(e_0 + (i + p^n z) e_1\big)$ for $z \in \mathbb{Z}_p$ is given by the values of a convergent power series $\sum_{m=0}^\infty a_{m,i} z^m$.

Define $D_k^o(T_0^\vee)[n] := \mathrm{Hom}_B\big(A_k^o(T_0^\vee)[n], B\big)$, the continuous dual of $A_k^o(T_0^\vee)[n]$ with respect to the weak topology of B. We write $A_k(T_0^\vee)[n] := A_k^o(T_0^\vee)[n] \otimes_{\mathbb{Z}} \mathbb{Q}$ and $D_k(T_0^\vee)[n] := D_k^o(T_0^\vee)[n] \otimes_{\mathbb{Z}} \mathbb{Q}$.

By [7, Lemma 3.1], the action of Iw_1 on $T_0^\vee$ induces actions of Iw_1 on $A_k^o(T_0^\vee)[n]$, $D_k^o(T_0^\vee)[n]$, $A_k(T_0^\vee)[n]$ and $D_k(T_0^\vee)[n]$. Moreover, [7, Def. 3.3 & Prop. 3.3] the B module $D_k^o(T_0^\vee)[n]$ admits a decreasing filtration $\mathrm{Fil}^\bullet D_k^o(T_0^\vee)[n]$ of B-modules, stable under the action of Iw_1, such that the graded pieces are finite and $D_k^o(T_0^\vee)[n]$ is the inverse limit $\lim\limits_{\infty \leftarrow m} D_k^o(T_0^\vee)[n]/\mathrm{Fil}^m D_k^o(T_0^\vee)[n]$.

3.5.1. An alternative description. For later purposes, we end this section by describing $A_k(T_0^\vee)[n]$ and $D_k(T_0^\vee)[n]$ using the formalism of VBMS. For every $\lambda = 0, \ldots, p^n - 1$ denote by $\mathbb{W}_k(T, s, t + \lambda s, p^n)$, or simply $\mathbb{W}_k(T, s, t + \lambda s)$ if the power of p we are working with is clear from the context, the sections $\mathbb{W}_k(T, s, t + \lambda s)(U)$ over the adic space $U = \mathrm{Spa}(B[1/p], B)$ associated to the rank 2, free $\mathcal{O}_U^+$-module $T \otimes \mathcal{O}_U^+$ and the two sections $s = f_0 \otimes 1$ and $t + \lambda s = f_1 \otimes 1 + \lambda (f_0 \otimes 1)$ modulo $\mathcal{I} := p^n \mathcal{O}_U^+$. Let Iw_n be the subgroup of matrices $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$ such that $\gamma \equiv 0$ modulo p^n. Then:

Proposition 3.8. *There is an Iw_n-equivariant isomorphism of B-modules*

$$\nu_n\colon \oplus_{\lambda=0}^{p^n-1} \mathbb{W}_k(T, s, t - \lambda s) \longrightarrow A_k^o(T_0^\vee)[n].$$

Then, taking duals with respect to the weak topology on B, we get a decomposition into a direct sum and a Iw_n-equivariant isomorphism

$$\nu_n^\vee\colon D_k^o(T_0^\vee)[n] \cong \oplus_{\lambda=0}^{p^n-1} \mathbb{W}_k(T, s, t - \lambda s)^\vee.$$

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Proof. We describe the isomorphism explicitly. First of all, notice that $A_k^o(T_0^\vee)[n]$ decomposes as a direct sum

$$A_k^o(T_0^\vee)[n] = \oplus_{\lambda=0}^{p^n-1} A_{k,\lambda}^o(T_0^\vee)$$

according to residue classes: We say that $f \in A_k^o(T_0^\vee)[n]$ lies in $A_{k,\lambda}^o(T_0^\vee)$ if an only if $f(1,w)$ is zero if $w \notin \lambda + p^n\mathbb{Z}_p$. In particular,

$$A_{k,\lambda}^o(T_0^\vee) = B\langle w_\lambda\rangle \cdot u^k,$$

where $\sum_{m=0}^\infty a_m w_\lambda^m \cdot u^k \colon T_0^\vee \to B$ sends $ue_0 + ve_1 \mapsto k(u) \cdot \sum_m a_m \left(\frac{v/u-\lambda}{p^n}\right)^m$ if $v/u \in \lambda + p^n\mathbb{Z}_p$ and to 0 otherwise. The standard left action of Iw_n on T is described as follows: Given $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Iw}_n$, we have $M(f_0) = \alpha f_0 + \gamma f_1$, $M(f_1) = \beta f_0 + \delta f_1$. This induces a right action given by $e_0 \cdot M = \alpha e_0 + \beta e_1$, $e_1 \cdot M = \gamma e_0 + \delta e_1$. We finally obtain the left action of Iw_n on $A_k^o(T_0^\vee)[n]$. Explicitly, as $(ue_0 + ve_1)\cdot M = (\alpha u + \gamma v)e_0 + (\beta u + \delta v)e_1$, then

$$M(w_0^m \cdot u^k) = k(\alpha + \gamma w_0)\left(\frac{\beta + \delta w_0}{\alpha + \gamma w_0}\right)^m \cdot u^k.$$

As $w_\lambda^m \cdot u^k = \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}(w_0^m \cdot u^k)$ we get the sought for action of Iw_n on $\oplus_{\lambda=0}^{p^n-1} A_{k,\lambda}^o(T_0^\vee)$.

On the other hand, consider the subfunctors $\amalg_{\lambda=0}^{p^n-1} \mathbb{V}_0(T,s,t-\lambda s) \to \mathbb{V}(T)$ over the adic space $U = \mathrm{Spa}(B[1/p],B)$. The action of Iw_n on T restricts to an action on this subfunctors and induces an action on $\oplus_{\lambda=0}^{p^n-1} \mathbb{W}_k(T,s,t-\lambda s)$. Explicitly,

$$\mathbb{W}_k(T,s,t-\lambda s) = B\langle \frac{W_\lambda}{1+p^n Z}\rangle k(1+p^n Z)$$

according to §3.3.1. It contains the B-submodule of the space of integral functions $B\langle X,Y\rangle$ of $\mathbb{V}(T\otimes \mathcal{O}_U^+)$, where $X = 1 + p^n Z$ and $Y = p^n W_\lambda + \lambda X$. Recall that we have a universal map $T \to B\langle X,Y\rangle$ defined by sending $mf_0 + rf_1 \mapsto mX + rY$. The left action of Iw_n on T defines by universality an action on $B\langle X,Y\rangle$: If $M = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$, then $M(f_0) = \alpha f_0 + \gamma f_1$, $M(f_1) = \beta f_0 + \delta f_1$ and $M(X) = \alpha X + \gamma Y$, $M(Y) = \beta X + \delta Y$. Denote by $\mathrm{Iw}_n^1 \subset \mathrm{Iw}_n$ the subgroup of matrices with $\alpha = 1$ modulo p^n. We then get an action of Iw_n^1 on the integral functions on $\amalg_{\lambda=0}^{p^n-1} \mathbb{V}_0(T,s,t-\lambda s)$, and hence on $\oplus_\lambda \mathbb{W}_k(T,s,t-\lambda s)$, determined on the variables Z and W_λ's by the formulas

$$M(Z) = \frac{(\alpha X - 1)}{p^n} + \gamma W_0, \quad M(W_0) = \frac{\beta}{p^n} X + \delta W_0, \quad \begin{pmatrix} 1 & -\lambda \\ 0 & 1 \end{pmatrix}(W_0) = W_\lambda.$$

Notice that if $M = \begin{pmatrix} \alpha & 0 \\ 0 & \delta \end{pmatrix} \in \mathrm{Iw}_n$, then $M\big(k(1+p^n Z)\big) = k(\alpha)k(1+p^n Z)$ and $M(W_0) = \delta W_0$ giving explicitly the action of diagonal matrices on each $\mathbb{W}_k(T,s,t-\lambda s)$. For every $\lambda = 0,\dots,p^n-1$ define the map

$$\nu_\lambda\colon \mathbb{W}_k(T,s,t-\lambda s) \longrightarrow A^o_{k,\lambda}(T_0^\vee)[n], \quad k(1+p^n Z)\sum_i a_i \left(\frac{W_\lambda}{1+p^n Z}\right)^i \mapsto \sum_i a_i w_\lambda^i \cdot u^k.$$

It is clearly an isomorphism of B-modules. We are left to show that

$$\nu_n := \sum_{\lambda=0}^{p^n-1} \nu_\lambda\colon \oplus_{\lambda=0}^{p^n-1} \mathbb{W}_k(T,s,t-\lambda s) \to \oplus_{\lambda=0}^{p^n-1} A^o_{k,\lambda}(T_0^\vee)[n] = A^o_k(T_0^\vee)[n]$$

is Iw_n-equivariant.

Consider the B-linear map $\xi\colon B\langle X,Y\rangle \to A(T^\vee)$, where $A(T^\vee)$ is the ring of analytic functions from $T^\vee$ to B, sending $f(X,Y) = \sum_{h,m} a_{h,m}X^hY^m$ to the function $\xi\big(f(X,Y)\big)\colon T^\vee \to B$, $ue_0 + ve_1 \to \sum_{h,m} a_{h,m}u^hv^m$. This map is Iw_n-equivariant. Indeed, given $M \in \mathrm{Iw}_n$ such that $M(f_0) = \alpha f_0 + \gamma f_1$, $M(f_1) = \beta f_0 + \delta f_1$ then $M(e_0) = \alpha e_0 + \beta e_1$, $M(e_1) = \gamma e_0 + \delta e_1$ so that $M(ue_0 + ve_1) = (u\alpha + v\gamma)e_0 + (u\beta + \delta v)e_1$. Hence, $M\big(\xi(f(X,Y))\big) = \xi\big(f\big(M(X),M(Y)\big)\big)$. As ν_n is determined by ξ using that $X = 1+p^n Z$ and $Y = p^n W_\lambda + \lambda X$, this implies that ν_n is Iw_n-equivariant as well. $\square$

Note that we have a Iw_n-equivariant map of functors, and hence of representing objects, $\amalg_{\lambda\in\mathbb{Z}/p^n\mathbb{Z}}\mathbb{V}_0(T,s,t-\lambda s) \to \mathbb{V}_0(T,s)$. This provides a Iw_n-equivariant map

$$\mathbb{W}_k(T,s) \to \oplus_{\lambda=0}^{p^n-1}\mathbb{W}_k(T,s,t-\lambda s).$$

Let $Q_n \subset T^\vee/p^nT^\vee$ be the $(\mathbb{Z}/p^n\mathbb{Z})$-dual of the quotient $\big(T/p^nT\big)/(\mathbb{Z}/p^n\mathbb{Z})s$. The duality between

$$\zeta_k\colon \mathbb{W}_k(T,s)^\vee \longrightarrow \mathbb{W}_k^D(T^\vee,s^\vee,Q_n)$$

of Definition 3.5 composed with the Iw_n-equivariant isomorphism

$$\nu_n^\vee\colon D^o_k(T_0^\vee)[n] \cong \oplus_{\lambda=0}^{p^n-1}\mathbb{W}_k(T,s,t-\lambda s)^\vee$$

of Proposition 3.8 give the following.

Corollary 3.9. *We have a* Iw_n*-equivariant,* B*-linear map*

$$D^o_k(T_0^\vee)[n] \cong \oplus_{\lambda=0}^{p^n-1}\mathbb{W}_k(T,s,t-\lambda s)^\vee \longrightarrow \mathbb{W}_k^D\big(T^\vee,s^\vee,Q_n\big).$$

3.5.2. The U_p operator. For $\rho = 0,\dots,p-1$, let $\pi_\rho\colon T \to T$ be the map defined by $\begin{pmatrix} 1 & \rho \\ 0 & p \end{pmatrix}$, that is, $f_0 \mapsto f_0$, $f_1 \mapsto pf_1 + \rho f_0$. It defines the map $\pi_\rho^\vee\colon T^\vee \to T^\vee$ that sends $ue_0 + ve_1 \mapsto ue_0 + (pv+\rho)e_1$. In particular, taking $(\pi_\rho^\vee)^*$ it induces a map $A^o_k(T_0^\vee)[n+1] \to A^o_k(T_0^\vee)[n]$ that is 0 on $A^o_{k,\lambda}(T_0^\vee)[n+1]$ for $\lambda \not\equiv \rho$ modulo p and it induces a map

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$A^o_{k,\lambda}(T_0^\vee)[n+1] \to A^o_{k,\lambda_0}(T_0^\vee)[n]$ if $\lambda = \rho + p\lambda_0$, with $\lambda_0 \in \{0,\ldots,p^n-1\}$. Taking the sum over the ρ's, we get a map $\pi_n = \sum_{\rho=0}^{p-1}(\pi_\rho^\vee)^*$, where

$$\begin{aligned}\pi_n \colon A^o_k(T_0^\vee)[n+1] = \oplus_{\rho=0}^{p-1}\oplus_{\lambda_0=0}^{p^n-1} A^o_{k,\rho+p\lambda_0}(T_0^\vee)[n+1] &\to \oplus_{\lambda_0\in\mathbb{Z}/p^n\mathbb{Z}} A^o_{k,\lambda_0}(T_0^\vee)[n] \\ &= A^o_k(T_0^\vee)[n].\end{aligned}$$

Notice that π_ρ defines by functoriality a map $\mathbb{V}_0(T,s,t-\lambda s,p^n) \to \mathbb{V}_0(T,s,t-(\rho+p\lambda)s,p^{n+1})$ (we have added the dependence on the power of p in the definition of $\mathbb{V}_0$ to avoid confusion). This gives a map $\mu_\rho \colon \mathbb{W}_k(T,s,t-(\rho+p\lambda)s,p^{n+1}) \to \mathbb{W}_k(s,t-\lambda s,p^n)$ and, summing over all ρ's,

$$\mu_n = \sum_{\rho=0}^{p-1}\mu_\rho \colon \oplus_{\rho=0}^{p-1}\oplus_{\lambda=0}^{p^n-1}\mathbb{W}_k(T,s,t-(\rho+p\lambda)s,p^{n+1}) \to \oplus_{\lambda\in\mathbb{Z}/p^n\mathbb{Z}}\mathbb{W}_k(T,s,t-\lambda s,p^n).$$

Lemma 3.10. *With the notation of Proposition 3.8, we have $\pi_n \circ \nu_{n+1} = \nu_n \circ \mu_n$ and similarly taking strong duals $\nu_{n+1}^\vee \circ \pi_n^\vee = \mu_n^\vee \circ \nu_n^\vee$.*

Proof. This is an explicit computation using the notation of the proof of Proposition 3.8 and follows from the fact that $(\pi_\rho^\vee)^*$ sends $w_{\rho+p\lambda_0} \mapsto w_{\lambda_0}$ and μ_λ sends $W_{\rho+p\lambda} \mapsto W_\lambda$. □

4. The modular curve setting

Let $p > 0$ be a prime integer. We fix once for all the p-adic completion $\mathbb{C}_p$of an algebraic closure of $\mathbb{Q}_p$. We denote by v the valuation on $\mathbb{C}_p$, normalized such that $v(p) = 1$.

Let $N \geq 5$ and $r \geq 0$ be integers with N prime to p, and let $X_0(p^r,N)$, resp. $X_1(p^r,N)$, resp. $X(p^r,N)$, be the modular curves over $\mathbb{C}_p$ of level $\Gamma_1(N)\cap\Gamma_0(p^r)$, resp. $\Gamma_1(N)\cap\Gamma_1(p^r)$, resp. $\Gamma_1(N)\cap\Gamma(p^r)$. Over the complement of the cusps of the modular curve $X_0(p^s,N)$, we have a universal elliptic curve E, a cyclic subgroup $H_s \subset E[p^s]$ of order p^s and an embedding $\Psi_N \colon \mu_N \hookrightarrow E[N]$. For $X_1(p^s,N)$, we further have a generator of H_s.

We denote the associated adic space over $\mathrm{Spa}(\mathbb{C}_p,\mathcal{O}_{\mathbb{C}_p})$ by $\mathcal{X}_0(p^r,N)$, resp. $\mathcal{X}_1(p^r,N)$, resp. $\mathcal{X}(p^r,N)$ considered as adic spaces with logarithmic structures given by the cusps, with reduced structure, as in [15, Ex. 2.3.17]. We simply write X, resp. $\mathcal{X}$ for $X_0(p^0,N)$, resp. $\mathcal{X}(p^0,N)$. Notice that the p^r-torsion of the universal elliptic curve E over the complement of the cusps in X defines a locally constant sheaf for the finite Kummer étale topology that we denote by $E[p^r]$ and $\mathcal{X}(p^r,N) \to \mathcal{X}$ is the finite Kummer étale Galois cover, with group $\mathbf{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$, defined by trivializing it. We let $T_p(E)$ be the sheaf on X_{profket}, resp. $\mathcal{X}_{\mathrm{profket}}$ defined by the inverse limt $\lim E[p^r]$. Thanks to [23, Thm. 3.1.2], we have

(i) a unique perfectoid space $\mathcal{X}(p^\infty,N)$ such that $\mathcal{X}(p^\infty,N) \sim \lim\limits_{\infty\leftarrow r}\mathcal{X}(p^r,N)$ in the sense of [24, Def. 2.4.1];

(ii) the Hodge–Tate period map $\pi_{\mathrm{HT}} \colon \mathcal{X}(p^\infty,N) \longrightarrow \mathbb{P}^1_{\mathbb{Q}_p}$.

In particular, we have morphisms of adic spaces $\pi_r \colon \mathcal{X}(p^\infty,N) \to \mathcal{X}(p^r,N)$, compatible for varying $r \geq 0$, inducing a homeomorphism of the underlying topological spaces $|\mathcal{X}(p^\infty,N)| \cong \lim\limits_{\infty\leftarrow r}|\mathcal{X}(p^r,N)|$.

4.1. On the pro-Kummer étale topology of modular curves

For every $s \in \mathbb{N}$, we write $\lim_{\infty \leftarrow r} \mathcal{X}(p^r,N)$ (for $r \geq s$) for the pro-finite Kummer étale cover of $\mathcal{X}_0(p^s,N)$ defined by the Kummer étale covers $\mathcal{X}(p^r,N) \to \mathcal{X}_0(p^s,N)$, for $r \geq s$. The following lemma provides a basis for the pro-Kumemr étale topology of $\mathcal{X}_0(p^s,N)$:

Lemma 4.1. *For every $s \in \mathbb{N}$, the site $\mathcal{X}_0(p^s,N)_{\mathrm{pke}}$ has a basis consisting of log affinoid perfectoid opens U that are pro-Kummer étale over an affinoid perfectoid open of $\lim_{\infty \leftarrow r} \mathcal{X}(p^r,N)$ (for $r \geq s$). In particular, for any such open U, $T_p(E)|_U$ is a constant sheaf and such basis is closed under fibre products over $\mathcal{X}_0(p^s,N)$.*

Proof. Due to [15, Prop. 5.3.12], the site $\mathcal{X}_0(p^s,N)_{\mathrm{pke}}$ admits a basis consisting of log affinoid perfectoid opens, and thanks to [15, Prop. 5.3.11], the category of such bases is closed under fibre products. Given any such W, consider the fibre product Z of W and $\lim_{\infty \leftarrow r} \mathcal{X}(p^r,N)$ over $\mathcal{X}_0(p^s,N)$. By [15, Cor. 5.3.9] if such a fibre product exists, it is pro-Kummer étale over $\lim_{\infty \leftarrow r} \mathcal{X}(p^r,N)$. As a cover of W, it can be represented as $\lim_{\infty \leftarrow r} W_r$ with $W_r := \mathcal{X}(p^r,N) \times_{\mathcal{X}_0(p^s,N)} W$. In particular, thanks to [15, Lemma 5.3.8], each W_r is a finite étale cover of W so that $Z = \lim_{\infty \leftarrow r} W_r \to W$ is a pro-finite étale cover of W. Since W is log affinoid perfectoid, we deduce from [15, Cor. 5.3.9] that also Z is log affinoid perfectoid. Recall from [23, Thm. 3.1.2] that $\lim_{\infty \leftarrow r} \mathcal{X}(p^r,N)$ is covered by perfectoid affinoid open subsets. Taking the fibre product over $\lim_{\infty \leftarrow r} \mathcal{X}(p^r,N)$ of Z with a cover of $\lim_{\infty \leftarrow r} \mathcal{X}(p^r,N)$ by perfectoid affinoid open subsets, the claim follows. □

Remark 4.2. Endow $\mathcal{X}(p^\infty,N)$ with the limit log structure. Then, open affinoid subsets of $\mathcal{X}(p^\infty,N)$ for the analytic topology are *not* log affinoid perfectoid opens as condition (d) of §2.3 is not satisfied.

This condition is used in [15, Cor. 5.3.8], an analogue of Abhyankar's lemma, stating that, for a log affinoid perfectoid, the finite Kummer étale site coincides with the finite étale site. This was already used in the proof of Lemma 4.1.

4.2. Standard opens

We start by defining certain opens of $\mathbb{P}^1 := \mathbb{P}^1_{\mathbb{Q}_p}$, namely let for every $n \geq 1$ $U_0, U_\infty, U_\infty^{(n)}, U_0^{(n)} \subset \mathbb{P}^1$ be defined as follows. Let T denote a parameter at 0 on $\mathbb{P}^1$, then

a) $U_\infty = \{x \in \mathbb{P}^1 \mid \|\frac{1}{T}\|_x \leq 1\}$, $U_0 = \{x \in \mathbb{P}^1 \mid \|T-\lambda\|_x \leq 1$, for some $\lambda \in \{0,1,\ldots,p-1\}\}$,

b) $U_\infty^{(n)} = \mathbb{P}^1\left(\frac{1}{p^nT}\right) = \{x \in \mathbb{P}^1 \mid \|\frac{1}{T}\|_x \leq \|p^n\|_x\}$,

c) $U_0^{(n)} = \cup_\lambda U_{0,\lambda}^{(n)}$ with $\lambda = \lambda_0 + \lambda_1 p + \ldots + \lambda_{n-1}p^{n-1}$, where $\lambda_0,\lambda_1,\ldots,\lambda_{n-1} \in \{0,1,\ldots,p-1\}$ and we have $U_{0,\lambda}^{(n)} := \mathbb{P}^1\left(\frac{T-\lambda}{p^n}\right)$.

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We remark that for every $n \geq 1$ we have $\mathbb{P}^1(\mathbb{Q}_p,\mathbb{Z}_p) \subset U_\infty^{(n)} \cup U_0^{(n)}$, and moreover, the family $\{U_\infty^{(n)} \cup U_0^{(n)}\}_{n\geq 1}$ is a fundamental system of open neighbourhoods of $\mathbb{P}^1(\mathbb{Q}_p,\mathbb{Z}_p)$ in $\mathbb{P}^1$.

We recall from [23, Thm. 3.3.18] that for every rational open subset $U' \subset U_0$ or $U' \subset U_\infty$ the inverse image $\mathcal{U}' := \pi_{\mathrm{HT}}^{-1}(U')$ is an affinoid perfectoid open subspace of $\mathcal{X}(p^\infty,N)$. In particular, it is quasi-compact so that it is also the inverse image of an open $\mathcal{U}'_r$ via π_r of $\mathcal{X}(p^r,N)$ for some r using the homeomorphism $|\mathcal{X}(p^\infty,N)| \cong \lim_{\infty\leftarrow n} |\mathcal{X}(p^r,N)|$. As the transition maps in the inverse limit are finite and surjective, $\mathcal{U}'_r$ is in fact the image of $\mathcal{U}'$ via π_r for r large enough. If U' is further invariant for the action of the Iwahori subgroup $\mathrm{Iw}_s \subset \mathbf{GL}_2(\mathbb{Z}_p)$ of matrices which are upper triangular modulo p^s, then also $\mathcal{U}'$ is Iw_s-invariant as π_{HT} is Iw_s-equivariant and $\mathcal{U}'$ is the inverse image of a unique open $\mathcal{U}'_{0,s}$ of $\mathcal{X}_0(p^s,N)$. Indeed, given $\mathcal{U}'_r \subset \mathcal{X}(p^r,N)$ for some $r \geq s$ such that its inverse image gives $\mathcal{U}'$, then $\mathcal{U}'_r$ is Iw_s-invariant. As the morphism $\mathcal{X}(p^r,N) \to \mathcal{X}_0(p^s,N)$ is finite Kummer étale and Galois with group G_s equal to the image of $\mathrm{Iw}_s \subset \mathbf{GL}_2(\mathbb{Z}_p) \to \mathbf{GL}_2(\mathbb{Z}/p^r\mathbb{Z})$ then $\mathcal{U}'_{0,s} := \mathcal{U}'_r/G_s$ is an open of $\mathcal{X}_0(p^s,N)$ with the required properties. Furthermore, $\mathcal{U}'_{0,s}$ defines the open $(\mathcal{U}'_r)_{r\geq s}$ for the pro-Kummer étale site of $\mathcal{X}_0(p^s,N)$, and hence of $\mathcal{X}$, given by $\mathcal{U}'_r := \mathcal{U}'_{0,s} \times_{\mathcal{X}_0(p^s,N)} \mathcal{X}(p^r,N)$. By construction, $\mathcal{U}' \sim \lim_{\infty\leftarrow r} \mathcal{U}'_r$.

In particular, for every $n \geq 1$, we consider the open rational subspaces $U_\infty^{(n)}$ of U_∞ and $U_0^{(n)}$ of U_0 defined above. We remark that $U_\infty^{(n)}$ is invariant under the left action of the subgroup Iw_n. Then we denote by $\mathcal{X}(p^\infty,N)_0^{(n)} := \pi_{\mathrm{HT}}^{-1}(U_0^{(n)})$ and $\mathcal{X}(p^\infty,N)_\infty^{(n)} := \pi_{\mathrm{HT}}^{-1}(U_\infty^{(n)})$ and recall that they define affinoid perfectoid open subspaces of $\mathcal{X}(p^\infty,N)$.

As explained above, they also define opens for the pro-Kummer étale site of $\mathcal{X}_0(p^n,N)$ and of $\mathcal{X}$ respectively. Namely, for $n \geq 1$, $\mathcal{X}(p^\infty,N)_\infty^{(n)}$, being invariant under Iw_n, descends to an affinoid open denoted $\mathcal{X}_0(p^m,N)_\infty^{(n)}$ of $\mathcal{X}_0(p^m,N)$, for all $m \geq n$. We also have variants $\mathcal{X}_1(p^m,N)_\infty^{(n)}$, resp. $\mathcal{X}(p^m,N)_\infty^{(n)}$, if we descend to $\mathcal{X}_1(p^m,N)$, resp. $\mathcal{X}(p^m,N)$.

In contrast, as $U_0^{(n)}$ is invariant with respect to Iw_1, the open $\mathcal{X}(p^\infty,N)_0^{(n)}$ descends to an open affinoid denoted $\mathcal{X}_0(p^m,N)_0^{(n)}$ of $\mathcal{X}_0(p^m,N)$, for all $m \geq 1$. In this case, we consider the variant $\mathcal{X}(p^m,N)_0^{(n)}$ open of $\mathcal{X}(p^m,N)$.

Lemma 4.3. *For every $h \in \mathbb{N}$, there exists $n = n(h) \geq 1$ such that for every $r \geq n$ the universal elliptic curve over $\mathcal{X}_0(p^r,N)_0^{(n)}$ and $\mathcal{X}_0(p^r,N)_\infty^{(n)}$ resp. admits a canonical subgroup of order p^h.*

Proof. This follows from [23, Lemma 3.3.14], stating that the preimage via π_{HT} of $\mathbb{P}^1(\mathbb{Q}_p,\mathbb{Z}_p)$ is, as a topological space, the closure of the inverse image in $\mathcal{X}(p^\infty,N)$ of the ordinary locus and the cusps of $\mathcal{X}$.

In particular, $\mathcal{X}(p^\infty,N)_0^{(n)}$ and $\mathcal{X}(p^\infty,N)_\infty^{(n)}$ define a fundamental system of open neighbourhoods of the ordinary locus in $\mathcal{X}(p^\infty,N)$. $\square$

Remark 4.4. Recall that the ordinary locus in $\mathcal{X}_0(p,N)$ has two connected components. Then $\mathcal{X}(p,N)_0^{(1)}$ and $\mathcal{X}(p,N)_\infty^{(1)}$ are neighbourhoods of these two components. The first is defined by requiring that the level subgroup is *not* the canonical one while the second is the component where the level subgroup *coincides* with the canonical one. Following

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conventions going back to Robert Coleman, we set up the notation so that the first is indexed by 0 and the second by ∞.

A reason to introduce the open subsets $\mathcal{X}_0(p^r,N)_\infty^{(n)}$ for $r\geq n$ and $\mathcal{X}_0(p^r,N)_0^{(n)}$, for any $r\geq 1$, is that they behave nicely under the U_p-correspondence, as we will explain below.

4.3. On the Hodge–Tate period map

Over $\mathcal{X}(p^\infty,N)$, the sheaf $T_p(E)$ admits a universal trivialization

$$T_p(E)=\mathbb{Z}_p a\oplus \mathbb{Z}_p b.$$

The map dlog defines a surjective map onto the sheaf of invariant differentials of E

$$T_pE^\vee\otimes_{\mathbb{Z}_p}\mathcal{O}_{\mathcal{X}(p^\infty,N)}\longrightarrow \omega_E$$

which is used to define the map π_{HT}: For every log affinoid perfectoid open $W=\mathrm{Spa}(R,R^+)$ of $\mathcal{X}(p^\infty,N)$ such that the universal elliptic curve extends to a (generalized) elliptic curve over $\mathrm{Spec}(R^+)$ and ω_E is generated as R^+-module by one element that we denote Ω_W, we write $\mathrm{dlog}(a^\vee)=\alpha\Omega_W$, $\mathrm{dlog}(b^\vee)=\beta\Omega_W$ with α, $\beta\in R$ generating the whole ring R. Then $\pi_{\mathrm{HT}}|_W: W\to\mathbb{P}^1$ is defined in homogeneous coordinates by $[\alpha;\beta]$. Namely, let $W_\infty\subset W$ be the rational open defined by $W(1/\alpha)$, and let $W_0\subset W$ be the rational open defined by $W(1/\beta)$. Then $\pi_{\mathrm{HT}}|_{W_0}: W_0\to U_0$ sends the standard coordinate T on the standard affinoid neighbourhood $U_0=\mathbb{A}^1$ of 0 to α/β and $\pi_{\mathrm{HT}}|_{W_\infty}: W_\infty\to U_\infty$ sends the standard coordinate T on the standard affinoid neighbourhood $U_\infty:=\mathbb{A}^1$ of ∞ to β/α.

For every $n\in\mathbb{N}$, we can refine such morphism to a morphism on $\mathcal{X}_0(p^n,N)_{\mathrm{pke}}$ as follows. Considering the smooth formal model $\mathfrak{X}$ of the modular curve X over $\mathcal{O}_{\mathbb{C}_p}$ given by moduli theory and the universal generalised elliptic curve E over $\mathfrak{X}$, the invariant differentials of E relative to $\mathfrak{X}$ and the fact that $\mathcal{X}$ is the adic generic fibre of $\mathfrak{X}$, give an invertible $\mathcal{O}^+_{\mathcal{X}}$-module ω_E^+ on $\mathcal{X}$. Pulling back via the projection map $\mathcal{X}_0(p^n,N)\to\mathcal{X}$, we get a $\mathcal{O}^+_{\mathcal{X}_0(p^n,N)_{\mathrm{pke}}}$-module for the pro-Kummer étale topology that we still abusively denote ω_E^+ and passing to p-adic completions we finally obtain an invertible $\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)}$-module $\widehat{\omega}_E^+$. Here, for simplicity, we write $\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)}$ for $\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)_{\mathrm{pke}}}$.

Consider the map dlog for the basis of $\mathcal{X}_0(p^n,N)_{\mathrm{pke}}$ given in Lemma 4.1. Let a modulo p^n be a generator in $T_p(E)/p^nT_p(E)$ of the level subgroup of order p^n. For every log affinoid perfectoid open U as in loc. cit., write $\widehat{U}=\mathrm{Spa}(R,R^+)$. Then $T_p(E)^\vee$ is constant on U; we have a universal generalized elliptic curve E over $\mathrm{Spec}(R^+)$ and $\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)}(U)=R^+$. We then have the map $\mathrm{dlog}: T_p(E)^\vee(U)\otimes_{\mathbb{Z}_p}R^+\longrightarrow\widehat{\omega}_E^+(U)$. Gluing, we obtain a map of sheaves on $\mathcal{X}_0(p^n,N)_{\mathrm{pke}}$:

$$\mathrm{dlog}: T_p(E)^\vee\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)}\longrightarrow\widehat{\omega}_E^+. \tag{2}$$

Proposition 4.5. *For every $r\in\mathbb{N}$ there exist $m\in N$ such that for every $n\in\mathbb{N}$ with $n\geq m$, r the following hold:*

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a. *There exists a canonical subgroup* C_r *of order* p^r *on* $\mathcal{X}_0(p^n,N)_\infty^{(m)}$ *and on* $\mathcal{X}_0(p^n,N)_0^{(m)}$ *which on* $\mathcal{X}_0(p^n,N)_\infty^{(m)}$ *coincides with* p^r*-torsion of the level* p^n *subgroup and which on* $\mathcal{X}_0(p^n,N)_0^{(m)}$ *is disjoint from the level* p^n *subgroup;*

b. *There exists an invertible* $\mathcal{O}^+_{\mathcal{X}_0(p^n,N)_\infty^{(m)}}$*-module* ω_E^{mod} *on* $\mathcal{X}_0(p^n,N)_\infty^{(m)}$ *(resp. a* $\mathcal{O}^+_{\mathcal{X}_0(p^n,N)_0^{(m)}}$*-module* ω_E^{mod} *on* $\mathcal{X}_0(p^n,N)_0^{(m)}$*) contained in* ω_E^+*;*

c. *The morphism* dlog *surjects onto the p-adic completion* $\widehat{\omega}_E^{\mathrm{mod}}$ *of the pullback of* ω_E^{mod} *to* $\mathcal{X}_1(p^n,N)_{\infty,\mathrm{pke}}^{(m)}$ *and its restriction to* $T_p(E)^\vee$ *modulo* p^r *factors via* $C_r^\vee$*. The kernel of* dlog *is isomorphic to the p-adic completion of* $\left(\widehat{\omega}_E^{\mathrm{mod}}\right)^{-1}$ *(here, we omit the Tate twist that usually appears as we work over* $\mathbb{C}_p$*);*

d. *The map* dlog *surjects onto the p-adic completion* $\widehat{\omega}_E^{\mathrm{mod}}$ *of* ω_E^{mod} *on* $\mathcal{X}(p^n,N)_{0,\mathrm{pke}}^{(m)}$ *and its restriction to* $T_p(E)^\vee$ *modulo* p^r *factors via* $C_r^\vee$*. The kernel of* dlog *is isomorphic to the p-adic completion of* $\left(\widehat{\omega}_E^{\mathrm{mod}}\right)^{-1}$*.*

Proof. It follows from [3] that the result holds true on strict neighbourhoods $\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right)$ of the ordinary loci in $\mathcal{X}$ defined by the points x, where $|p|_x < |\mathrm{Ha}^{p^s}|_x$ for s large enough; here, Ha is a (any) local lift of the Hasse invariant. Thanks to [23, Lemma 3.3.8], there exists $m \in \mathbb{N}$ such that $\mathcal{X}(p^n,N)_\infty^{(m)}$ and $\mathcal{X}(p^n,N)_0^{(m)}$ are contained in $\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right)$. See Lemma 4.3. The claim follows. □

Remark 4.6. See [12] for similar results in the case of Shimura curves. The notation ω_E^{mod} is taken from [20].

From Proposition 4.5, we get an integral version of the Hodge–Tate exact sequence:

$$0 \to \left(\widehat{\omega}_E^{\mathrm{mod}}\right)^{-1} \longrightarrow T_p(E)^\vee \otimes_{\mathbb{Z}_p} \widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)_\infty^{(m)}} \longrightarrow \widehat{\omega}_E^{\mathrm{mod}} \to 0.$$

This will be useful to compute the cohomology $\mathrm{H}^1\left(\mathcal{X}_0(p^n,N)_{\infty,\mathrm{pke}}^{(m)}, \widehat{\mathcal{O}}_{\mathcal{X}_0(p^n,N)_\infty^{(n)}}\right)$. In fact, tensoring the exact sequence with $\widehat{\omega}_E^{\mathrm{mod},2}$ and taking the long exact sequence in cohomology we obtain a map

$$\mathrm{H}^0\left(\mathcal{X}_0(p^n,N)_{\infty,\mathrm{pke}}^{(m)}, \widehat{\omega}_E^{\mathrm{mod},2}\right) \longrightarrow \mathrm{H}^1\left(\mathcal{X}_0(p^n,N)_{\infty,\mathrm{pke}}^{(m)}, \widehat{\mathcal{O}}^+_{\mathcal{X}_1(p^n,N)_\infty^{(m)}}\right) \tag{3}$$

and similarly for $\mathcal{X}_0(p^n,N)_0^{(m)}$, or their covers $\mathcal{X}_1(p^n,N)_\infty^{(m)}$, $\mathcal{X}(p^n,N)_0^{(m)}$.

4.4. The sheaf ω_E^k

Take r, m and $n \in \mathbb{N}$ as in Proposition 4.5. The map dlog provides $\omega_E^{\mathrm{mod}}/p^r\omega_E^{\mathrm{mod}}$ with a marked section s over $\mathcal{X}_1(p^n,N)_\infty^{(m)}$ as the image of the tautological generator of $C_r^\vee$.

Similarly, recall that we have a decomposition $\mathcal{X}(p^n,N)_0^{(m)} = \amalg_{\lambda \in \mathbb{Z}/p^n\mathbb{Z}} \mathcal{X}(p^n,N)_{0,\lambda}^{(m)}$, where over $\mathcal{X}(p^n,N)_{0,\lambda}^{(m)}$, using the trivialization $T_p(E)/p^nT_p(E) = (\mathbb{Z}/p^n\mathbb{Z})a \oplus (\mathbb{Z}/p^n\mathbb{Z})b$, we have $\mathrm{dlog}(a^\vee) = \lambda \mathrm{dlog}(b^\vee)$. In particular, the canonical subgroup C_r is generated by $b+\lambda a$ and $\omega_E^{\mathrm{mod}}/p^r\omega_E^{\mathrm{mod}}$ acquires a marked section $s := \mathrm{dlog}(b^\vee)$.

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Suppose we are given an h-analytic character $k\colon \mathbb{Z}_p^* \to B^*$, for $h \le r$, as in Definition 3.6. Using the formalism of VBMS from §3.3 for ω_E^{mod} and the section s modulo p^r, we get the invertible $\mathcal{O}^+_{\mathcal{X}_1(p^n,N)_\infty^{(m)}}\widehat{\otimes}B$-module, resp. $\mathcal{O}_{\mathcal{X}(p^n,N)_0^{(m)}}\widehat{\otimes}B$-module

$$\omega_E^k := \mathbb{W}_k(\omega_E^{\mathrm{mod}}, s).$$

Since ω_E^{mod} and the section s modulo p^r are stable under the action of the automorphism group $\Delta_n := (\mathbb{Z}/p^n\mathbb{Z})^*$ of $j_n\colon \mathcal{X}_1(p^n,N)_\infty^{(m)} \to \mathcal{X}_0(p^n,N)_\infty^{(m)}$, then $(\mathbb{Z}/p^n\mathbb{Z})^*$ acts on $j_{n,*}\big(\omega_E^k[1/p]\big)$ and, taking the subsheaf of $j_{n,*}\big(\omega_E^k[1/p]\big)$ on which $\mathbb{Z}_p^*$ acts via the character k, then $j_{n,*}\big(\omega_E^k[1/p]\big)$ descends to an invertible $\mathcal{O}_{\mathcal{X}_0(p^n,N)_\infty^{(m)}}\widehat{\otimes}B[1/p]$-module that we denote $\omega_E^k[1/p]$.

Similarly, the Galois group of $j_n'\colon \mathcal{X}(p^n,N)_0^{(m)} \to \mathcal{X}_0(p^n,N)_0^{(m)}$, which is identified with the standard Borel subgroup of $\mathrm{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$, acts compatibly on $(\omega_E^{\mathrm{mod}}, s)$ and hence it acts on $j'_{n,*}\big(\omega_E^k[1/p]\big)$. In this case we let $\omega_E^k[1/p]$ be the invertible $\mathcal{O}_{\mathcal{X}_0(p^n,N)_0^{(m)}}\widehat{\otimes}B[1/p]$-module defined as the subsheaf of $j'_{n,*}\big(\omega_E^k[1/p]\big)$ on which the standard Borel subroup of $\mathrm{GL}_2(\mathbb{Z}_p)$ acts via the projection onto the lower right entry $\mathbb{Z}_p^*$ composed with the character k.

4.5. The U_p-correspondence

Given the modular curve $\mathcal{X}_0(p^s,N)$ for $s \ge 1$, we have correspondences T_ℓ, for ℓ not dividing pN, and U_p. They are defined by the analytification $\mathcal{X}_0(p^s,N,\ell)$, resp. $\mathcal{X}_0(p^s,N,p)$, of the modular curve $X_0(p^s,N,\ell)$, resp. $X_0(p^s,N,p)$, classifying, at least away from the cusps, subgroups D of order ℓ of the universal elliptic curve E, resp. subgroups of order p of E complementary to the p-torsion H_1 of the cyclic subgroup H_s of order p^s defined by the level structure. We have two maps $q_1,q_2\colon, \mathcal{X}_0(p^s,N,\ell) \to \mathcal{X}_0(p^s,N)$ defined by the analytification of the maps $q_1,q_2\colon X_0(p^s,N,\ell) \to X_0(p^s,N)$, where q_1 sends the universal object (E,H_s,Ψ_N,D) to (E,H_s,Ψ_N) (the forgetful map) and q_2 sends the universal object (E,H_s,Ψ_N,D) to $(E/D,H_s',\Psi_N')$, where H_s' is the image of H_s via the isogeny $E \to E/D$ and Ψ_N' is Ψ_N composed with this isogeny.

These maps induce morphisms of sites $\mathcal{X}_0(p^s,N,\ell)_{\mathrm{pke}} \to \mathcal{X}_0(p^s,N)_{\mathrm{pke}}$, resp. $\mathcal{X}_0(p^s,N,p)_{\mathrm{pke}} \to \mathcal{X}_0(p^s,N)_{\mathrm{pke}}$. As q_1,q_2 are finite Kummer étale, the following follows from the discussion in [5, Cor. 2.6] or [15, Prop. 4.5.2].

Lemma 4.7. *There is a natural isomorphism of functors $q_{i,*} \cong q_{i,!}$. In particular, $q_{i,*}$ is exact and we have a natural transformation $\mathrm{Tr}_{q_i}\colon q_{i,*}q_i^* \to \mathrm{Id}$, called the trace map.*

The maps q_1 and q_2 induce maps of perfectoid spaces. Take ℓ prime to N but possibly equal to p. The fibre product $\mathcal{X}(p^\infty,N,\ell) := \mathcal{X}(p^\infty,N) \times^{q_i}_{\mathcal{X}_0(p^s,N)} \mathcal{X}_0(p^s,N,\ell)$ exists and is independent of the choice of maps q_1 or q_2. We then have the two projections $q_1,q_2\colon \mathcal{X}(p^\infty,N,\ell) \to \mathcal{X}(p^s,N)$. Notice that $\mathcal{X}(p^\infty,N,p)$ splits completely as

$$\mathcal{X}(p^\infty,N,p) = \amalg_{\lambda=0,\ldots,p-1}\mathcal{X}(p^\infty,N),$$

where over the copy labeled by $\lambda = 0,\dots,p-1$ the isogeny $E \to E' := E/D$ produces the map of $\mathbb{Z}_p$-modules

$$u_\lambda\colon T_pE = \mathbb{Z}_p a \oplus \mathbb{Z}_p b \to T_p(E') := T_p(E_\lambda) := \mathbb{Z}_p a' \oplus \mathbb{Z}_p b',$$
$$a' = a, b' = \frac{b+\lambda a}{p} \in T_pE \otimes \mathbb{Q}_p.$$

Using this description, the maps q_1 and q_2 restricted to the component labeled λ define maps $q_{1,\lambda}$ and $q_{2,\lambda}$, where $q_{1,\lambda}$ is the identity map and $q_{2,\lambda}\colon \mathcal{X}(p^\infty,N) \to \mathcal{X}(p^\infty,N)$ is an isomorphism such that the pullback of T_pE is $T_p(E_\lambda)$.

We consider the maps $t_1,t_2\colon \amalg_{\lambda=0,\dots,p-1}\mathbb{P}^1 \to \mathbb{P}^1$ where on the component labeled by λ the map $t_{1,\lambda}$ induced by t_1 is the identity while the map t_2 is the isomorphism $t_{2,\lambda}\colon \mathbb{P}^1 \to \mathbb{P}^1$ defined on points by $[\alpha,\beta] \mapsto [\alpha - \lambda\beta, p\beta]$. We then have the following diagram:

$$\begin{array}{ccccc} \mathcal{X}(p^\infty,N) & \xleftarrow{q_2} & \mathcal{X}(p^\infty,N,p) = \amalg_{\lambda=0,\dots,p-1}\mathcal{X}(p^\infty,N) & \xrightarrow{q_1} & \mathcal{X}(p^\infty,N) \\ \pi_{\mathrm{HT}}\big\downarrow & & \amalg_{\lambda=0,\dots,p-1}\pi_{\mathrm{HT}}\big\downarrow & & \pi_{\mathrm{HT}}\big\downarrow \\ \mathbb{P}^1 & \xleftarrow{t_2} & \amalg_{\lambda=0,\dots,p-1}\mathbb{P}^1 & \xrightarrow{t_1} & \mathbb{P}^1. \end{array}$$

In fact, the squares are commutative. This follows from the functoriality of dlog with respect to isogenies and by computing $u_\lambda^\vee$: The map u_λ sends $a \mapsto a'$ and $b \mapsto pb' - \lambda a'$ so that on the dual basis $u_\lambda^\vee$ sends $(a')^\vee \mapsto a^\vee - \lambda b^\vee$ and $(b')^\vee \mapsto pb^\vee$.

Remark 4.8. Let us observe that, with the notations above, if we denote by U_p the correspondence on $\mathcal{X}(p^\infty,N)$ given by $U_p := q_2 \circ q_1^{-1}$ and if we denote by $\widetilde{U}$ the correspondence on $\mathbb{P}^1$ defined by $\widetilde{U} := t_2 \circ t_1^{-1}$, then we have: $\pi_{\mathrm{HT}} \circ U_p = \widetilde{U} \circ \pi_{\mathrm{HT}}$.

We conclude this section with a lemma on the dynamic of the operators $t_{2,\lambda}$. For $\lambda = 0,\dots,p-1$ we write t_λ in place of $t_{2,\lambda}$ in the next lemma.

Lemma 4.9. a) *Let $\lambda = \lambda_0 + p\lambda_1 + \cdots + \lambda_n p^n$ with $\lambda_i \in \{0,\dots,p-1\}$. Write $t_\lambda := t_{\lambda_n} \circ \cdots \circ t_{\lambda_0}$. Then $t_\lambda\big(U_{0,\lambda}^{(n+1)}\big) = U_0$, $t_\lambda\big(\mathbb{P}^1 \backslash U_{0,\lambda}^{(n+1)}\big) = U_\infty^{(1)}$.*

b) *If $\mu := \lambda_0 + \lambda_1 p + \ldots + \lambda_{n-1}p^{n-1}$ with λ_i as above for $0 \le i \le n-1$ and $t_\mu := t_{\lambda_{n-1}} \circ \ldots \circ t_{\lambda_0}$, then we have $t_\mu\big(\mathbb{P}^1 \backslash U_{0,\lambda}^{(n+1)}\big) \subset U_\infty \backslash U_0^{(1)}$ and $t_\mu(U_\infty^{(1)}) \subset U_\infty^{(n+1)}$.*

Proof. It is enough to prove the statement for $\mathrm{Spa}(K,K^+)$-valued points for an affinoid field (K,K^+). This is determined by a K-valued point of $\mathbb{P}^1$, or equivalently a K^+-valued point $[\alpha;\beta]$ as K^+ is a valuation ring, whose normalized valuation is denoted v.

a) We prove the statement for $\lambda = \lambda_0$ leaving the inductive process to the reader. If $[\alpha;\beta]$ is a point of $U_\infty \backslash U_0$, then we can assume that $\alpha = 1$ and that β is in the maximal ideal of K^+ and $t_\lambda([1;\beta]) = [1 - \lambda\beta; p\beta]$ defines a point of $U_\infty^{(1)}$.

If β is a unit, we can assume that $\beta = 1$ and then $t_\lambda([\alpha;1]) = [\alpha - \lambda; p]$. This is a point of U_0 if and only if $\frac{\alpha-\lambda}{p} \in K^+$, that is, if and only if $[\alpha;1]$ defines a point of $U_{0,\lambda}^{(1)}$. Else $\frac{p}{\alpha-\lambda}$ lies in the maximal ideal of K^+ and then $t_\lambda([\alpha;1])$ defines a point of $U_\infty \backslash U_0$.

b) If $[\alpha,\beta] \in \mathbb{P}^1 \backslash U_0^{(n+1)}$, $\alpha,\beta \in K^+$, we may assume that one of α,β is 1.

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If $\beta = 1$, we have $t_\mu([\alpha,1]) = [\alpha - \mu, p^n]$. Moreover, $[\alpha,1] \notin U_0^{(n+1)}$ implies either that $r := v(\alpha - \mu) < n$ and in this case $[\alpha - \mu, p^n] \in U_\infty \backslash U_0^{(1)}$, or that $v(\alpha - \mu) = n$ and $\alpha = \mu + p^n\gamma$, with $\gamma \in K^+$, $\gamma \notin \mathbb{F}_p (\mathrm{mod}\ pK^+)$. Then $t_\mu([\alpha,\beta]) = [\gamma,1] \in U_\infty \backslash U_0^{(1)}$.

If now $\alpha = 1$ and β is in the maximal ideal of K^+, we have $t_\mu([1,\beta]) = [1 - \mu\beta, p^n\beta]$. Since $1 - \mu\beta \in (K^+)^*$, we have $t_\mu([1,\beta]) = [1, p^n\beta/(1-\mu\beta)] \in U_\infty^{(n)} \subset U_\infty \backslash U_0^{(1)}$.

Let us notice that if $x \in U_\infty \backslash U_0^{(1)}$ and $\nu \in \{0,1,\ldots,p-1\}$, then $t_\nu(x) \in U_\infty^{(1)}$. This observation and claim (b) imply the part $t_\lambda(\mathbb{P}^1 \backslash U_0^{(n+1)}) \subset U_\infty^{(1)}$ of claim (a). $\square$

4.6. Étale sheaves

Let $H \subset \mathbf{GL}_2(\mathbb{Z}_p)$ be a finite index subgroup. In this section, we recall the tensor functor from the category of profinite H-representations to the category of sheaves on the pro-Kummer étale site of the modular curve $X(H,N)$ defined by H or, equivalently, of the associated adic space $\mathcal{X}(H,N)$. We work with the latter.

We fix log geometric points ζ_i, one for every connected component Z_i of $\mathcal{X}(H,N)$. Due to [15, Prop. 5.1.12], the sites $Z_{i,\mathrm{fket}}$ are Galois categories with underlying profinite group, the Kummer étale fundamental group $\pi_1^{\mathrm{ket}}(Z_i,\zeta_i)$. In particular, the pro-Kummer finite étale cover $\big(\mathcal{X}(p^r,N)\big)_r$ of $\mathcal{X}(H,N)$ for r big enough, restricted to each Z_i defines a homomorphism

$$\pi_1(Z_i,\zeta_i) \to \lim_{\infty \leftarrow r} \mathrm{Aut}\big(\mathcal{X}(p^r,N)/\mathcal{X}(H,N)\big) = H.$$

Given a finite representation L_n of H, we view it as a representation of $\pi_1(Z_i,\zeta_i)$, for every i, and hence as a local system on each $Z_{i,\mathrm{ket}}$ and, hence, a local system $\mathbb{L}_n$ on $\mathcal{X}(H,N)_{\mathrm{ket}}$. In fact, $\mathbb{L}_n$ is a sheaf on $\mathcal{X}(H,N)_{\mathrm{ket}}$ such that there exists a finite Kummer étale cover $\mathcal{X}(p^r,N) \to \mathcal{X}(H,N)$, for $r \gg 0$, on which the sheaf $\mathbb{L}_n$ is constant.

Given a profinite representation L of H, that is, an inverse limit $L = \lim_{\infty \leftarrow n} L_n$ of finite representations L_n for $n \in \mathbb{N}$, we let $\mathbb{L}$ be the inverse limit $\mathbb{L} = \lim_{\infty \leftarrow n} \mathbb{L}_n$. It is a sheaf on $\mathcal{X}(H,N)_{\mathrm{ket}}$. Notice that using the scheme $X(H,N)$ one gets, as mentioned before, a sheaf on $X(H,N)_{\mathrm{ket}}$, that we will denote $\mathbb{L}$. We have the following GAGA type of results:

Theorem 4.10. *For every $i \in \mathbb{N}$, the maps*

$$\mathrm{H}^i\big(X(H,N)_{\mathrm{pke}},\mathbb{L}\big) \longrightarrow \mathrm{H}^i\big(\mathcal{X}(H,N)_{\mathrm{pke}},\mathbb{L}\big)$$

are isomorphisms. Analogously, the natural map

$$\mathrm{H}^i\big(\mathcal{X}(H,N)_{\mathrm{pke}},\mathbb{L}\big)\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p} \longrightarrow \mathrm{H}^i\big(\mathcal{X}(H,N)_{\mathrm{pke}},\mathbb{L}\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}(H,N)}\big)$$

is an almost isomorphism.

Proof. The result for each $\mathbb{L}_n$ follows from the discussion in §2.4 and from Proposition 2.1.

Consider the natural map $\lim\limits_{\leftarrow}\colon \mathrm{Sh}^{\mathbb{N}}(Z) \to \mathrm{Sh}(Z)$ from inverse systems of sheaves on $Z = X(H,N)_{\mathrm{pke}}$, and $Z = \mathcal{X}(H,N)_{\mathrm{pke}}$, respectively. Using the existence of bases of log affinoid perfectoid opens with the properties recalled in §2.3, it follows from [22, lemma 3.18] that we have $\mathrm{R}^i\lim\limits_{\leftarrow}(\mathbb{L}) = 0$, both in the algebraic and

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in the adic setting, and $\mathrm{R}^i\varprojlim\big(\mathbb{L}\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}(H,N)}\big)=0$ for $i\geq 1$. Hence, $\mathrm{H}^j\big(Z,\mathbb{L}\big)$ and $\mathrm{H}^j\big(\mathcal{X}(H,N)_{\mathrm{pke}},\mathbb{L}\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}(H,N)}\big)$ coincide with the derived functors $\mathrm{H}^j\big(Z,(\mathbb{L}_n)_{n\in\mathbb{N}}\big)$, resp. $\mathrm{H}^j\big(\mathcal{X}(H,N)_{\mathrm{pke}},(\mathbb{L}_n\otimes\mathcal{O}^+_{\mathcal{X}(H,N)_{\mathrm{pke}}})_{n\in\mathbb{N}}\big)$ of $\varprojlim\mathrm{H}^0(Z,\_)$ introduced by [19] on the inverse system $\mathrm{Sh}^{\mathbb{N}}(Z)$. Due to [19, Prop. 1.6], these cohomology groups sit in exact sequences

$$0\longrightarrow\varprojlim{}^{(1)}\mathrm{H}^{j-1}\big(Z,\mathbb{L}_n\big)\longrightarrow\mathrm{H}^j\big(Z,(\mathbb{L}_n)_{n\in\mathbb{N}}\big)\longrightarrow\varprojlim\mathrm{H}^j\big(Z,\mathbb{L}_n\big)\longrightarrow 0$$

and similarly for the inverse system $(\mathbb{L}_n\otimes\mathcal{O}^+_{\mathcal{X}(H,N)_{\mathrm{pke}}})_{n\in\mathbb{N}}$. The maps in the theorem are compatible with these sequences, and the claims follow from the finite case, that is, for the sheaves $\mathbb{L}_n$. □

Remark 4.11. The sheaf $\mathbb{L}$ has the property that its pullback to the perfectoid space $\mathcal{X}(p^\infty,N)$ is constant and coincides, together with the its H-action, with $\pi_{\mathrm{HT}}^{-1}(L)$ where we view L as a constant, H-equivariant sheaf on $\mathbb{P}^1$ (compare with [10, §2.3] for the case of p-adic automorphic étale sheaves).

Example 4.12. Consider on $\mathcal{X}(p^\infty,N)$ trivializations $T_pE=\mathbb{Z}_pa\oplus\mathbb{Z}_pb$. The group $\mathbf{GL}_2(\mathbb{Z}_p)$ acts on the left on the family of trivializations: Given such a basis $\mathcal{A}:=\{a,b\}$ as above and a matrix $M\in\mathbf{GL}_2(\mathbb{Z}_p)$, we get a new basis $\mathcal{A}':=(a',b'):=(a,b)M$. If we think of a trivialization as an isomorphism $\psi_{\mathcal{A}}\colon T_pE\cong\mathbb{Z}_p^2$, then $\psi_{\mathcal{A}'}$ is $\psi_{\mathcal{A}}$ times left multiplication by M. Thus, $T_p(E)$ corresponds to the standard representation $T=\mathbb{Z}_p\oplus\mathbb{Z}_p$ of $\mathbf{GL}_2(\mathbb{Z}_p)$.

This action of M induces a map on dual basis ${}^t(a^\vee,b^\vee)=M^t(a^{',\vee},b^{',\vee})$. Then the trivializations $\psi_{\mathcal{A}^\vee}:T_pE^\vee\cong\mathbb{Z}_p^2$ and $\psi_{\mathcal{A}',\vee}:T_pE^\vee\cong\mathbb{Z}_p^2$ induced by the dual bases are such that $\psi_{\mathcal{A}^\vee}$ is $\psi_{\mathcal{A}',\vee}$ times the right multiplication by M. To make the map π_{HT} equivariant for the $\mathbf{GL}_2(\mathbb{Z}_p)$-action we take on $\mathbb{P}^1_{\mathbb{Q}_p}$ the standard action. If $\pi_{\mathrm{HT}}(\psi_{\mathcal{A}^\vee})=[\alpha;\beta]$ and $\pi_{\mathrm{HT}}(\psi_{\mathcal{A}',\vee})=[\alpha';\beta']$, then ${}^t[\alpha;\beta]=M\,{}^t[\alpha';\beta']$ (where ${}^t[\alpha;\beta]$ means $[\alpha;\beta]$ viewed as column vector).

Consider an s-analytic character $k\colon\mathbb{Z}_p^\times\longrightarrow B^*$ as in Definition 3.6. Consider the $\mathbb{Q}_p$-module $D_k(T_0^\vee)[n]$, for $n\geq s$, with action of the Iwahori subroup Iw_1, defined in §3.7. As $D_k(T_0^\vee)[n]=\big(D_k^o(T_0^\vee)[n]\big)[1/p]$ and $D_k^o(T_0^\vee)[n]$ admits a Iw_1-equivariant filtration with finite graded pieces, we get an associated sheaf $\mathbb{D}_k(T_0^\vee)[n]$ on the pro-Kummer étale site of $\mathcal{X}_0(p,N)$. Then:

Proposition 4.13. *For every $i\in\mathbb{N}$, we have isomorphisms*

$$\mathrm{H}^i\big(\Gamma_0(p)\cap\Gamma(N),D_k(T_0^\vee)[n]\big)\widehat{\otimes}\mathbb{C}_p\cong\mathrm{H}^i\big(\mathcal{X}_0(p,N)_{\mathrm{pke}},\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p,N)}\big).$$

Proof. The first group is identified with $\mathrm{H}^i\big(X_0(p,N)_{\mathrm{pke}},\mathbb{D}_k(T_0^\vee)[n]\big)\widehat{\otimes}\mathbb{C}_p$ arguing as in [5, Thm. 3.15] using the filtration $\mathrm{Fil}^\bullet\mathbb{D}_k(T_0^\vee)[n]$ discussed in §3.5. As $\widehat{\mathcal{O}}_{\mathcal{X}_0(p,N)}=\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p,N)}[1/p]$ and cohomology commutes with direct limits, the conclusion follows from Theorem 4.10 and inverting p. □

For every $s\geq 1$, we have actions of Hecke operators on $\mathrm{H}^i\big(\mathcal{X}_0(p^s,N)_{\mathrm{pke}},\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}\big)$ as follows. Let ℓ be a prime integer not dividing N, and let $q_1,q_2\colon X_0(p,N,\ell)\to$

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$X_0(p,N)$ be as in §4.5. For $\ell \neq p$, we have natural isomorphisms $q_1^*\big(\mathbb{D}_k^o(T_0^\vee)[n]\big) \to q_2^*\big(\mathbb{D}_k^o(T_0^\vee)[n]\big)$ inducing a map

$$\mathcal{T}_\ell\colon q_1^*\big(\mathbb{D}_k^o(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^s,N)}\big) \to q_2^*\big(\mathbb{D}_k^o(T_0^\vee)[n]\big)\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^s,N,\ell)}.$$

Inverting p, taking $q_{2,*}$ and using the trace map $\mathrm{Tr}\colon q_{2,*}q_2^* \to \mathrm{Id}$ of Lemma 4.7, we get a map

$$q_{2,*}q_1^*\big(\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}\big) \to \mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}.$$

For $\ell = p$, it follows by taking the dual of the Iw_s-equivariant map π_n of §3.5.2 that we have a map $\mathcal{U}_p\colon q_1^*\big(\mathbb{D}_k^o(T_0^\vee)[n]\big) \to q_2^*\big(\mathbb{D}_k^o(T_0^\vee)[n+1]\big)$ and, hence, a map

$$\mathcal{U}_p\colon q_1^*\big(\mathbb{D}_k^o(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^s,N)}\big) \to q_2^*\big(\mathbb{D}_k^o(T_0^\vee)[n+1]\big)\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^s,N,p)}.$$

Inverting p, taking $q_{2,*}$ and using the trace map $\mathrm{Tr}\colon q_{2,*}q_2^* \to \mathrm{Id}$, we get a map

$$q_{2,*}q_1^*\big(\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}\big) \to \mathbb{D}_k(T_0^\vee)[n+1]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}.$$

We have a restriction map $D_k(T_0^\vee)[n+1] \to D_k(T_0^\vee)[n]$ which is Iw_s-equivariant and defines a map $\mathbb{D}_k(T_0^\vee)[n+1] \to \mathbb{D}_k(T_0^\vee)[n]$. We finally get a morphism

$$q_{2,*}q_1^*\big(\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}\big) \to \mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}.$$

Taking cohomology groups over $\mathcal{X}_0(p^s,N)_{\mathrm{pke}}$ and using the map

$$\mathrm{H}^i\big(\mathcal{X}_0(p^s,N)_{\mathrm{pke}},\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}\big) \to \mathrm{H}^i\big(\mathcal{X}_0(p^s,N,\ell)_{\mathrm{pke}},q_1^*\big(\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}\big)\big),$$

we get Hecke operators

$$\begin{aligned} T_\ell,U_p^{\mathrm{naive}}\colon \mathrm{H}^i\big(\mathcal{X}_0(p^s,N)_{\mathrm{pke}},\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}\big)& \\ \to \mathrm{H}^i\big(\mathcal{X}_0(p^s,N)_{\mathrm{pke}},\mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^s,N)}\big).& \end{aligned} \tag{4}$$

(Here we introduce the unnormalized operator U_p^{naive} which is p times the standard operator U_p as it preserves integral structures, a fact that will be of crucial importance in section §5.)

4.7. A comparison result on $\mathcal{X}_0(p^n,N)_\infty^{(m)}$

Consider an r-analytic weight $k\colon \mathbb{Z}_p^* \to B^*$ as in Definition 3.6 and integers $n \geq m$ as in Proposition 4.5, and define the sheaf

$$\mathfrak{D}_{k,\infty}^{o,(m)}[n] := \mathbb{D}_k^o(T_0^\vee)[n]|_{\mathcal{X}_0(p^n,N)_\infty^{(m)}}\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)_\infty^{(m)}}.$$

The aim of this section is to prove that for m large enough it admits a decreasing filtration $\mathrm{Fil}^h\mathfrak{D}_{k,\infty}^{o,(m)}[n]$ for $h \geq -1$ with the following properties.

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Proposition 4.14. *The following hold:*

i. *For $n \geq m' \geq m$, we have* $\mathrm{Fil}^{\bullet}\mathfrak{D}^{o,(m)}_{k,\infty}[n]\widehat{\otimes}_{\widehat{\mathcal{O}}^{+}_{\mathcal{X}_0(p^n,N)^{(m)}_{\infty}}}\widehat{\mathcal{O}}^{+}_{\mathcal{X}_0(p^n,N)^{(m')}_{\infty}} \cong \mathrm{Fil}^{\bullet}\mathfrak{D}^{o,(m')}_{k,\infty}[n]$;

ii. *We have a surjective map* $\mathrm{Gr}^{-1}\mathfrak{D}^{o,(m)}_{k,\infty}[n] = \mathfrak{D}^{o,(m)}_{k,\infty}[n]/\mathrm{Fil}^0\mathfrak{D}^{o,(m)}_{k,\infty}[n] \longrightarrow \omega_E^k \otimes_{\mathcal{O}^{+}_{\mathcal{X}_0(p^n,N)^{(m)}_{\infty}}}\widehat{\mathcal{O}}^{+}_{\mathcal{X}_0(p^n,N)^{(m)}_{\infty}}$, *where ω_E^k is the sheaf defined in §4.4;*

iii. *We have an isomorphism*

$$\mathrm{H}^1\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}},\omega_E^k \otimes_{\mathcal{O}^{+}_{\mathcal{X}_0(p^s,N)^{(m)}_{\infty}}}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^n,N)^{(m)}_{\infty}}\big) \cong \mathrm{H}^0\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty},\omega_E^{k+2}\big)[1/p];$$

iv. *The map*

$$\begin{aligned}&\mathrm{H}^1\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}},\mathfrak{D}^{o,(m)}_{k,\infty}[n]/\mathrm{Fil}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)\\ &\quad\to \mathrm{H}^1\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}},\omega_E^k\otimes_{\mathcal{O}^{+}_{\mathcal{X}_0(p^s,N)^{(m)}_{\infty}}}\widehat{\mathcal{O}}^{+}_{\mathcal{X}_0(p^n,N)^{(m)}_{\infty}}\big),\end{aligned}$$

induced by the projection onto $\mathrm{Gr}^{-1}\mathfrak{D}^{o,(m)}_{k,\infty}[n]$ and (ii), has kernel and cokernel annihilated by a power p^{ah} of p (with a depending on n);

v. $\mathrm{H}^i\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}},\mathfrak{D}^{o,(m)}_{k,\infty}[n]/\mathrm{Fil}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$ *is equal to* $\mathrm{H}^0\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty}, \mathrm{Gr}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$ *for $i=0$, and it is 0 for $i \geq 2$.*

Proof. Recall the surjective map

$$\mathrm{dlog}\colon T_p(E)^{\vee}\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^{+}_{\mathcal{X}_0(p^n,N)^{(m)}_{\infty}} \longrightarrow \omega_E^{\mathrm{mod}}\widehat{\otimes}\widehat{\mathcal{O}}^{+}_{\mathcal{X}_0(p^n,N)^{(m)}_{\infty}}$$

from equation (2) and Proposition 4.5. It is defined for every m large enough and ω_E^{mod} is an invertible $\mathcal{O}^{+}_{\mathcal{X}_0(p^n,N)^{(m)}_{\infty}}$-module. We also assume that over $\mathcal{X}_0(p^n,N)^{(m)}_{\infty}$ we have a canonical subgroup C_r of order p^r. Consider the natural projection $j_n\colon \mathcal{X}(p^n,N)^{(m)}_{\infty} \to \mathcal{X}_0(p^n,N)^{(m)}_{\infty}$; then j_n is Kummer étale and Galois with group Δ_n the subgroup of $\mathbf{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ of upper triangular matrices. In particular, to define a sheaf on $\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}}$ is equivalent to define a sheaf on $\mathcal{X}(p^n,N)^{(m)}_{\infty,\mathrm{pke}}$ with an action of Δ_n compatible with the action on $\mathcal{X}(p^n,N)^{(m)}_{\infty}$. In order to define $\mathrm{Fil}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]$, we define a Δ_n-equivariant filtration on $j_n^*\big(\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$.

Over $\mathcal{X}(p^n,N)^{(m)}_{\infty}$, we have a trivialization

$$T_p(E)/p^nT_p(E) = (\mathbb{Z}/p^n\mathbb{Z})a \oplus (\mathbb{Z}/p^n\mathbb{Z})b$$

such that $\mathrm{dlog}(a^{\vee})$ is a basis for $\omega_E^{\mathrm{mod}}/p^r\omega_E^{\mathrm{mod}}$ as $\mathcal{O}^{+}_{\mathcal{X}(p^n,N)^{(m)}_{\infty}}$-module. Dualizing, we get an injective map

$$(\mathrm{dlog})^{\vee}: \omega_E^{\mathrm{mod},-1}\otimes_{\mathcal{O}^{+}_{\mathcal{X}_0(p,N)^{(m)}_{\infty}}}\widehat{\mathcal{O}}^{+}_{\mathcal{X}(p^n,N)^{(m)}_{\infty}} \longrightarrow T_p(E)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^{+}_{\mathcal{X}(p^n,N)^{(m)}_{\infty}}$$

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with quotient $\mathcal{Q}$ isomorphic to $\omega_E^{\mathrm{mod}}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p,N)^{(m)}_\infty}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty}$ and such that modulo p^r the section a generates $\omega_E^{\mathrm{mod},-1}/p^n\omega_E^{\mathrm{mod},-1}$ as $\mathcal{O}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty}$-module. Then $(\mathrm{dlog})^\vee$ induces, for every $\lambda\in\mathbb{Z}/p^n\mathbb{Z}$, an affine map

$$\rho_\lambda\colon \mathbb{V}_0\big(T_p(E),a,b-\lambda a\big)\times_{\mathbb{Z}_p}\mathcal{X}(p^n,N)^{(m)}_\infty\longrightarrow \mathbb{V}_0\big(\omega_E^{\mathrm{mod},-1},a\big)\times_{\mathcal{X}_0(p,N)^{(m)}_\infty}\mathcal{X}(p^n,N)^{(m)}_\infty$$

on the pro-Kummer étale site of $\mathcal{X}(p^n,N)^{(m)}_\infty$. Here and below, the formalism f VBMS is applied with respect to the ideal $\mathcal{I}$ generated by p^r. Notice that $\mathbb{V}_0\big(T_p(E),a,b-\lambda a\big)\times_{\mathbb{Z}_p}\mathcal{X}(p^n,N)^{(m)}_\infty\cong\mathbb{V}_0\big(T_p(E)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty},a,b-\lambda b\big)$ is a principal homogeneous space under the formal vector group $\mathbb{V}'\big(\mathcal{Q}\big)\subset\mathbb{V}\big(\mathcal{Q}\big)$ classifying sections of $\mathcal{Q}^\vee$ which are zero modulo p^r. We have the invertible $\mathcal{O}^+_{\mathcal{X}_0(p,N)^{(m)}_\infty}\widehat{\otimes}B$-module $\omega_E^{-k}:=\mathbb{W}_k(\omega_E^{\mathrm{mod},-1},a)$. Set

$$\mathfrak{W}^{(m)}_{k,\infty,\lambda}:=\mathbb{W}_k\big(T_p(E),a,b-\lambda a\big)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty}.$$

Applying the formalism of VBMS, we get the map of sheaves on the pro-Kummer étale site of $\mathcal{X}(p^n,N)^{(m)}_\infty$

$$\rho^k_\lambda\colon \omega_E^{-k}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p,N)^{(m)}_\infty}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty}\longrightarrow\mathfrak{W}^{(m)}_{k,\infty,\lambda}.$$

Using that $\mathbb{V}_0\big(T_p(E)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty},a,b+\lambda a\big)$ is a principal homogeneous space under $\mathbb{V}'\big(\mathcal{Q}\big)$, we obtain a $\mathbb{V}'\big(\mathcal{Q}\big)$-stable increasing filtration $\mathrm{Fil}_h\mathfrak{W}^{(m)}_{k,\infty,\lambda}$ for $h\geq 0$ with

$$\begin{aligned}\mathrm{Fil}_0\mathfrak{W}^{(m)}_{k,\infty,\lambda}&=\omega_E^{-k}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p,N)^{(m)}_\infty}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty},\\ \mathrm{Gr}_h\mathfrak{W}^{(m)}_{k,\infty}&\cong\omega_E^{-k+2h}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p,N)^{(m)}_\infty}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty}.\end{aligned}\tag{5}$$

See [2] and [6, Prop. 5.2]. Taking $\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty}$-duals, we get a sheaf and a decreasing filtration

$$\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda}=W_k\big(T_p(E),a,b-\lambda b\big)^\vee\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m}_\infty},\qquad \mathrm{Fil}^h\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda},h\geq -1$$

on the pro-Kummer étale site of $\mathcal{X}(p^n,N)^{(m)}_\infty$. Here, $\mathrm{Fil}^h\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda}$ consists of those sections of $\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda}$ which are zero on $\mathrm{Fil}_h\mathfrak{W}^{(m)}_{k,\infty,\lambda}$ (where we set $\mathrm{Fil}_{-1}\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda}=0$ so that $\mathrm{Fil}^{-1}\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda})=\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda})$. Then

$$\mathrm{Gr}^h\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda}\cong\omega_E^{k-2h-2}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p,N)^{(m)}_\infty}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)^{(m)}_\infty}.\tag{6}$$

Due to Proposition 3.8, we have a Δ_n-equivariant isomorphism

$$j_n^*\big(\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)\cong\oplus_{\lambda\in\mathbb{Z}/p^n\mathbb{Z}}\mathfrak{W}^{(m),\vee}_{k,\infty,\lambda}$$

and we set $\mathrm{Fil}^h j_n^*\big(\mathfrak{D}_{k,\infty}^{o,(m)}[n]\big)$ to be the filtration corresponding to $\oplus_{\lambda\in\mathbb{Z}/p^n\mathbb{Z}}\mathrm{Fil}^h\mathfrak{W}_{k,\infty,\lambda}^{(m),\vee}$. The map $\oplus_\lambda\rho_\lambda^k$ defines a map $\rho^k\colon \omega_E^{-k}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p,N)_\infty^{(m)}}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_\infty^{(m)}}\longrightarrow\oplus_\lambda\mathfrak{W}_{k,\infty,\lambda}^{(m)}$ and, hence, a map

$$\nu\colon \mathrm{Gr}^{-1}j_n^*\big(\mathfrak{D}_{k,\infty}^{o,(m)}[n]\big)\to\omega_E^k\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p,N)_\infty^{(m)}}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_\infty^{(m)}}.$$

By construction, (i) and (ii) hold over $\mathcal{X}(p^n,N)_\infty^{(m)}$, and to prove those claims we need to show that the given filtration and the map ν are Δ_n-equivariant. Also, given a sheaf $\mathcal{F}$ on $\mathcal{X}_0(p,N)_\infty^{(m)}$, we have a spectral sequence

$$\mathrm{H}^i\big(\Delta_n,\mathrm{H}^j\big(\mathcal{X}(p^n,N)_{\infty,\mathrm{pke}}^{(m)},j_n^*(\mathcal{F})\big)\Rightarrow\mathrm{H}^{i+j}\big(\mathcal{X}_0(p^n,N)_{\infty,\mathrm{pke}}^{(m)},\mathcal{F}\big).$$

Since the cohomology groups $\mathrm{H}^j\big(\Delta_n,\mathcal{G}\big)$ are annihilated by the order of Δ_n for $j\geq 1$, to conclude (iv)–(v) it suffices to prove those claims for $\mathcal{X}(p^n,N)_\infty^{(m)}$.

The rest of the proof is a computation using the log affinoid perfectoid cover by opens U of the adic space $\mathcal{X}_0(p^n,N)_{\infty,\mathrm{pke}}^{(m)}$ defined by trivializing the full Tate module $\prod_\ell T_\ell(E)$. It is Galois over $\mathcal{X}_0(p^n,N)_{\infty,\mathrm{pke}}^{(m)}$ with group G_U. Let $\widehat{U}:=\mathrm{Spa}(R,R^+)$ be the associated affinoid perfectoid space as in §2.3. Write $T_p(E)^\vee(U)\otimes R^+=e_0R^+\oplus e_1R^+$ with e_0 mapping to a generator of ω_E^{mod} and e_1 in the kernel of $d\log$ generating $\omega_E^{\mathrm{mod},-1}$. Recall from Proposition 3.8 that $\mathfrak{W}_{k,\infty,\lambda}^{(m),\vee}(U)$ is the dual of $\mathfrak{W}_{k,\infty,\lambda}^{(m)}(U)\cong R^+\otimes B\langle\frac{W_\lambda}{1+p^rZ}\rangle\cdot k(1+p^rZ)$ with increasing filtration defined by $\mathrm{Fil}^h=\oplus_{i=0}^h R^+\otimes B\big(\frac{W_\lambda}{1+p^rZ}\big)^i\cdot k(1+p^rZ)$.

For every $\sigma\in G_U$, we then have $\sigma(e_1)=e_1$ and $\sigma(e_0)=e_0+\xi(\sigma)e_1$. Then, $\sigma(W_\lambda)=W_\lambda+\frac{\xi(\sigma)}{p^n}(1+p^rZ)$ and $\sigma(Z)=Z$. If $\xi(\sigma)=\alpha+p^r\beta$ then $\sigma(W_\lambda)=W_{\lambda+\alpha}+\beta(1+p^rZ)$. Thus, the increasing filtration on $\oplus_{\lambda\in\mathbb{Z}/p^n\mathbb{Z}}\mathfrak{W}_{k,\infty,\lambda}^{(m)}$ and the diagonal embedding of $R^+\otimes B\to\oplus_{\lambda\in\mathbb{Z}/p^n\mathbb{Z}}\mathrm{Fil}^0\mathfrak{W}_{k,\infty,\lambda}^{(m)}$ are both stable for the action of G_U. This concludes the proof of (i) and (ii).

We pass to claims (iii)–(v). As ω_E^{k+2} is a locally free $\mathcal{O}^+_{\mathcal{X}_0(p,N)_\infty^{(m)}}\widehat{\otimes}B$-module, we are left to show that

$$\mathrm{H}^1\big(\mathcal{X}_0(p^n,N)_{\infty,\mathrm{pke}}^{(m)},\widehat{\mathcal{O}}_{\mathcal{X}_0(p^n,N)_\infty^{(m)}}\big)\cong\mathrm{H}^0\big(\mathcal{X}_0(p^n,N)_\infty^{(m)},\omega_E^2\big)[1/p].$$

So to prove claim (iii), we are left to show that that the map (3) is an isomorphism. The étale cohomology of the structure sheaf on U is trivial as recalled in §2.3. Thanks to equation (6), the sheaves $\mathfrak{D}_{k,\infty}^{o,(m)}[n]/\mathrm{Fil}^h\mathfrak{D}_{k,\infty}^{o,(m)}[n]|_U$ are extensions of the structure sheaf $\widehat{\mathcal{O}}_U^+$ so that also their cohomology over U is trivial. Then the cohomology groups in equation (3) and those of $\mathfrak{D}_{k,\infty}^{o,(m)}[n]/\mathrm{Fil}^h\mathfrak{D}_{k,\infty}^{o,(m)}[n]$ appearing in (iv) and (v) coincide with the continuous cohomology of G_U of the sections of the relevant sheaves over U. This reduces the proof of claims (iii), (iv) and (v) to a Galois cohomology computation for which we refer to [6, Thm. 5.4]. □

We have Hecke operators acting on the cohomology of $\mathfrak{D}_{k,\infty}^{o,(m)}[n]$; see equation (4). We assume the hypothesis in the proof of Proposition 4.14.

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Proposition 4.15. *The operator $U_p^{\mathrm{naive}} = pU_p$ is defined on $\mathrm{H}^i\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}}, \mathrm{Fil}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$ for every $h \geq -1$. There exists an operator $U_{p,h}$*

$$U_{p,h}\colon \mathrm{H}^i\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}},\mathrm{Fil}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big) \to \mathrm{H}^i\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}},\mathrm{Fil}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$$

such that on $\mathrm{H}^i\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}},\mathrm{Fil}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$ we have $U_p^{\mathrm{naive}} = p^{h+1}U_{p,h}$ for $i = 0,\ 1$.

Moreover, for every positive integer h the cohomology group $\mathrm{H}^1\big(\mathcal{X}_0(p^n,N)^{(n)}_{\infty,\mathrm{pke}}, \mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)[1/p]$ admits a slope $\leq h$-decomposition with respect to the U_p-operator and we have an isomorphism of Hecke modules:

$$\Psi\colon \mathrm{H}^1\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty,\mathrm{pke}},\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)[1/p]^{(h)} \cong \mathrm{H}^0\big(\mathcal{X}_0(p^n,N)^{(m)}_{\infty},\omega_E^{k+2}\big)^{(h)}\widehat{\otimes}\mathbb{C}_p,$$

where the tensor product is over the finite extension of $\mathbb{Q}_p$ over which $\mathcal{X}_0(p^n,N)^{(m)}_{\infty}$ is defined.

Proof. Recall the construction of the U_p-operator in equation (4); all steps are defined integrally on $\mathfrak{D}^{o,(m)}_{k,\infty}[n]$ except for the trace $\mathrm{Tr}\colon q_{2,*}q_2^* \sim \mathrm{Id}$, and all steps are defined for $\mathrm{Fil}^h\mathfrak{D}^{o,(m)}_{k,\infty}[n]$ except possibly for

$$\mathcal{U}\colon q_1^*\big(\mathbb{D}^o_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p,N)}\big) \to q_2^*\big(\mathbb{D}^o_k(T_0^\vee)[n+1]\big)\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p,N,p)}.$$

We claim that $\mathcal{U}$ restricts to a map on $q_1^*\big(\mathrm{Fil}^h\mathbb{D}^o_k(T_0^\vee)[n]\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p,N)}\big) \to q_2^*\big(\mathrm{Fil}^h\mathbb{D}^o_k(T_0^\vee)[n+1]\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p,N,p)}\big)$ which can be written as p^{h+1} times an operator $\mathcal{U}'$. Both statements can be checked upon passing to sections over a log affinoid perfectoid. These statements are then proven in [6, Thm. 5.5].

The statement on the existence of slope decomposition and the displayed isomorphism are proven as in [6, Thm. 5.1] using Proposition 4.14. □

4.8. A comparison result on $\mathcal{X}_0(p^n,N)^{(m)}_0$

As in the previous section, we fix an r-analytic weight $k\colon \mathbb{Z}_p^* \to B^*$, as in definition 3.6, and we write $u_k \in B[1/p]$ for the element such that $k(a) = \exp u_k \log a$ for $a \in 1 + p^r\mathbb{Z}_p$. Note that $p^r u_k \in B$. Fix an integer $n \geq r$, and define the sheaf $\mathfrak{D}^{o,(m)}_{k,0}[n] := \mathbb{D}^o_k(T_0^\vee)[n]|_{\mathcal{X}_0(p^n,N)^{(m)}_0}\widehat{\otimes}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)^{(m)}_0}$. In this case, we have the following.

Proposition 4.16. *For m large enough, there exists an increasing filtration $\big(\mathrm{Fil}_s\mathfrak{D}^{o,(m)}_{k,0}[n]\big)_{s\geq 0}$ with the following properties:*

i. *For $m' \geq m$, we have $\mathrm{Fil}_\bullet\mathfrak{D}^{o,(m)}_{k,0}[n]\widehat{\otimes}_{\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)^{(m)}_0}}\widehat{\mathcal{O}}^+_{\mathcal{X}_0(p^n,N)^{(m')}_0} \cong \mathrm{Fil}_\bullet\mathfrak{D}^{o,(m')}_{k,0}[n]$;*

ii. *For every $s \geq 0$ the image of the map*

$$\mathrm{H}^1\big(\mathcal{X}_0(p^n,N)^{(m)}_{0,\mathrm{pke}},\mathrm{Fil}_s\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big) \to \mathrm{H}^1\big(\mathcal{X}_0(p^n,N)^{(m)}_{0,\mathrm{pke}},\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$$

is annihilated by the product $p^{(r+c)s}\binom{u_k}{s}$ (with c depending on n).

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Proof. We follow closely the proof of Proposition 4.14. We take m large enough so that E admits a canoncial subgroup C_n of level p^n over $\mathcal{X}_0(p^n,N)_0^{(m)}$. Note that the canonical subgroup C_n and the level subgroup H_n are distinct.

The natural projection $j_n\colon \mathcal{X}(p^n,N)_0^{(m)} \to \mathcal{X}_0(p^n,N)_0^{(m)}$ is Kummer étale with automorphism group Δ_n isomorphic to the subgroup of $\mathbf{GL}_2(\mathbb{Z}/p^n\mathbb{Z})$ of matrices that are upper triangular modulo p^n. It is not a Galois cover as $\mathcal{X}(p^n,N)_0^{(m)}$ is not connected. In fact, we have a trivialization $T_pE/p^nT_pE=(\mathbb{Z}/p^n\mathbb{Z})a\oplus(\mathbb{Z}/p^n\mathbb{Z})b$ over $\mathcal{X}(p^n,N)_0^{(m)}$. As recalled in 4.2, we have a decomposition $\mathcal{X}(p^n,N)_0^{(m)}=\amalg_{\xi\in\mathbb{Z}/p^n\mathbb{Z}}\mathcal{X}(p^n,N)_{0,\xi}^{(m)}$ where over $\mathcal{X}(p^n,N)_{0,\xi}^{(m)}$ we have $\mathrm{dlog}(a^\vee)=\xi\mathrm{dlog}(b^\vee)$ modulo p^n. We recall that to give a sheaf on $\mathcal{X}_0(p,N)_{0,\mathrm{pke}}^{(m)}$ is equivalent to give a sheaf on $\mathcal{X}(p^n,N)_{0,\mathrm{pke}}^{(m)}$ endowed with a compatible action of Δ_n. In particular to define $\mathrm{Fil}_s\mathfrak{D}_{k,\infty}^{o,(m)}[n]$, we define the filtration over $\mathcal{X}(p^n,N)_{0,\mathrm{pke}}^{(m)}$ and we prove that it is stable for the action of Δ_n. The maps dlog and $\mathrm{dlog}^\vee$ define an exact sequence of sheaves on the pro-Kummer étale site of $\mathcal{X}(p^n,N)_0^{(m)}$:

$$
\begin{aligned}
Q:=\omega_E^{\mathrm{mod},-1}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p^n,N)_0^{(m)}}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_0^{(m)}} &\longrightarrow T_p(E)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_0^{(m)}}\\
&\longrightarrow \omega_E^{\mathrm{mod}}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p^n,N)_0^{(m)}}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_0^{(m)}}.
\end{aligned}
$$

Via the trivialization $T_pE/p^nT_pE=(\mathbb{Z}/p^n\mathbb{Z})a\oplus(\mathbb{Z}/p^n\mathbb{Z})b$ over $\mathcal{X}(p^n,N)_0^{(m)}$, over the component $\mathcal{X}(p^n,N)_{0,\xi}^{(m)}$ the canonical subgroup C_r, and hence, Q modulo p^r, is generated by the section $b+\xi a$ and a maps to a generator of the quotient $\omega_E^{\mathrm{mod}}\otimes_{\mathcal{O}^+_{\mathcal{X}_0(p,N)_0^{(m)}}}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_0^{(m)}}$ modulo p^r. In particular, we get an affine map

$$
\begin{aligned}
\rho\colon \mathbb{V}_0\big(\omega_E^{\mathrm{mod},-1},a\big) &\subset \mathbb{V}_0\big(T_p(E)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_{0,-\lambda}^{(m)}},a,b-\lambda a\big)\\
&\cong \mathbb{V}_0\big(T_p(E),a,b-\lambda a\big)\times_{\mathbb{Z}_p}\mathcal{X}(p^n,N)_{0,-\lambda}^{(m)}
\end{aligned}
$$

on the pro-Kummer étale site of $\mathcal{X}(p^n,N)_{0,-\lambda}^{(m)}$. Considering the underlying sheaves of functions of weight 0, the map ρ^* induces a surjective map of sheaves of rings

$$
\rho_0^*\colon \mathbb{W}_0\big(T_p(E),a,b-\lambda a\big)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_{0,-\lambda}^{(m)}}\to\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_{0,-\lambda}^{(m)}}.
$$

We let $\mathcal{I}$ be its kernel. We set

$$
\mathfrak{W}_{k,0,\lambda}^{(m)}:=\mathbb{W}_k\big(T_p(E),a,b-\lambda a\big)\otimes_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_{0,-\lambda}^{(m)}},\qquad \mathrm{Fil}^s\mathfrak{W}_{k,0,\lambda}^{(m)}:=\mathcal{I}^s\mathfrak{W}_{k,0,\lambda}^{(m)}.
$$

Taking $\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_\infty^{(m)}}$-duals we get a sheaf and an increasing filtration

$$
\mathfrak{W}_{k,0,\lambda}^{(m),\vee}=\mathbb{W}_k\big(T_p(E),a,b-\lambda b\big)^\vee\hat{\otimes}_{\mathbb{Z}_p}\widehat{\mathcal{O}}^+_{\mathcal{X}(p^n,N)_{0,-\lambda}^{(m}},\qquad \mathrm{Fil}_s\mathfrak{W}_{k,0,\lambda}^{(m),\vee},s\geq -1
$$

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on the pro-Kummer étale site of $\mathcal{X}(p^n,N)^{(m)}_{0,-\lambda}$. Here, $\mathrm{Fil}_s\mathfrak{W}^{(m),\vee}_{k,0,\lambda}$ consists of those sections vanishing on $\mathrm{Fil}^s\mathfrak{W}^{(m)}_{k,0,\lambda}$. Due to Proposition 3.8, we have a Δ_n-equivariant isomorphism

$$j_n^*\big(\mathfrak{D}^{o,(m)}_{k,0}[n]\big) \cong \oplus_{\lambda\in\mathbb{Z}/p^n\mathbb{Z}}\mathfrak{W}^{(m),\vee}_{k,0,\lambda},$$

and we set $\mathrm{Fil}_s j_n^*\big(\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$ to be the filtration given by $\oplus_{\lambda\in\mathbb{Z}/p^n\mathbb{Z}}\mathrm{Fil}_s\mathfrak{W}^{(m),\vee}_{k,0,\lambda}$.

In order to get a well-defined filtration on $\mathfrak{D}^{o,(m)}_{k,0}[n]$, we need to prove that $\mathrm{Fil}_s j_n^*\big(\mathfrak{D}^{o,(m)}_{k,\infty}[n]\big)$ is Δ_n-equivariant. If this holds, claim (i) is then clear by construction. Arguing as in Proposition 4.14, to prove claim (ii) it suffices to prove it for $\mathrm{H}^1\big(\mathcal{X}(p^n,N)^{(m)}_{0,\mathrm{pke}},\text{-}\big)$. As in loc. cit. one reduces the proof of this statement and the statement of the Δ_n-equivariance after passing to a log affinoid perfectoid cover U of $\mathcal{X}(p^n,N)^{(m)}_0$, with group of automorphisms G_U relatively to $\mathcal{X}_0(p^n,N)^{(m)}_0$ and with $\widehat{U} := \mathrm{Spa}(R,R^+)$ the associated affinoid perfectoid space. Write $T_p(E)(U)\otimes R^+ = R^+f_0 + R^+f_1$ and $T_p(E)^\vee(U)\otimes R^+ = e_0R^+\oplus e_1R^+$ with $e_0 = f_0^\vee \equiv b^\vee$ mapping to a generator of ω_E^{mod} and $e_1 = f_1^\vee \equiv a^\vee - \lambda b^\vee$ in the kernel of dlog generating $\omega_E^{\mathrm{mod},-1}$. Recall from Proposition 3.8 that $\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)$ is the dual of $\mathfrak{W}^{(m)}_{k,0,\lambda}(U) \cong (R^+\otimes B)\langle\frac{W_\lambda}{1+p^rZ}\rangle\cdot k(1+p^rZ)$ where $p^rW_\lambda = Y$ and $X = 1+p^rZ$, and we have the universal map $\alpha f_0+\beta f_1 \mapsto \alpha Y+\beta X$ (note the roles of X and Y are interchanged compared to loc. cit. as in this case we are looking for functions on T that are 0 on b modulo p^r and 1 on a modulo p^r, or equivalently that are 1 on e_1 modulo p^r and 0 on $e_0-\lambda e_1$ modulo p^r). The decreasing filtration is defined by $\mathrm{Fil}^s = \oplus_{i\geq s}(R^+\otimes B)\big(\frac{W_\lambda}{1+p^rZ}\big)^i\cdot k(1+p^rZ)$.

Given $\sigma\in G_U$, we have $\sigma(e_0) = e_0$ and $\sigma(e_1) = e_1+\xi(\sigma)e_0$ so that $\sigma(Y) = Y$ and $\sigma(X) = X+\xi(\sigma)Y = X+p^r\xi(\sigma)W_\lambda$. Here, ξ is an R^+-valued continuous 1-cocycle on G_U. In particular, write $k(t) = \exp(u_k\log(t))$ for $t\equiv 1$ modulo p^r. Then

$$\sigma\big(k(X)X^{-i}\big) = k\big(\sigma(X)\big)\sigma(X)^{-i} = \exp\big((u_k-i)\log(X(1+p^r\xi(\sigma)\frac{W_\lambda}{X}))\big) =$$
$$= k(X)X^{-i}\exp\big((u_k-i)\log(1+p^r\xi(\sigma)\frac{W_\lambda}{X})\big) = k(X)X^{-i}\sum_{m=0}^{\infty}p^{rm}\binom{u_k-i}{m}\big(\xi(\sigma)\frac{W_\lambda}{X}\big)^m.$$

Notice that the term $m=0$ is $(u_k-i)k(X)X^{-i}$. Thus,

$$\sigma\Big(\big(\frac{W_\lambda}{1+p^rZ}\big)^i\cdot k(1+p^rZ)\Big) = \big(\frac{W_\lambda}{1+p^rZ}\big)^i\cdot k(1+p^rZ)\sum_m p^{rm}\binom{u_k-i}{m}\left(\xi(\sigma)\frac{W_\lambda}{1+p^rZ}\right)^m. \tag{7}$$

This implies first of all that G_U preserves the filtration. Claim (ii) follows from the following lemma. $\square$

Lemma 4.17. *For every s, consider the short exact sequence of G_U-modules*

$$0\to \mathrm{Fil}_s\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)\to \mathrm{Fil}_{2s}\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)\to \mathrm{Fil}_{2s}\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)/\mathrm{Fil}_s\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)\to 0.$$

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Then, the connecting homomorphism

$$\mathrm{H}^0\big(G_U, \mathrm{Fil}_{2s}\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)/\mathrm{Fil}_s\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)\big) \to \mathrm{H}^1\big(G_U, \mathrm{Fil}_s\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)\big)$$

has cokernel annihilated by $p^{(r+c)s}s!\binom{u_k}{s}$ *for some* c *depending on* n.

Proof. We argue by induction on s. We are reduced to prove that for any s the short exact sequence

$$0 \to \mathrm{Gr}_s\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U) \to \mathrm{Fil}_{s+1}\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U)/\mathrm{Fil}_{s-1}\mathfrak{W}^{(m),\vee}_{k,0,\lambda}(U) \to \mathrm{Gr}_{s+1}\mathfrak{W}^{(m)}_{k,0,\lambda}(U) \to 0$$

the connecting homomorphism

$$\mathrm{H}^0\big(G_U, \mathrm{Gr}_{s+1}\mathfrak{W}^{(m)}_{k,0,\lambda}(U)\big) \to \mathrm{H}^1\big(G_U, \mathrm{Gr}_s\mathfrak{W}^{(m)}_{k,0,\lambda}(U)\big)$$

has cokernel annihilated by $p^{(r+c)}\frac{(u_k-s)}{s}$ for some c depending on n. Note that

$$\mathrm{H}^0\big(G_U, \mathrm{Gr}_{s+1}\mathfrak{W}^{(m)}_{k,0,\lambda}(U)\big) = \big((R^+)^{G_U}\otimes B\big)\left(\left(\frac{W_\lambda}{1+p^rZ}\right)^{s+1}\cdot k(1+p^rZ)\right)^\vee$$

and

$$\mathrm{H}^1\big(G_U, \mathrm{Gr}_s\mathfrak{W}^{(m)}_{k,0,\lambda}(U)\big) = \big((R^+)^{G_U}\otimes B\big)\otimes_{(R^+)^{G_U}}\left(\left(\frac{W_\lambda}{1+p^rZ}\right)^{s}\cdot k(1+p^rZ)\right)^\vee \mathrm{H}^1\big(G_U, R^+\big).$$

Thanks to equation (7), the map sends $\left(\left(\frac{W_\lambda}{1+p^rZ}\right)^{s+1}\cdot k(1+p^rZ)\right)^\vee$ to the cocyle $G_U \ni \sigma \mapsto (u_k-s)p^r\xi(\sigma)\left(\left(\frac{W_\lambda}{1+p^rZ}\right)^{s}\cdot k(1+p^rZ)\right)^\vee$. The quotient of $\mathrm{H}^1\big(G_U, R^+\big)$ by the $(R^+)^{G_U}$-span of the cocycle ξ is torsion and hence killed by a power p^c of p; see [6, Prop. 5.2]. The conclusion follows. □

5. The Hodge–Tate Eichler–Shimura map revisited

Let $k\colon \mathbb{Z}_p^* \longrightarrow B^*$ be a B-valued weight, as in Definition 3.6, which is r-analytic for some $r \in \mathbb{N}$ (definition 3.6), that is, there is $u_k \in B[1/p]$ such that $k(t) = \exp\big(u_k\log(t)\big)$ for all $t \in 1+p^r\mathbb{Z}_p$. In this section, we fix an integer $n \geq r$ and denote $\mathbb{D}^o_k(T_0^\vee)[n]$ the integral pro-Kummer étale sheaf of distributions on the base-change of $\mathcal{X} := \mathcal{X}_0(p^n,N)$ over $\mathrm{Spa}(B[1/p],B)$. We will simply denote this sheaf by $\mathbb{D}^o_k$ in this section and also set $\mathbb{D}_k := \mathbb{D}^o_k \otimes_{\mathbb{Z}_p}\mathbb{Q}_p$.

Let us fix a slope $h \in \mathbb{N}$, and recall that if M is a $\mathbb{Q}_p$-vector space with a linear endomorphism U_p, we denote $M^{(h)}$ the subvector space of M of elements $x \in M$ such that $P(U_p)(x) = 0$ for all polynomials $P(X) \in \mathbb{Q}_p[X]$, whose roots in $\mathbb{C}_p$ have all valuations in $[0,h]\cap\mathbb{Q}$. Up to localization of B and for s large enough, both $\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}}, \mathbb{D}_k\big)$ and $\mathrm{H}^0\big(\mathcal{X}\langle\frac{p}{\mathrm{Ha}^{p^s}}\rangle, \omega_E^{k+2}\big)$ admit slope h decompositions; here, Ha is a (any) local lift of the Hasse invariant. Then the main result of this section is as follows.

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Theorem 5.1. *For s large enough, there is a canonical $\mathbb{C}_p$-linear, Galois and Hecke equivariant map:*

$$\Psi_{\mathrm{HT}}\colon \mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k(1)\big)^{(h)}\widehat{\otimes}\mathbb{C}_p \longrightarrow \mathrm{H}^0\big(\mathcal{X}\langle\frac{p}{\mathrm{Ha}^{p^s}}\rangle,\omega_E^{k+2}\big)^{(h)}\widehat{\otimes}\mathbb{C}_p.$$

Moreover, if $\prod_{i=0}^{h-1}(u_k-i)\in\big(B[1/p]\big)^*$, then Ψ_{HT} is surjective.

The map Ψ_{HT} was defined in [5] using Faltings' sites of $\mathcal{X}$ and $\mathcal{X}\langle\frac{p}{\mathrm{Ha}^{p^s}}\rangle$, respectively. To relate it to the language developed in this paper, first of all we allow ourselves to increase n. In fact, the map $\mathcal{X}_0(p^{n_1},N)\to\mathcal{X}_0(p^{n_2},N)$ for $n_1\geq n_2$ is finite étale, and one can obtain the map Ψ_{HT} for $n=n_1$ in the theorem upon taking traces from the map Ψ_{HT} for $n=n_2$; on the left-hand side the trace is defined using Lemma 4.7. We also replace $\mathcal{X}_0(p^n,N)\langle\frac{p}{\mathrm{Ha}^{p^s}}\rangle$ with $\mathcal{X}_0(p^n,N)_\infty^{(m)}$ for arbitrary large integers $n\geq m$, as the first is contained in the latter for m large enough thanks to [23, Lemma 3.3.15].

In particular, we choose n large enough such that there is $m\leq n$ with the property that the restriction of $\mathbb{D}_k^o$ to the pro-Kummer étale site of $\mathcal{X}_0(p^n,N)_\infty^{(m)}$ has the property that the (pro-Kummer étale) sheaf $\mathbb{D}_k^o\widehat{\otimes}\widehat{\mathcal{O}}_{\mathcal{X}_0(p^n,N)_\infty^{(m)}}$ has the decreasing filtration defined in Proposition 4.14. We fix such n, m. Let us denote by $\mathfrak{D}_k^o:=\mathbb{D}_k^o\widehat{\otimes}\mathcal{O}^+_{\mathcal{X}_{\mathrm{pke}}}$, where $\mathcal{O}^+_{\mathcal{X}_{\mathrm{pke}}}$ is the structure sheaf of the pro-Kummer étal site of $\mathcal{X}$. As in [5], we let Ψ_{HT} be the composition of the following maps:

$$\mathrm{H}^1\big(\mathcal{X}_0(p^n,N)_{\mathrm{pke}},\mathbb{D}_k(1)\big)^{(h)}\widehat{\otimes}\mathbb{C}_p\cong\Big(\mathrm{H}^1\big(\mathcal{X}_0(p^n,N)_{\mathrm{pke}},\mathfrak{D}_k^o(1)\big)[1/p]\Big)^{(h)}\xrightarrow{\mathcal{R}}$$
$$\xrightarrow{\mathcal{R}}\Big(\mathrm{H}^1\big((\mathcal{X}_0(p^n,N)_\infty^{(m)})_{\mathrm{pke}},\mathfrak{D}_k^o(1)\big)[1/p]\Big)^{(h)}\xrightarrow{\Phi}\mathrm{H}^0\big(\mathcal{X}_0(p^n,N)_\infty^{(m)},\omega_E^{k+2}\big)^{(h)}\widehat{\otimes}\mathbb{C}_p,$$

where $\mathcal{R}$ is the restriction map while the map Φ is defined in Proposition 4.15 and it is proved in loc. cit. that it is an isomorphism. Therefore, in order to prove Theorem 5.1, it is enough to prove the following.

Theorem 5.2. *In the notations above, if $\prod_{i=0}^{h-1}(u_k-i)\in\big(B[1/p]\big)^*$, then the map $\mathcal{R}$ is surjective.*

To simplify the notation, in the rest of the section, we write $\mathcal{X}$ instead of $\mathcal{X}_0(p^n,N)$, $\mathcal{X}_\infty^{(m)}$ instead of $\mathcal{X}_0(p^n,N)_\infty^{(m)}$, and so on. Before starting the proof of Theorem 5.2, we'll describe the dynamic of the U_p-operator on the modular curve $\mathcal{X}$. We think about U_p as correspondences on $\mathcal{X}$, and we have:

Lemma 5.3. *For every integer $u\geq 1$, we have*

i) $U_p^{u+1}\big(\mathcal{X}\backslash\mathcal{X}_0^{(u+1)}\big)\subset\mathcal{X}_\infty^{(1)}$ *and* $U_p^u\big(\mathcal{X}\backslash\mathcal{X}_0^{(u+1)}\big)\subset\mathcal{X}_\infty\backslash\mathcal{X}_0^{(1)}$.

ii) $U_p^u\big(\mathcal{X}_\infty^{(1)}\big)\subset\mathcal{X}_\infty^{(u+1)}$.

Proof. This is a direct consequence of Lemma 4.9. □

We also have:

Lemma 5.4. *For every $u\geq 1$, there is a canonical decomposition of correspondences $U_p^u|_{\mathcal{X}_0^{(u)}}=\big(U_p^u\big)^{\mathrm{good}}\amalg\big(U_p^u\big)^{\mathrm{bad}}$ such that:*

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a) $(U_p^u)^{\mathrm{good}}\big(\mathcal{X}_0^{(u)}\big) \subset \mathcal{X}_\infty^{(1)}$.

b) $(U_p^u)^{\mathrm{bad}}\big(\mathcal{X}_0^{(u)}\backslash\mathcal{X}_0^{(u+1)}\big) \subset \mathcal{X}_\infty\backslash\mathcal{X}_0^{(1)}$.

We remark that for $u=1$, $U_p^1 = U_p$ *and so we obtain the decomposition of* $U_p|_{\mathcal{X}_0(1)} = (U_p)^{\mathrm{good}} \amalg (U_p)^{\mathrm{bad}}$.

Proof. In view of Lemma 4.9 and Remark 4.8, it is enough to define this decomposition for the correspondence $\widetilde{U}^u := \cup_\mu t_\mu \colon U_0^{(u)} \longrightarrow \mathcal{P}(\mathbb{P}^1)$, where $\mu = \lambda_0 + \lambda_1 p + \ldots + \lambda_{u-1}p^{u-1}$, $\lambda_i \in \{0,1,\ldots,p-1\}$, for $i=0,1,\ldots,u-1$. Namely, we define $(\widetilde{U}^u)^{\mathrm{bad}}|_{U_{0,\mu}^{(u)}} := t_\mu$ and $(\widetilde{U}^u)^{\mathrm{good}}|_{U_{0,\mu}^{(u)}} := \cup_{\lambda\neq\mu} t_\lambda$. With these definitions, the rest of Lemma 5.4 follows from Lemma 4.9. □

We have the following consequence of Lemma 5.3 and Lemma 5.4. Let us recall our notation $\mathfrak{D}_k^o := \mathbb{D}_k^o \widehat{\otimes} \mathcal{O}_{\mathcal{X}}^+$ (this is a sheaf on the pro-Kummer étale site of the base-change of $\mathcal{X}$ to $\mathrm{Spa}(B[1/p],B)$) base-change which is not shown in the notations. Also, we recall that we use the operator U_p^{naive} induced on cohomology by the U_p-correspondence and not its normalized version $p^{-1}U_p^{\mathrm{naive}}$. With this understanding, to ease the notation, we simply write U_p instead of U_p^{naive}.

Corollary 5.5. *Let* $P(T) \in (B\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})[T]$ *be such that* $P(T) = TR(T)$ *and for every* $u \geq 1$ *denote* $\big(P(U_p)^u\big)^{\mathrm{good}} := (U_p^u)^{\mathrm{good}} R(U_p)^u$ *and* $\big(P(U_p)^u\big)^{\mathrm{bad}} := \big(U_p^u\big)^{\mathrm{bad}} R(U_p)^u$. *Then*

i) $P(U_p)^u \colon \mathrm{H}^1\big((\mathcal{X}_\infty^{(1)})_{\mathrm{pke}},\mathfrak{D}_k^o\big) \longrightarrow \mathrm{H}^1\big((\mathcal{X}\backslash\mathcal{X}_0^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)$.

ii) $P(U_p)^{u-1} \colon \mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big) \longrightarrow \mathrm{H}^1\big((\mathcal{X}_\infty^{(1)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)$.

iii) $\big(P(U_p)^u\big)^{\mathrm{good}} P(U_p) \colon \mathrm{H}^1\big((\mathcal{X}_\infty\backslash\mathcal{X}_0^{(1)})_{\mathrm{pke}},\mathfrak{D}_k^o\big) \longrightarrow \mathrm{H}^1\big((\mathcal{X}_0^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)$.

iv) $\big(P(U_p)^u\big)^{\mathrm{bad}} \colon \mathrm{H}^1\big((\mathcal{X}_\infty\backslash\mathcal{X}_0^{(1)})_{\mathrm{pke}},\mathfrak{D}_k^o\big) \longrightarrow \mathrm{H}^1\big((\mathcal{X}_0^{(u)}\backslash\mathcal{X}_0^{(u+1)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)$.

Proof. In view of the fact that, for any polynomial $Q(T)$ with $(B\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})$-coeffcients, $Q(U_p)$ maps $\mathrm{H}^1\big((\mathcal{X}_\infty^{(1)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)$ to itself, i) and, respectively, ii) are immediate consequences of Lemma 5.3 a) and b), respectively, while iii) is a consequence of Lemma 5.4 a) and iii) follows from Lemma 5.4 b). □

Before we start the actual proof of Theorem 5.2, it seems natural to recall and gather here the main ingredients in the proof, namely the properties of the cohomology of the filtrations of the sheaf $\mathfrak{D}_k^o$ on $\mathcal{X}_\infty^{(m)}$ and respectively on $\mathcal{X}_0^{(v)}$, for m as fixed at the beginning of this section and for the moment $v \geq 1$ such that the pro-Kummer étale sheaf $\mathfrak{D}_k^o$ has the increasing filtration $\mathfrak{Fil}_\bullet$ of Proposition 4.16 when restricted to $\mathcal{X}_0^{(v)}$.

Recall that we have denoted by $\mathfrak{D}_k^o := \mathbb{D}_k^o \widehat{\otimes} \widehat{\mathcal{O}}_{\mathcal{X}}^+$ for $\widehat{\mathcal{O}}_{\mathcal{X}}^+$ the completion of the structure sheaf of the pro-Kummer étale site of $\mathcal{X}$. We have:

i) The image of the morphism $\mathrm{H}^1\big((\mathcal{X}_0^{(v)})_{\mathrm{pke}},\mathfrak{Fil}_h\big) \longrightarrow \mathrm{H}^1\big((\mathcal{X}_0^{(v)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)$ is annihilated by $p^{(r+c)h}\prod_{i=0}^{h-1}(u_k - i)/h!$, where let us recall r is the degree of analyticity of the weight on B, $\mathcal{X} = \mathcal{X}_0(p^n,N)$ and c is constant independent of h. In particular, if $\Big(\prod_{i=0}^{h-1}(u_k - i)/h!\Big)\gamma = p^q$ for some $\gamma \in B$, then $p^{(r+c)h+q}$ annihilates the image of the above map.

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ii) The U_p correspondence on $\mathcal{X}_0^{(m)}$ can be written as the disjoint union of $U_p^{\rm good}$, mapping $\mathcal{X}_0^{(m)}$ to $\mathcal{X}_\infty^{(1)}$ and $U_p^{\rm bad}\colon \mathcal{X}_0^{(m)} \to \mathcal{X}_0^{(m-1)}$ and corresponding to the p-isogeny $E \to E'$ given by modding out by the canonical subgroup. To keep track of where our sheaves are defined, we denote $\mathfrak{D}_{k,0}^{o,(s)}$ the restriction of the pro-Kummer étale sheaf $\mathfrak{D}_k^o$ to $(\mathcal{X}_0^{(s)})_{\rm pke}$. Suppose $1 \le u \le m$ is such that $\mathfrak{D}_{0,k}^{o,(u-1)}$ has the increasing filtration $\mathfrak{Fil}_\bullet^{(u-1)}$ on $(\mathcal{X}_0^{(u-1)})_{\rm pke}$. We have:

Proposition 5.6. *The map $U_p^{\rm bad}$ induces a map $U_p^{\rm bad}\colon \mathfrak{D}_{k,0}^{o,(u)} \to \gamma^*\big(\mathfrak{D}_{k,0}^{o,(u-1)}\big)$ that preserves the filtration. Furthermore, for every b, there exists an operator*

$$U_p^{\rm bad,b}\colon \mathfrak{D}_{k,0}^{o,(u)}/\mathfrak{Fil}_b^{(u)} \to \gamma^*\big(\mathfrak{D}_{k,0}^{o,(u-1)}/\mathfrak{Fil}_b^{(u-1)}\big)$$

such that $U_p^{\rm bad} = p^b U_p^{\rm bad,b}$ on $\mathfrak{D}_{k,0}^{o,(u)}/\mathfrak{Fil}_b^{(u)}$. Here, $\gamma : \mathcal{X}_0^{(u)} \subset \mathcal{X}_0^{(u-1)}$ is the inclusion.

Proof. We use the notation of Proposition 4.16. It suffices to prove both statements after passing to a log affinoid perfectoid cover U of $\mathcal{X}_{0,\lambda}^{(u)}$ for $\lambda = \lambda_0 + \lambda_1 p + \ldots + \lambda_{n-1}p^{n-1}$, $\lambda_i \in \{0,1,\ldots,p-1\}$, for $i = 0,1,\ldots,n-1$, with associated affinoid perfectoid space $\widehat{U} := \mathrm{Spa}(R,R^+)$. Recall from loc. cit. that $T_p(E)$ is trivialized $T_p(E)(U) = \mathbb{Z}_p\alpha \oplus \mathbb{Z}_p\beta$, where the level subgroup is generated by α and the canonical subgroup is generated by the section $\beta + \lambda\alpha$. For every λ as above, write $\lambda = \lambda_0 + p\lambda'$, and then $U_p^{\rm bad}$ restricts to a map $u_{\lambda_0}\colon \mathcal{X}_0(p^n,N)_{0,\lambda}^{(m)} \to \mathcal{X}_0(p^n,N)_{0,\lambda'}^{(m-1)}$. At the level of universal elliptic curves over U, it corresponds to the p-isogeny defined by $u_{\lambda_0}\colon T_p(E)(U) \to T_p(E')(U)$ with $T_p(E')(U) = \mathbb{Z}_p\alpha \oplus \mathbb{Z}_p\beta'$ with $\beta' = \frac{\beta+\lambda_0\alpha}{p}$ in $T_p(E)\otimes\mathbb{Q}$. Notice that $\frac{\beta+\lambda\alpha}{p} = \beta' + \lambda'\alpha$ defines a generator of the canonical subgroup on E'. The map u_{λ_0} induces by functoriality the map $\mathfrak{D}_{k,0,\lambda}^{o,(u)} \to u^*\big(\mathfrak{D}_{k,0,\lambda_0}^{o,(u-1)}\big)$. We describe it explicitly.

Write $T_p(E)(U)\otimes R^+ = R^+ f_0 \oplus R^+ f_1$ as in Proposition 4.16 so that $f_0 \equiv \beta + \lambda\alpha$ and $f_1 \equiv a$ modulo p^n and $T_p(E)^\vee(U)\otimes R^+ = e_0R^+ \oplus e_1R^+$ with $e_0 = f_0^\vee \equiv \beta^\vee$ mapping to a generator of $\omega_E^{\rm mod}$ and $e_1 = f_1^\vee \equiv \alpha^\vee - \lambda\beta^\vee$ in the kernel of dlog generating $\omega_E^{{\rm mod},-1}$. Then $\mathfrak{W}_{k,0,\lambda}^{(u),\vee}(U)$ is the dual of $\mathfrak{W}_{k,0,\lambda}^{(u)}(U) \cong R^+ \otimes B\langle \frac{W_\lambda}{1+p^rZ}\rangle \cdot k(1+p^rZ)$. Similarly, write $T_p(E')(U)\otimes R^+ = R^+ f_0' \oplus R^+ f_1'$ with $f_0' = \frac{f_0}{p} = \beta' + \lambda'\alpha$ and $f_1' = f_1$. Then $T_p(E')^\vee(U) \otimes R^+ = e_0'R^+ \oplus e_1'R^+$ with $e_0' = (f_0')^\vee = pe_0$ mapping to a generator of $\omega_{E'}^{\rm mod}$ and $e_1' = (f_1')^\vee = f_1$ in the kernel of dlog for E' generating $\omega_{E'}^{{\rm mod},-1}$. In particular, $\mathfrak{W}_{k,0,\lambda_0}^{(u-1),\vee}(U)$ is the dual of $\mathfrak{W}_{k,0,\lambda_0}^{(u-1)}(U) \cong R^+ \otimes B\langle \frac{W_{\lambda_0}'}{1+p^rZ'}\rangle \cdot k(1+p^rZ')$. The map $T_p(E) \to T_p(E')$ induces a map $T_p(E')^\vee \to T_p(E)^\vee$ on the duals and, hence, a map

$$v_\lambda\colon \mathfrak{W}_{k,0,\lambda}^{(u)}(U) \to \mathfrak{W}_{k,0,\lambda_0}^{(u-1)}(U), \qquad Z \mapsto Z', W_\lambda \mapsto pW_{\lambda_0}.$$

As the decreasing filtrations are defined by

$$\mathrm{Fil}^b\mathfrak{W}_{k,0,\lambda}^{(u)}(U) = \mathfrak{W}_{k,0,\lambda}^{(u)}(U)\cdot\Big(\frac{W_\lambda}{1+p^rZ}\Big)^b, \quad \mathrm{Fil}^b\mathfrak{W}_{k,0,\lambda_0}^{(u-1)}(U) = \mathfrak{W}_{k,0,\lambda_0}^{(u-1)}(U)\cdot\Big(\frac{W_{\lambda_0}'}{1+p^rZ'}\Big)^b,$$

we see that v_λ respects the filtrations and on Fil^b can be written as $p^b v_\lambda'$ with $v_\lambda'\colon \mathrm{Fil}^b\mathfrak{W}_{k,0,\lambda}^{(u)}(U) \to \mathrm{Fil}^b\mathfrak{W}_{k,0,\lambda_0}^{(u-1)}(U)$. The claim follows upon taking strong R^+-duals. $\square$

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iii) Given $h \in \mathbb{N}$, there exists an integer d such that p^d annihilates $\mathrm{Ker}\Big(\mathrm{H}^1\big((\mathcal{X}_\infty^{(m)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big) \longrightarrow \mathrm{H}^1\big((\mathcal{X}_\infty^{(m)})_{\mathrm{pke}}, \mathfrak{D}_k\big)\Big)$ and $p^d e_h$ is integral, where e_h is an idempotent projecting onto the slope $\leq h$ parts of $\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}}, \mathfrak{D}_k\big)$ and of $\mathrm{H}^1\big((\mathcal{X}_\infty^{(m)})_{\mathrm{pke}}, \mathfrak{D}_k\big)$. This is the main result of §7.

Proof of Theorem 5.2. We now start the proof of Theorem 5.2. We work with a weight k as in Definition 3.6, which will be assumed in this proof to be the universal weight of some (wide) open disk of the weight space. The case of a finite extension of $\mathbb{Q}_p$ is obtained by specialization. In particular, $u_k - i$ is not 0 in B for every $i \in \mathbb{N}$. This will be used in step 3.

We recall that we have fixed a slope $h \in \mathbb{N}$ and consider the module

$$\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k\big)^{(h)} \cong \Big(\mathrm{H}^1\big(\mathcal{X}_\infty^{(u)}, \mathbb{D}_k\big)\widehat{\otimes}\mathbb{C}_p\Big)^{(h)}.$$

Let $Q(T) \in (B\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})[T]$ be the polynomial with $\deg(Q(T)) \geq 1$ and having the property: $y \in \mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k\big)^{(h)}$ if and only if $Q(U_p)y = 0$. Such a polynomial exists as by [6] we have an isomorphism $\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k\big)^{(h)} \cong \mathrm{H}^0\big(\mathcal{X}_\infty^{(u)}, \omega_E^{k+2}\big)^{(h)}$ and on $\mathrm{H}^0\big(\mathcal{X}_\infty^{(u)}, \omega_E^{k+2}\big)$ the operator U_p is compact and has a Fredholm determinant which is an entire power series. $Q(T)$ is obtained from a factor of this Fredholm determinant. We write $Q(T) = P(T) - \alpha$, with $P(T) = TR(T)$ and remark that there is nonnegative $a \in \mathbb{Q}$ such that $\alpha \in p^a(B\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})^*$. Then $a \leq h \cdot \deg(Q(T))$. Now, we choose integers b with $b \geq 2a+2$, $s := (r+c)b+q$ (see i) above) and d as in iii) above. Then it is easy to verify that there exist integers m and $\theta \geq 1$ such that

$$mb \geq \theta \geq \frac{\theta}{2} + s + d + 1 + (m+u+1)a.$$

We will work with the open affinoids $\mathcal{X}_\infty^{(u)} := \mathcal{X}_0(p^n,N)_\infty^{(u)}$, $\mathcal{X}_0^{(m)} := \mathcal{X}_0(p^n,N)_0^{(m)}$, $\mathcal{X}_0^{m+1)} := \mathcal{X}_0(p^n,N)_0^{(m+1)} \subset \mathcal{X}_0(p^n,N)$and their pro-Kummer étale sites.

Step 1. Fix $x \in \mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k\big)^{(h)}$, then $P(U_p)x = \alpha x$. Without loss of generality, we may consider $x \in \mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big)^{\mathrm{tf}}$ such that $P(U_p)x = \alpha x$ (this notation was introduced in Section §7) as $\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k\big) = \mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big)[1/p]$. Then there is a unique $x' \in \mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big)$ with $P(U_p)x' = \alpha x'$ and $(x')^{\mathrm{tf}} = p^d x$, and using Corollary 5.5 ii), we have: $P(U_p)^u(x') \in \mathrm{H}^1\big((\mathcal{X}_\infty^{(1)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big)$, and by using Corollary 5.5 i), we have that $P(U_p)^{m+u+1}(x') \in \mathrm{H}^1\big((\mathcal{X}\backslash\mathcal{X}_0^{(m+1)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big)$. Denote $\tilde{x}$ the image of $P(U_p)^{m+u+1}(x')$ in $\mathrm{H}^1\big((\mathcal{X}\backslash\mathcal{X}_0^{(m+1)})_{\mathrm{pke}}, \mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big)$.

We recall that $P(U_p)^{u+1}(x') = P(U_p)\big(P(U_p)^u(x')\big) \in \mathrm{H}^1((\mathcal{X}_\infty\backslash\mathcal{X}_0^{(1)})_{\mathrm{prok}}, \mathfrak{D}_k^o)$ by Corollary 5.5 i) and so denote $\mathcal{P}(x)$ the image of

$$\big(P(U_p)^m\big)^{\mathrm{good}}\big(P(U_p)^{u+1}(x')\big) \in \mathrm{H}^1\big((\mathcal{X}_0^{(m)})_{\mathrm{pke}}, \mathfrak{D}_k^o\big)$$

in $\mathrm{H}^1\big((\mathcal{X}_0^{(m)})_{\mathrm{pke}}, \mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big)$.

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Now, let us observe that the opens $(\mathcal{X}\backslash\mathcal{X}_0^{(m+1)})$ and $\mathcal{X}_0^{(m)}$ constitute an open covering of $\mathcal{X}$, and so we have a Mayer–Vietoris exact sequence:

$$\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big)\xrightarrow{\varphi}\mathrm{H}^1\big((\mathcal{X}\backslash\mathcal{X}_0^{(m+1)})_{\mathrm{pke}},\mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big)\oplus\mathrm{H}^1\big((\mathcal{X}_0^{(m)})_{\mathrm{pke}},\mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big)\xrightarrow{\psi}$$
$$\xrightarrow{\psi}\mathrm{H}^1\big((\mathcal{X}_0^{(m)}\backslash\mathcal{X}_0^{(m+1)})_{\mathrm{pke}},\mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big),$$

together with an almost isomorphism $\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\big)\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}\xrightarrow{\rho}\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathfrak{D}_k^o\big)$. We consider the element

$$(A,B):=\big(p^s\tilde{x},p^s\mathcal{P}(x)\big)\in\mathrm{H}^1\big((\mathcal{X}\backslash\mathcal{X}_0^{(m+1)})_{\mathrm{pke}},\mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big)\oplus\mathrm{H}^1\big((\mathcal{X}_0^{(m)})_{\mathrm{pke}},\mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big),$$

and we first claim that $\psi(A,B)=0$.

Indeed, let us denote by $\mathcal{V}$ the pro-Kummer étale site $(\mathcal{X}_0^{(m)}\backslash\mathcal{X}_0^{(m+1)})_{\mathrm{pke}}$. Thanks to Corollary 5.5 iv), we have

$$\psi(A,B)=A|_{\mathcal{V}}-B|_{\mathcal{V}}=\big(P(U_p)^m\big)^{\mathrm{bad}}\big(p^sP(U_p)^{u+1}(x')\big)\in p^s\mathrm{H}^1\big(\mathcal{V},\mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big).$$

Now, let us recall that we have an exact sequence

$$\mathrm{H}^1\big(\mathcal{V},\mathfrak{Fil}_b/p^\theta\mathfrak{Fil}_b\big)\xrightarrow{f}\mathrm{H}^1\big(\mathcal{V},\mathfrak{D}_k^o/p^\theta\mathfrak{D}_k^o\big)\xrightarrow{g}\mathrm{H}^1\big(\mathcal{V},\mathfrak{D}_k^o/(\mathfrak{Fil}_b+p^\theta\mathfrak{D}_k^o)\big).$$

Using Proposition 5.6 and the fact that $mb\geq\theta$, we have $g\big((P(U_p)^m)^{\mathrm{bad}}\big(P(U_p)^{u+1}(x')\big)=0$ which implies that there is $\beta\in\mathrm{H}^1\big(\mathcal{V},\mathfrak{Fil}_b/p^\theta\mathfrak{Fil}_h\big)$ such that $\big(P(U_p)^m\big)^{\mathrm{bad}}\big(P(U_p)^{u+1}(x')\big)=f(\beta)$. But by i) above, the image of f is annihilated by p^s, therefore $\psi(A,B)=p^sf(\beta)=0$. This proves the claim.

We continue the proof of the theorem. The Mayer–Vietoris exact sequence implies that there is $y\in\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\big)\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ such that

$$\rho(y)|_{(\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}}\equiv p^{s+1}P(U_p)^{m+u+1}x'=p^{s+1}\alpha^{m+u+1}x'\ \Big(\mathrm{mod}\ p^\theta\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)\Big).$$

Let $z_0\in\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\big)\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ be $z_0:=p^de_h(y)$. Then $z_0\in\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\big)^{(h)}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ and $\rho(z_0)|_{(\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}}\equiv p^{s+d+1}\alpha^{m+u+1}x'\ \Big(\mathrm{mod}\ p^\theta\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)^{(b)}\Big)$, where we write $\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)^{(h)}$ for the image $p^de_h\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)$.

Step 2. Let $x_1\in\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)^{(h)}$ be such that $\rho(z_0)|_{(\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}}=p^{s+d+1}\alpha^{m+u+1}\big(x'-p^{\theta/2}x_1\big)$. Such x_1 exists indeed because $x_1\in p^{-s-d-1+\theta/2}\alpha^{-m-u-1}\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)^{(h)}\subset\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)^{(h)}$.

Now, we apply to x_1 the **Step 1** we used on x and obtain $z_1\in\mathrm{H}^1\big(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\big)^{(h)}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ such that $\rho(z_1)\equiv p^{s+d+1}\alpha^{m+u+1}x_1\Big(\mathrm{mod}\ p^\theta\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)^{(b)}\Big)$.

We remark that $\rho(z_0+p^{\theta/2}z_1)|_{(\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}}\equiv p^{s+d+1}\alpha^{m+u+1}x'\ \Big(\mathrm{mod}\ p^{3\theta/2}\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)^{(b)}\Big)$.

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Repeating **Step 2**, we construct inductively a sequence $(z_v)_{v\geq 0}$ in $\mathrm{H}^1\left(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\right)^{(h)}\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}$ such that for every $v\geq 0$ we have

$$\rho\Big(\sum_{j=0}^{v}p^{j\theta/2}z_j\Big)|_{(\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}}\equiv p^{s+d+1}\alpha^{m+u+1}x' \left(\mathrm{mod}\ p^{(2v+1)\theta/2}\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}},\mathfrak{D}_k^o\big)^{(b)}\right).$$

If we denote by

$$z:=p^{-s-d-1}\alpha^{-m-u-1}\sum_{j=0}^{\infty}p^{j\theta/2}z_j\in\mathrm{H}^1\left(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\right)^{(h)}\widehat{\otimes}\mathbb{C}_p,$$

then $\rho\big(p^{-d}z\big)|_{(\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}}=p^{-d}\big((x')^{\mathrm{tf}}\big)=x$. To see that the written series converges, we recall (see [5]) that $\mathrm{H}^1\left(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\right)$ is profinite, therefore compact, and so $\mathrm{H}^1\left(\mathcal{X}_{\mathrm{pke}},\mathbb{D}_k^o\right)^{(h)}\widehat{\otimes}\mathbb{C}_p$ is complete for the p-adic topology.

Step 3. Using step (2), Theorem 5.2 is proven under the condition that $\left(\prod_{i=0}^{b-1}(u_k-i)\right)\in B[1/p]^*$. As no u_k-i is 0 in B due to our assumptions, we have $B\big[p^{-1}\prod_{i=h}^{h-1}(u_k-i)^{-1}\big]=\cap_{n\in\mathbb{N}}B_n[p^{-1}]$, with $B_n=B\big[\big[p^n/\prod_{i=h}^{b-1}(u_k-i)\big]\big]$ satisfying the requirements of Definition 3.6 for every $n\geq 0$. The theorem then holds for each B_n. Since the map $\mathcal{R}$ is a map of finite and projective $B[1/p]$-modules, we get that Theorem 5.2 holds after inverting $\prod_{i=h}^{b-1}(u_k-i)$.

Consider the reduction of Ψ_{HT} modulo u_k-i, for $h\leq i\leq b-1$. It is compatible with the classical p-adic Hodge–Tate decomposition, which provides a surjective map

$$\mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathrm{Sym}^i(T_0^\vee)\right)\otimes\mathbb{C}_p\longrightarrow\mathrm{H}^0\left(\mathcal{X},\omega_E^{i+2}\right);$$

see [5]. The map $\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}_i^o(T_0^\vee)[n]\big)\to\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathrm{Sym}^i(T_0^\vee)\big)$ is induced by the surjective map $\mathbb{D}_k^o(T_0^\vee)[n]/(u_k-i)\cong\mathbb{D}_i^o(T_0^\vee)[n]\to\mathrm{Sym}^i(T_0^\vee)$ and induces a surjective morphism on the slope $\leq h$-part $\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\text{-}\big)^{(h)}$, by results of Stevens. The map $\mathrm{H}^0\big(\mathcal{X},\omega_E^{i+2}\big)\to\mathrm{H}^0\big(\mathcal{X}_\infty^{(u)},\omega_E^{i+2}\big)\cong\mathrm{H}^0\big(\mathcal{X}_\infty^{(u)},\omega_E^{k+2}\big)/(u_k-i)$ is the restriction map, and it is an isomorphism on the slope $\leq h$-part by the classicity result of Coleman. We conclude that Ψ_{HT}, and hence $\mathcal{R}$, is surjective modulo u_k-i for every $h\leq i\leq b-1$. This implies that $\mathcal{R}$ is surjective if $\left(\prod_{i=0}^{h-1}(u_k-i)\right)\in B[1/p]^*$, as claimed. □

6. The B_{dR}-comparison

In this section, we consider the modular curve $\mathcal{X}=\mathcal{X}(N)$ of full-level N and the modular curve $\mathcal{X}_0(p^n,N)$ over the ring of integers $\mathcal{O}_K$ of an unramified extension K of $\mathbb{Q}_p$. Let $\widehat{X}$ be the p-adic formal scheme over $\mathcal{O}_K$ defined by formally completing the modular curve $X(N)$ over $\mathcal{O}_K$ along the special fibre $X(N)_0$. Let E be the universal generalised elliptic curve over $X(N)$, and denote its modulo p-reduction by E_0. As $X(N)_0$ is smooth, the crystalline cohomology $\mathrm{H}^1_{\mathrm{crys}}\big(E_0/\widehat{X}\big)$ is identified with the de Rham cohomology $\mathrm{H}_E:=\mathrm{H}^1_{\mathrm{dR}}\big(E/\widehat{X}\big)$ (as a module with log connection; see its description below).

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6.1. Period sheaves

There is a map of sites $w\colon \mathcal{X}_{\rm pke}\to \widehat{X}_{\rm et}$ defined by associating to an étale map $U\to \widehat{X}$ of formal schemes its adic generic fibre. We set $\mathcal{O}^{\rm unr,+}_{\mathcal{X}_{\rm pke}}:=w^{-1}\big(\mathcal{O}_{\widehat{X}}\big)$ (see [25, §2.2]). It is a subsheaf of $\mathcal{O}^+_{\mathcal{X}_{\rm pke}}$. As $\widehat{X}$ is endowed with the log structure defined by the cusps, this induces a log structure $\widehat{\alpha}\colon \widehat{\mathcal{M}}\to \mathcal{O}_{\widehat{X}}$; more precisely, $\widehat{\mathcal{M}}$ is the inverse limit $\lim\limits_{\infty\leftarrow n}\mathcal{M}_n$, where $\mathcal{M}_n\to \mathcal{O}_{\widehat{X}}/p^n\mathcal{O}_{\widehat{X}}$ is the log structure defined by the cusps. This induces a log structure $\alpha^{\rm unr}\colon \mathcal{M}^{\rm unr,+}:=w^{-1}\big(\widehat{\mathcal{M}}\big)\to \mathcal{O}^{\rm unr,+}_{\mathcal{X}}$ and a log structure $\alpha^+\colon \mathcal{M}^+\to \mathcal{O}^+_{\mathcal{X}_{\rm pke}}$ which defines the log structure $\alpha\colon \mathcal{M}\to \mathcal{O}_{\mathcal{X}_{\rm pke}}$ of §2.2. In particular, we can refine the prelog structure $\alpha^\flat\colon \mathcal{M}^\flat\to \widehat{\mathcal{O}}_{\mathcal{X}^\flat_{\rm pke}}$ of loc. cit. to a prelog structure $\alpha^{\flat,+}\colon \mathcal{M}^{\flat,+}\to \widehat{\mathcal{O}}^+_{\mathcal{X}^\flat_{\rm pke}}$, where $\mathcal{M}^{\flat,+}$ is the inverse limit $\lim\limits_{\leftarrow}\mathcal{M}^+$, indexed by $\mathbb{N}$, with transition maps given by raising to the p-th power. We write $a\mapsto a^\sharp$ for the first projection $\mathcal{M}^{\flat,+}\to\mathcal{M}^+$.

Recall that we have the period sheaf $\mathbb{A}_{\rm inf}$ over $\mathcal{X}_{\rm pke}$; see §2.2. We define the morphism of multiplicative monoids $\alpha_{\rm inf}\colon \mathcal{M}^{\flat,+}\to\mathbb{A}_{\rm inf}$ by composing $\alpha^{\flat,+}$ with the Teichmüller lift. Then the map $\vartheta\colon \mathbb{A}_{\rm inf}\to\widehat{\mathcal{O}}^+_{\mathcal{X}_{\rm pke}}$ is compatible with prelog structures, namely $\vartheta\circ\alpha_{\rm inf}$ coincides with the first projection $\mathcal{M}^{+,\flat}\to\mathcal{M}^+$ composed with α^+ and the natural map $\mathcal{O}^+_{\mathcal{X}_{\rm pke}}\to\widehat{\mathcal{O}}^+_{\mathcal{X}_{\rm pke}}$. The map ϑ defines a map

$$\vartheta_{\mathcal{X}}:=1\otimes\vartheta\colon \mathcal{O}\mathbb{A}_{\rm inf}:=\mathcal{O}^{\rm unr,+}_{\mathcal{X}_{\rm pke}}\otimes_{\mathbb{Z}_p}\mathbb{A}_{\rm inf}\longrightarrow\widehat{\mathcal{O}}^+_{\mathcal{X}_{\rm pke}}$$

of $\mathcal{O}^{\rm unr,+}_{\mathcal{X}_{\rm pke}}$-algebras. Furthermore, $\vartheta_{\mathcal{X}}$ is a map of sheaves compatible with the prelog structures $\alpha^{\rm unr}\times\alpha_{\rm inf}\colon \mathcal{M}^{\rm unr,+}\times\mathcal{M}^{\flat,+}\to\mathcal{O}\mathbb{A}_{\rm inf}$ and α^+; namely, $\vartheta_{\mathcal{X}}\circ\big(\alpha^{\rm unr}\times\alpha_{\rm inf}\big)$ coincides with α^+ composed with the homomorphism of monoids $\tau:\mathcal{M}^{\rm unr,+}\times\mathcal{M}^{\flat,+}\to\mathcal{M}^+$ provided by the natural map $\mathcal{M}^{\rm unr,+}\to\mathcal{M}^+$, the projection $\mathcal{M}^{\flat,+}\to\mathcal{M}^+$ given by $a\mapsto a^\sharp$ and the multiplication map $\mathcal{M}^+\times\mathcal{M}^+\to\mathcal{M}^+$.

Define $\mathcal{O}\mathbb{A}_{\rm log}$ to be the sheaf on $\mathcal{X}_{\rm pke}$ given by the p-adic completion of the log-divided powers (DP) envelope of $\mathcal{O}\mathbb{A}_{\rm inf}$ with respect to the product prelog structure $\alpha^{\rm unr}\times\alpha_{\rm inf}$ and with respect to $\vartheta_{\mathcal{X}}$; see [1, Lemma 2.16] for the definition. More precisely, let $\mathcal{M}'\subset(\mathcal{M}^{\rm unr,+})^{\rm gp}\times(\mathcal{M}^{\flat,+})^{\rm gp}$ be the sheaf of monoids defined as the inverse image of $\mathcal{M}^+\subset(\mathcal{M}^+)^{\rm gp}$ via the map $\tau^{\rm gp}\colon(\mathcal{M}^{\rm unr,+})^{\rm gp}\times(\mathcal{M}^{\flat,+})^{\rm gp}\to(\mathcal{M}^+)^{\rm gp}$ associated to τ defined above; here, the superscript gp is the sheaf of groups associated to a sheaf of monoids. Let $\mathcal{O}\mathbb{A}'_{\rm inf}:=\mathcal{O}\mathbb{A}_{\rm inf}\otimes_{\mathbb{Z}\left[\mathcal{M}^{\rm unr,+}\times\mathcal{M}^{\flat,+}\right]}\mathbb{Z}\big[\mathcal{M}'\big]$ be the log envelope of $\mathcal{O}^{\rm unr+}_{\mathcal{X}}\otimes_{\mathbb{Z}_p}\mathbb{A}_{\rm inf}$, with respect to prelog structure $\mathcal{M}^{\rm unr,+}\times\mathcal{M}^{\flat,+}$ and the map $\vartheta_{\mathcal{X}}$. It is endowed with a tautological prelog structure $\alpha'\colon\mathcal{M}'\to\mathcal{O}\mathbb{A}'_{\rm inf}$. The map $\vartheta_{\mathcal{X}}$ extends uniquely to a map $\vartheta'_{\mathcal{X}}\colon\mathcal{O}\mathbb{A}'_{\rm inf}\longrightarrow\widehat{\mathcal{O}}^+_{\mathcal{X}_{\rm pke}}$ compatible with the prelog structures α' and α^+, that is, such that $\vartheta'_{\mathcal{X}}\circ\alpha'$ is α^+ composed with the natural morphism $\mathcal{M}'\to\mathcal{M}^+$ induced by $(\alpha^{\rm unr})^{\rm gp}\times(\alpha_{\rm inf})^{\rm gp}$. Let $\mathcal{I}$ be the kernel of $\vartheta'_{\mathcal{X}}$.

Then $\mathcal{O}\mathbb{A}_{\rm log}$ is the p-adic completion of the DP envelope of $\mathcal{O}\mathbb{A}'_{\rm inf}$ with respect to the ideal $\mathcal{I}$. For every positive integer n, we also define $\mathcal{O}\mathbb{A}^{\rm log}_{{\rm max},n}$ to be the p-adic completion of the subsheaf $\mathcal{O}\mathbb{A}'_{\rm inf}\big[\frac{\mathcal{I}}{p^n}\big]$ of $\mathcal{O}\mathbb{A}'_{\rm inf}[p^{-1}]$. The map $\vartheta'_{\mathcal{X}}$ extends to a map $\vartheta_{{\rm max},n}\colon\mathcal{O}\mathbb{A}^{\rm log}_{{\rm max},n}\to\widehat{\mathcal{O}}^+_{\mathcal{X}_{\rm pke}}$. As $\mathcal{I}$ admits DP powers in $\mathcal{O}\mathbb{A}^{\rm log}_{{\rm max},n}$ and $\mathcal{O}\mathbb{A}_{\rm log}$ is the p-adic completion of the

DP envelope of $\mathcal{O}\mathbb{A}'_{\inf}$ with respect to $\mathcal{I}$, we have a natural map $\mathcal{O}\mathbb{A}_{\log} \to \mathcal{O}\mathbb{A}^{\log}_{\max,n}$ by the universal property of the DP envelope.

The derivation $d\colon \mathcal{O}_{\widehat{X}} \to \Omega^{\log}_{\widehat{X}/\mathcal{O}_K}$ defines $\mathbb{A}_{\inf}$-linear connections

$$\nabla\colon \mathcal{O}\mathbb{A}_{\log} \longrightarrow \mathcal{O}\mathbb{A}_{\log} \otimes_{\mathcal{O}^{\mathrm{unr},+}_{\mathcal{X}_{\mathrm{pke}}}} \Omega^{\mathrm{unr}}_{\mathcal{X}_{\mathrm{pke}}}$$

and

$$\nabla\colon \mathcal{O}\mathbb{A}^{\log}_{\max,n} \longrightarrow \mathcal{O}\mathbb{A}^{\log}_{\max,n} \otimes_{\mathcal{O}^{\mathrm{unr},+}_{\mathcal{X}_{\mathrm{pke}}}} \Omega^{\mathrm{unr}}_{\mathcal{X}_{\mathrm{pke}}},$$

where $\Omega^{\mathrm{unr}}_{\mathcal{X}_{\mathrm{pke}}} := w^{-1}\big(\Omega^{\log}_{\widehat{X}/\mathcal{O}_K}\big)$ is the inverse image of the logarithmic differentials on $\widehat{X}$.

Remark 6.1. On the complement of the cusps, where the log structure is trivial, the sheaf $\mathcal{O}\mathbb{A}_{\log}$ is the p-adic completion of the DP envelope of $\mathcal{O}\mathbb{A}_{\inf}$ with respect to the kernel of $\vartheta_{\mathcal{X}}$; it is the sheaf denoted $\mathcal{O}\mathbb{A}_{\mathrm{cris}}$ in [25, Def. 2.9].

Take an affine open neighbourhood $\widehat{U}_0 \subset \widehat{X}$ of a cusp, with local coordinate Y at the given cusp. Define $U_0 \subset \mathcal{X}$ to be the associated affinoid with induced log structure. Take $U = \lim_i U_i$ with $U_i = \big(\mathrm{Spa}(R_i,R_i^+),\mathcal{M}_i\big)$ to be a log affinoid perfectoid with initial object U_0, and let (R,R^+) be the p-adic completion of $\lim_i(R_i,R_i^+)$. By assumption, we have a compatible system of p^n-th roots Y_n of Y so that the system $\underline{Y} := [Y,Y_1,Y_2,\cdots] \in \alpha^{\flat,+}\big(\mathcal{M}^{\flat,+}\big)(U)$. As shown in [1, Lemma 3.25], we have $\mathcal{O}\mathbb{A}_{\log}(U) = \mathbb{A}_{\mathrm{cris}}(R,R^+)\{\langle w-1\rangle\}$, which is the p-adic completion of the DP algebra $\mathbb{A}_{\mathrm{cris}}(R,R^+)\langle w-1\rangle$. The structure as R_0-algebra is provided by sending $Y \mapsto [\underline{Y}]w$. Here, $\mathbb{A}_{\mathrm{cris}}$ is the p-adic completion of the DP envelope of $\mathbb{A}_{\inf}$ with respect to the kernel of ϑ. There is a similar description for $\mathcal{O}\mathbb{A}^{\log}_{\max,n}(U)$.

There are also the geometric de Rham sheaves $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ and $\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ defined in [16, Def. 2.2.10], with a map $\vartheta_{\mathrm{dR},\log}\colon \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log} \to \widehat{\mathcal{O}}^+_{\mathcal{X}_{\mathrm{pke}}}$ and logarithmic connection $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log} \longrightarrow \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log} \otimes_{\mathcal{O}^{\mathrm{unr},+}_{\mathcal{X}_{\mathrm{pke}}}} \Omega^{\mathrm{unr}}_{\mathcal{X}_{\mathrm{pke}}}$ (see [16, §2.2]). As $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ is an $\mathcal{O}_{\mathcal{X}_{\mathrm{pke}}} \otimes \mathbb{A}_{\inf}$-algebra by construction, there is a natural map of sheaves $\mathcal{O}\mathbb{A}_{\inf} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ whose composite with $\vartheta_{\mathrm{dR},\log}$ is $\vartheta_{\mathcal{X}}$.

Lemma 6.2. *The map $\mathcal{O}\mathbb{A}_{\inf} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ extends to morphisms $\mathcal{O}\mathbb{A}_{\log} \to \mathcal{O}\mathbb{A}^{\log}_{\max,n} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$, for every n, such that the composite with $\vartheta_{\mathrm{dR},\log}$ is the map $\vartheta_{\max,n}$ and it is compatible with connections.*

Proof. First of all, we show how to get a map $\mathcal{O}\mathbb{A}'_{\inf} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ of $\mathcal{O}\mathbb{A}_{\inf}$-algebras.

It suffices to construct this for log affinoid perfectoid objects of $\mathcal{X}_{\mathrm{pke}}$ arising from toric charts, as those form a basis of $\mathcal{X}_{\mathrm{pke}}$. Take any such object that we denote by U. It follows from [16, Lemma 2.3.12] that there exists a morphism of monoids $\beta\colon \mathcal{M}^{\flat,+}(U)^* \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}(U)$ such that for every $a \in \mathcal{M}^{\flat,+}(U)$ we have $\alpha^+(a^\sharp) = \big[\alpha^{\flat,+}(a)\big]\beta(a)$. Since the map $\mathcal{M}^{\flat,+}(U) \to \mathcal{M}^+(U)$ is an isomorphism, any element in the kernel H of $(\alpha^{\mathrm{unr}})^{\mathrm{gp}}(U) \times (\alpha_{\inf})^{\mathrm{gp}}(U)\colon (\mathcal{M}^{\mathrm{unr},+})^{\mathrm{gp}}(U) \times (\mathcal{M}^{\flat,+})^{\mathrm{gp}}(U) \to (\mathcal{M}^+)^{\mathrm{gp}}(U)$ is of the form $\big(a^\sharp(b^\sharp)^{-1}, ab^{-1}\big)$ with $a,b \in \mathcal{M}^\flat(U)$ such that $a^\sharp, b^\sharp \in \mathcal{M}^{\mathrm{unr},+}(U)$. Any such element can be sent to $\beta(a)\beta(b)^{-1}$, and this defines a group homomorphism $\beta\colon H \to \mathcal{M}^{\flat,+}(U)^*$. As

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$\mathcal{M}'(U) = H \cdot (\mathcal{M}^{\mathrm{unr},+}(U) \times \mathcal{M}^{\flat,+}(U)) \subset (\mathcal{M}^{\mathrm{unr},+})^{\mathrm{gp}}(U) \times (\mathcal{M}^{\flat,+})^{\mathrm{gp}}(U) \to (\mathcal{M}^{+})^{\mathrm{gp}}(U)$, such map extends to a map of monoids $\beta\colon \mathcal{M}'(U) \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}(U)$ which is compatible with the map $\alpha^{\mathrm{unr}}(U) \times \alpha_{\mathrm{inf}}(U)\colon \mathcal{M}^{\mathrm{unr},+}(U) \times \mathcal{M}^{\flat,+} \to \mathcal{O}\mathbb{A}_{\mathrm{inf}}(U) \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}(U)$. Using this we get a unique morphism $\mathcal{O}\mathbb{A}'_{\mathrm{inf}}(U) \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}(U)$ of $\mathcal{O}\mathbb{A}_{\mathrm{inf}}(U)$-algebras which coincides with β on $\mathcal{M}'(U)$.

In this way, we get the claimed morphism $\mathcal{O}\mathbb{A}'_{\mathrm{inf}} \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ whose composite with $\vartheta_{\mathrm{dR},\log}$ is $\vartheta'_{\mathcal{X}}$. In particular, the kernel $\mathcal{I}$ of $\vartheta'_{\mathcal{X}}$ is mapped to the kernel of $\vartheta_{\mathrm{dR},\log}$. As p is invertible in $\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$, it also extends to a morphism $\mathcal{O}\mathbb{A}'_{\mathrm{inf}}[\frac{\mathcal{I}}{p^n}] \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ and $\frac{\mathcal{I}}{p^n}$ maps to the kernel of $\vartheta_{\mathrm{dR},\log}$. Passing to log affinoid perfectoid objects of $\mathcal{X}_{\mathrm{pke}}$ one shows that such map extends to the p-adic completion $\mathcal{O}\mathbb{A}^{\log}_{\max,n}$ of $\mathcal{O}\mathbb{A}'_{\mathrm{inf}}[\frac{\mathcal{I}}{p^n}] \to \mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$. The compatibility with $\vartheta_{\max,n}$ is clear. The compatibility of the connections is also clear as both are defined using the derivation $d\colon \mathcal{O}_{\widehat{X}} \to \Omega^{\log}_{\widehat{X}/\mathcal{O}_K}$. □

6.2. Crystalline comparison morphisms

We have a crystalline comparison morphism over $\mathcal{X}_{\mathrm{pke}}$:

$$\alpha_{\log}\colon (T_p(E))^\vee \longrightarrow \left(\mathrm{H}^1_{\mathrm{crys}}\big(E_0/\widehat{X}\big) \otimes_{\mathcal{O}^+_{\mathcal{X}}} \mathcal{O}\mathbb{A}_{\log}\right)^{\nabla'=0}, \tag{8}$$

where $(T_p(E))^\vee$ is the $\mathbb{Z}_p$-dual of $T_p(E)$ and ∇' is the natural connection on $\mathrm{H}^1_{\mathrm{crys}}\big(E_0/\widehat{X}\big) \otimes \mathcal{O}\mathbb{A}_{\log}$ determined by the connections on the factors. We also write

$$\big(\mathrm{H}^1_{\mathrm{crys}}(E_0/\widehat{X}) \otimes_{\mathcal{O}^+_{\mathcal{X}}} -\big) \text{ for } \Big(w^{-1}\big(\mathrm{H}^1_{\mathrm{crys}}(E_0/\widehat{X})\big) \otimes_{\mathcal{O}^{\mathrm{unr}+}_{\mathcal{X}}} -\Big)$$

to ease the notation.

We describe $\alpha_{\log}$ for a log affinoid perfectoid open cover of $\mathcal{X}$. Consider an étale open $U = \mathrm{Spf}(S) \subset \widehat{X}$. Let $W = \mathrm{Spa}(R,R^+)$ be a log affinoid perfectoid cover of $U_{\overline{\mathbb{Q}}_p}$, the adic geometric generic fibre of U. We assume that the universal elliptic curve extends to a (generalised) elliptic curve $\widetilde{E}$ over $\mathrm{Spec}(S)$ and that $T_p(E)$ is trivialized over W. Consider the rings $A_{\mathrm{cris}}(R^+)$ (resp. $A_{\log}(R^+)$) defined by taking the p-adic completion of the DP envelope (resp. the log DP envelope) of $S \otimes \mathbb{A}_{\mathrm{inf}}(W)$ with respect to the kernel of the map $1 \otimes \vartheta(W)\colon S \otimes \mathbb{A}_{\mathrm{inf}}(W) \to \widehat{\mathcal{O}}^+_{\mathcal{X}}(W) = R^+$. It naturally maps to $\mathcal{O}\mathbb{A}_{\log}(W)$. We define $\alpha_{\log}(W)$ with values in $\mathrm{H}^1_{\mathrm{crys}}\big(\widetilde{E}_0/S\big) \otimes_S A_{\log}(R^+)$ as follows.

Away from the cusps: Assume first that $\widetilde{E}$ is an elliptic curve so that $\widetilde{E}[p^n](R^+) = E[p^n](R)$ for every $n \in N$ and $T_p(\widetilde{E})(R^+) = T_p(E)(R)$. Equivalently, the log structure is trivial and $A_{\log}(R^+) = A_{\mathrm{cris}}(R^+)$. Then to give $a \in T_p(E^\vee)(R) = T_p\big(E\big)^\vee(R^+)(1)$ is equivalent to give a map of p-divisible groups $\gamma_a\colon \mathbb{Q}_p/\mathbb{Z}_p \to E^\vee[p^\infty]$ over R^+ and γ_a defines a map of covariant Dieudonné modules

$$\mathbf{D}_{\mathrm{cris}}(\gamma_a)\colon \mathbf{D}_{\mathrm{cris}}(\mathbb{Q}_p/\mathbb{Z}_p)\big(A_{\mathrm{cris}}(R^+)\big) \to \mathbf{D}_{\mathrm{cris}}\big(E^\vee[p^\infty]\big)\big(A_{\mathrm{cris}}(R^+)\big).$$

Note that $A_{\mathrm{cris}}(R^+) = \mathbf{D}_{\mathrm{cris}}(\mathbb{Q}_p/\mathbb{Z}_p)\big(A_{\mathrm{cris}}(R^+)\big)$ and $\mathbf{D}_{\mathrm{cris}}\big(\widetilde{E}^\vee[p^\infty]\big)\big(A_{\mathrm{cris}}(R^+)\big) = \mathrm{H}^1_{\mathrm{crys}}\big(E_0^\vee/S\big)^\vee \otimes_S A_{\mathrm{cris}}(R^+)$, where $E_0^\vee$ is the modulo p reduction of $E^\vee$. Thus $\mathbf{D}_{\mathrm{cris}}(\gamma_a)$ defines a map

$$\mathbf{D}_{\mathrm{cris}}(\gamma_a)\colon A_{\mathrm{cris}}(R^+) \longrightarrow \mathrm{H}^1_{\mathrm{crys}}\big(E_0^\vee/S\big)^\vee \otimes_S A_{\mathrm{cris}}(R^+)$$

and setting

$$\alpha_{\log}(W)(a) = \alpha_{\mathrm{cris}}(W)(a) := \mathbf{D}_{\mathrm{cris}}(\gamma_a)(1)$$

(see [24]§3.5) and using Weil, respectively Poincaré dualities gives the map

$$\alpha_{\log}(W) : T_p(E)^\vee(R^+) \longrightarrow \mathrm{H}^1_{\mathrm{crys}}(E_0/S) \otimes_s a : \mathrm{cris}(R^+).$$

As $\mathbf{D}_{\mathrm{cris}}(\gamma_a)$ is a map of crystals; therefore, it is compatible with connections. Since $\nabla(1) = 0$ for $1 \in A_{\mathrm{cris}}(R^+)$, then $\nabla'\Big(\mathrm{Im}\big(\alpha_{\mathrm{cris}}(W)\big)\Big) = 0$.

Around the cusps: Assume next that U does not contain supersingular points. Then the connected part $\widetilde{E}[p^\infty]^0$ of $\widetilde{E}[p^\infty]$ is a p-divisible group of multiplicative type. Let $\widetilde{E}[p^\infty]^{0,\vee}$ be its Cartier dual; it is an étale p-divisible group over U. We write $\mathrm{H}^1_{\mathrm{crys}}(E_0/S)$ for the direct sum of $\omega_{E/S} \cong \mathbf{D}_{\mathrm{cris}}\big(\widetilde{E}[p^\infty]^0\big)(S)^\vee$ and $\omega^{-1}_{E/S} \cong \mathbf{D}_{\mathrm{cris}}\big(\widetilde{E}[p^\infty]^{0,\vee}\big)(S)^\vee$. The connection $\mathrm{H}^1_{\mathrm{crys}}(E_0/S) \to \mathrm{H}^1_{\mathrm{crys}}(E_0/S) \otimes_S \Omega^{1,\log}_{S/\mathcal{O}_K}$ is the sum of the connections induced from those on $\mathbf{D}_{\mathrm{cris}}\big(\widetilde{E}[p^\infty]^0\big)(S)^\vee$ and $\mathbf{D}_{\mathrm{cris}}\big(\widetilde{E}[p^\infty]^{0,\vee}\big)(S)^\vee$ plus the S-linear isomorphism $\omega_{E/S} \cong \omega^{-1}_{E/S} \otimes_S \Omega^{1,\log}_{S/\mathcal{O}_K}$ provided by the Kodaira–Spencer isomorphism, which we denote KS. In other words, as a module $\mathbf{D}_{\mathrm{cris}}(\widetilde{E}[p^\infty])$ is isomorphic to the direct sum $\mathbf{D}_{\mathrm{cris}}(\widetilde{E}[p^\infty]^0)(S)^\vee \oplus \mathbf{D}_{\mathrm{cris}}(\widetilde{E}[p^\infty]^{0,\vee})(S)^\vee \cong \omega_{E/S} \oplus \omega^{-1}_{E/S}$, while the logarithmic connection $\nabla_{\widetilde{E}[p^\infty]}$ is given with respect to this decomposition by

$$\begin{pmatrix} \nabla_{\widetilde{E}[p^\infty]^0} & 0 \\ \mathrm{KS} & \nabla_{\widetilde{E}[p^\infty]^{0,\vee}} \end{pmatrix}.$$

Lemma 6.3. *If U does not contain supersingular points and $\widetilde{E}$ is an elliptic curve, the two definitions of $\mathrm{H}^1_{\mathrm{crys}}(E_0/S)$ coincide. Moreover, a splitting of the Tate module $T_p\big(\widetilde{E}[p^\infty]\big)(R) = T_p\big(\widetilde{E}[p^\infty]^0\big)(R^+) \oplus T_p\big(\widetilde{E}[p^\infty]^{0,\vee}\big)(R^+)$ uniquely defines*

(1) *a splitting $\widetilde{E}[p^\infty]_{R^+} = \widetilde{E}[p^\infty]^{0,\vee}_{R^+} \oplus \widetilde{E}[p^\infty]^0_{R^+}$ of the connected-étale sequence for $\widetilde{E}[p^\infty]_{R^+}$;*

(2) *a splitting of $\boldsymbol{D}_{\mathrm{cris}}\big(\widetilde{E}^\vee[p^\infty]\big)\big(A_{\mathrm{cris}}(R^+)\big)^\vee \cong \mathrm{H}^1_{\mathrm{crys}}\big(E_0/S\big)^\vee \otimes_S A_{\mathrm{cris}}(R^+)$;*

(3) *a splitting of $\alpha_{\log}(W)$ as the direct sum of the crystalline comparison maps for the p-divisible groups $\widetilde{E}[p^\infty]^0_{R^+}$ and $\widetilde{E}[p^\infty]^{0,\vee}_{R^+}$, compatibly with (1) ad (2).*

Proof. Notice that the quotient map $\widetilde{E}^\vee[p^\infty] \to \widetilde{E}^\vee[p^\infty]^{\mathrm{et}}$ onto the étale part induces a map from $\mathrm{H}^1_{\mathrm{crys}}\big(\widetilde{E}_0/S\big)$ to the Dieudonné module of $\widetilde{E}^\vee[p^\infty]^{\mathrm{et}}(S)$ which splits canonically via the unit root decomposition. As $\widetilde{E}^\vee[p^\infty]^{\mathrm{et}} \cong \widetilde{E}[p^\infty]^{0,\vee}$ and similarly $\widetilde{E}^\vee[p^\infty]^0 \cong \widetilde{E}[p^\infty]^{\mathrm{et},\vee} \cong \widetilde{E}[p^\infty]^0$ we conclude that $\mathrm{H}^1_{\mathrm{crys}}\big(\widetilde{E}_0/S\big)$ splits canonically, as a module, as the direct sum of $\omega_{E/S} \cong \mathbf{D}_{\mathrm{cris}}\big(\widetilde{E}[p^\infty]^0\big)(S)^\vee$ and $\omega^{-1}_{E/S} \cong \mathbf{D}_{\mathrm{cris}}\big(\widetilde{E}[p^\infty]^{0,\vee}\big)(S)^\vee$, with

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connection given by the connection on each factor plus the Kodaira–Spencer isomorphism. Thus, the two descriptions of $\mathrm{H}^1_{\mathrm{crys}}(\widetilde{E}_0/S)$ coincide. Claims (2) and (3) follow from the functoriality of $\mathbf{D}_{\mathrm{cris}}$ and of $\alpha_{\log}$. □

Lemma 6.3 suggests how to define $\alpha_{\log}(W)$ in the case that U does not contain supersingular points but can possibly contain the cusps. Namely, we write $T_p(\widetilde{E}[p^\infty])(R)$ as a split extension of $T_p(\widetilde{E}[p^\infty]^{0,\vee})(R^+)$ by $T_p(\widetilde{E}[p^\infty]^0)(R^+)$ and $\alpha_{\log}(W)$ as the direct sum of the crystalline comparison maps for the p-divisible groups $\widetilde{E}[p^\infty]^0_{R^+}$ and $\widetilde{E}[p^\infty]^{0,\vee}_{R^+}$. Then Lemma 6.3 implies that this definition agreed with the definition given away from the cusps. Thus, $\alpha_{\log}(W)$ are functorial in the pair (U,W), and hence, they glue to a morphism $\alpha_{\log}$ on $\mathcal{X}_{\mathrm{pke}}$. We also consider the composite map of sheaves

$$\alpha^{\log}_{\mathrm{max,n}}\colon (T_p(E))^\vee \longrightarrow \left(\mathrm{H}^1_{\mathrm{crys}}(E_0/\widehat{X})\otimes_{\mathcal{O}^+_{\mathcal{X}}}\mathcal{O}\mathbb{A}^{\log}_{\mathrm{max,n}}\right)^{\nabla'=0},$$

induced by composing $\alpha_{\log}$ with the map of sheaves $\mathcal{O}\mathbb{A}_{\log}\to\mathcal{O}\mathbb{A}^{\log}_{\mathrm{max,n}}$ described at the beginning of the section.

6.3. The sheaves $\mathbf{W}_{k,\mathrm{dR}}$

We consider strict neighbourhoods $\mathcal{X}(p/\mathrm{Ha}^{p^s})$ of the ordinary locus in $\mathcal{X}$, where Ha is a (any) local lift of the Hasse invariant. It follows from [23, Lemma 3.3.8 & Lemma 3.3.15] that the neighbourhoods $\mathcal{X}(p/\mathrm{Ha}^{p^s})\otimes_K\mathbb{C}_p$ and $\mathcal{X}^{(m)}_\infty$ of §4.2 for varying s, respectively m, are fundamental systems of open neighbourhoods of the ordinary locus of $\mathcal{X}_{\mathbb{C}_p}$. We then take s and m large enough so that the conclusions of Proposition 4.5 hold for $\mathcal{X}(p/\mathrm{Ha}^{p^s})$ and for $\mathcal{X}^{(m)}_\infty$. Namely, we require that a canonical subgroup C_n of order p^n exists, and this defines a section

$$\mathcal{X}(p/\mathrm{Ha}^{p^s})\to\mathcal{X}_0(p^n,N)(p/\mathrm{Ha}^{p^s}) \tag{9}$$

of the natural forgetful map $\mathcal{X}_0(p^n,N)(p/\mathrm{Ha}^{p^s})\longrightarrow\mathcal{X}(p/\mathrm{Ha}^{p^s})$. Write

$$\nu\colon\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})\longrightarrow\mathcal{X}(p/\mathrm{Ha}^{p^s})$$

for the $(\mathbb{Z}/p^n\mathbb{Z})^*$-Galois cover classifying trivializations of $C_n^\vee$.

Let E be the universal elliptic curve over the normalization of $\widehat{X}$ in $\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})$. Its invariant differentials ω_E and relative de Rham cohomology H_E define locally free $\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}$-modules with the Hodge filtration $\omega_E\subset\mathrm{H}_E$. Write $\underline{\delta}$ for the invertible $\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}$-module defined by $\underline{\delta}:=\omega_E(\omega_E^{\mathrm{mod}})^{-1}$. Note that $\underline{\delta}^{p-1}=\nu^*(\mathrm{Hdg})$, where Hdg is the ideal of $\mathcal{O}^+_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}$ generated by the local lifts Ha of the Hasse invariant (due to the blowup, it does not depend on the choice of the local lifts).

From the tautological section P of $C_n^\vee$, we get a canonical section t of $\omega_E^{\mathrm{mod}}/p^r\omega_E^{\mathrm{mod}}$ generating it as $\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}$-module. Recall that $k\colon\mathbb{Z}_p^*\to B^*$ is an r-analytic character

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for $r \le n$, where B is a ring as in definition 3.6. We denote by $\mathrm{H}_E^{\#}$ the pushout in the category of $\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}$-modules of the diagram

$$\begin{array}{ccc} \underline{\delta}^p\omega_E & \longrightarrow & \underline{\delta}^p\mathrm{H}_E \\ \cap & & \\ \omega_E^{\mathrm{mod}}. & & \end{array}$$

We then have the following commutative diagram of sheaves with exact rows:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \underline{\delta}^p\omega_E & \longrightarrow & \underline{\delta}^p\mathrm{H}_E & \longrightarrow & \underline{\delta}^p\omega_E^{-1} & \longrightarrow & 0 \\ & & \cap & & \downarrow & & \| & & \\ 0 & \longrightarrow & \omega_E^{\mathrm{mod}} & \longrightarrow & \mathrm{H}_E^{\#} & \longrightarrow & \underline{\delta}^p\omega_E^{-1} & \longrightarrow & 0 \\ & & \cap & & \cap & & \cap & & \\ 0 & \longrightarrow & \omega_E & \longrightarrow & \mathrm{H}_E & \longrightarrow & \omega_E^{-1} & \longrightarrow & 0. \end{array}$$

It follows that $\mathrm{H}_E^{\#}$ is a locally free $\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^r})}$-module of rank two and $(\omega_E^{\mathrm{mod}},t) \subset (\mathrm{H}_E^{\#},t)$ is a compatible inclusion of locally free sheaves with marked sections.

Using the formalism of the dual VBMS (see §3.3), we get sheaves of $\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}\widehat{\otimes}B$-modules $\omega_E^k \subset \mathbf{W}_{k,\mathrm{dR}}$. We have a residual action of the Galois group $(\mathbb{Z}/p^n\mathbb{Z})^*$ of $j_n\colon \mathcal{I}g_n(p/\mathrm{Ha}^{p^s}) \to \mathcal{X}(p/\mathrm{Ha}^{p^s})$ on $(\omega_E^{\mathrm{mod}},t)$ and $(\mathrm{H}_E^{\#},t)$ on which it acts by scalar multiplication. We then get sheaves $\omega_E^k \subset \mathbf{W}_{k,\mathrm{dR}}$ of $\mathcal{O}_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}\widehat{\otimes}B$-modules by taking the subsheaves of $j_{n,*}(\omega_E^k) \subset j_{n,*}(\mathbf{W}_{k,\mathrm{dR}})$ on which $\mathbb{Z}_p^*$ acts via k. We refer to [2, §3.2& 3.3] for details.

Proposition 6.4. *The base change of $\omega_E^k[1/p]$ to $\mathcal{X}(p/\mathrm{Ha}^{p^s})_{\mathbb{C}_p}$ coincides with the restriction of the sheaf $\omega_E^k[1/p]$ defined in §4.4 over $\mathcal{X}_0(p^n,N)_\infty^{(m)}$.*

The sheaf $\mathbf{W}_{k,\mathrm{dR}}$ has a natural, increasing filtration $\left(\mathrm{Fil}_n\mathbf{W}_{k,\mathrm{dR}}\right)_{n\ge 0}$ such that $\omega_E^k[1/p] = \mathrm{Fil}_0\mathbf{W}_{k,\mathrm{dR}}[1/p]$. The Gauss–Manin connection $\nabla\colon \mathrm{H}_E \to \mathrm{H}_E \otimes \Omega^{\log}_{\mathcal{X}(p/\mathrm{Ha}^{p^s})/K}$ induces a connection

$$\nabla_k\colon \mathbf{W}_{k,\mathrm{dR}}[1/p] \longrightarrow \mathbf{W}_{k,\mathrm{dR}}\widehat{\otimes}\Omega^{\log}_{\mathcal{X}(p/\mathrm{Ha}^{p^s})/K}$$

satisfying Griffiths' transversality, that is, $\nabla_k\left(\mathrm{Fil}_n\mathbf{W}_{k,\mathrm{dR}}[1/p]\right) \subset \mathrm{Fil}_{n+1}\mathbf{W}_{k,\mathrm{dR}}[1/p]\widehat{\otimes}\Omega^{\log}_{\mathcal{X}(p/\mathrm{Ha}^{p^s})/K}$.

The cohomology groups $\mathrm{H}^0\left(\mathcal{X}(p/\mathrm{Ha}^{p^s}),\mathbf{W}_{k,\mathrm{dR}}\right)$ and $\mathrm{H}^0\left(\mathcal{X}(p/\mathrm{Ha}^{p^s}),\mathrm{Fil}_n\mathbf{W}_{k,\mathrm{dR}}\right)$ are endowed with an action of the U_p-operator, and for every integer h they admit slope $\le h$ decompositions. Furthermore, we have $p\nabla_k \circ U_p = U_p \circ \nabla_k$, and for $n \gg 0$ we have

$$\mathrm{H}^0\left(\mathcal{X}(p/\mathrm{Ha}^{p^s}),\mathrm{Fil}_n\mathbf{W}_{k,\mathrm{dR}}\right)^{(h)} = \mathrm{H}^0\left(\mathcal{X}(p/\mathrm{Ha}^{p^s}),\mathbf{W}_{k,\mathrm{dR}}\right)^{(h)}.$$

Proof. The first statement follows from the fact that the two constructions coincide on $\mathcal{X}(p/\mathrm{Ha}^{p^s})_{\mathbb{C}_p}$. The other statements are proven in [2]. Namely, the filtration is constructed

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in Theorem 3.11, the connection in Theorem 3.18, the U_p-operator is defined in §3.6 and the statements about the slope decomposition follow from Correlation 3.26. $\square$

Using the Kodaira–Spencer isomorphism $\omega_E^2|_{\mathcal{X}(p/\mathrm{Ha}^{p^s}}\Omega^{\log}_{\mathcal{X}(p/\mathrm{Ha}^{p^s})/K}$, we can and will view the connection ∇_k as a map $\nabla_k\colon \mathbf{W}_{k,\mathrm{dR}}[p^{-1}]\to \mathbf{W}_{k+2,\mathrm{dR}}[p^{-1}]$.

6.4. The de Rham comparison map

Fix an r-analytic weight $k\colon \mathbb{Z}_p^*\to B^*$ as in definition 3.6. Let B_{dR}^+ and $B_{\mathrm{dR}}=B_{\mathrm{dR}}^+[t^{-1}]$ be the classical period rings of Fontaine with the canonical topology so that, for example, the quotient topology on $B_{\mathrm{dR}}^+/tB_{\mathrm{dR}}^+=\mathbb{C}_p$ is the p-adic topology on $\mathbb{C}_p$. They are endowed with filtrations such that $\mathrm{Fil}^iB_{\mathrm{dR}}=t^iB_{\mathrm{dR}}^+$ for every $i\in\mathbb{Z}$. We write $\mathbf{W}_{k,\mathrm{dR},\bullet}$ for the complex $\mathbf{W}_{k,\mathrm{dR}}[p^{-1}]\to \mathbf{W}_{k+2,\mathrm{dR}}[p^{-1}]$, where the map is defined by ∇_k and $\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}$ for the subcomplex $\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR}}[p^{-1}]\to \mathrm{Fil}_{m+1}\mathbf{W}_{k+2,\mathrm{dR}}[p^{-1}]$. In this section, we use the map $\alpha_{\log}$ to get the following result.

Theorem 6.5. *We have a Hecke equivariant, $B\widehat{\otimes}B_{\mathrm{dR}}^+$-linear, $\mathrm{Gal}(\overline{K}/K)$-equivariant map*

$$\rho_k\colon \mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}_k^o(T_0^\vee)[n]\big)^{(h)}\widehat{\otimes}B_{\mathrm{dR}}^+\longrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k,\mathrm{dR},\bullet}\big)^{(h)}\widehat{\otimes}\mathrm{Fil}^{-1}B_{\mathrm{dR}},$$

where the completed tensor product is taken considering the canonical topology on B_{dR}^+. Moreover,

$$\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k,\mathrm{dR},\bullet}\big)^{(h)}\cong\frac{\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k+2,\mathrm{dR}}[1/p]\big)^{(h)}}{\nabla_k\Big(\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k,\mathrm{dR}}[1/p]\big)^{(h-1)}\Big)}.$$

Furthermore:

i. *If $u_k(u_k-1)\cdots(u_k-h+1)$ is invertible in $B[1/p]$ the map ρ_k is surjective;*
ii. *The map $\omega_E^{k+2}\to\mathbf{W}_{k+2,\mathrm{dR}}$ induces a surjective map, which is an isomorphism if (i) above holds:*

$$\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\omega_E^{k+2}[1/p]\big)^{(h)}\longrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k,\mathrm{dR},\bullet}[1/p]\big)^{(h)}.$$

iii. *For specializations, $B[1/p]\to\mathbb{Q}_p$ so that the composite weight k_0 is classical, ρ_k is compatible with the classical de Rham comparison map*

$$\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathrm{Sym}^{k_0}(T_p(E)^\vee)\big)\otimes B_{\mathrm{dR}}\cong \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X},\mathrm{Sym}^{k_0}(\mathrm{H}_E)\big)\otimes_K B_{\mathrm{dR}}$$

via the map induced by taking on the left-hand side the pro-Kummer étale cohomology via the projection $\mathbb{D}_k^o(T_0^\vee)[n]\to\mathbb{D}_{k_0}^o(T_0^\vee)[n]\to\mathrm{Sym}^{k_0}(T_p(E)^\vee)$ and on the right-hand side the restriction map to the open $\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)$

$$\begin{aligned}\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X},\mathrm{Sym}^{k_0}(\mathrm{H}_E)\big)\longrightarrow &\ \mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\omega_E^{k_0+2}[1/p]\big)/\vartheta^{k_0+1}\\ &\times\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\omega_E^{-k_0}[1/p]\big).\end{aligned}$$

(Here, ϑ is the classical theta operator on modular forms.)

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The proof of Theorem 6.5 will be given in Section §6.4.2.

We now consider the Galois cohomology for the group $G := \mathrm{Gal}(\overline{K}/K)$. Recall that $\mathrm{H}^1\left(G, \mathrm{Fil}^{-1}B_{\mathrm{dR}}(1)\right) = K[\log\chi]$, where χ is the cyclotomic character. More precisely, we see $\log\chi$ as a 1-cocycle $\log\chi\colon G \longrightarrow \mathbb{Z}_p \subset \mathrm{Fil}^{-1}B_{\mathrm{dR}}(1) = t(t^{-1}B_{\mathrm{dR}}^+) = B_{\mathrm{dR}}^+$, and we denoted $[\log\chi]$ its cohomology class. We then obtain from Theorem 6.5 the following corollary:

Corollary 6.6. *We have a Hecke equivariant, B-linear map*

$$\mathrm{Exp}_k^*\colon \mathrm{H}^1\left(G, \mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}_k^o(T_0^\vee)[n](1)\right)^{(h)}\right) \longrightarrow \mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right), \mathbf{W}_{k,\mathrm{dR},\bullet}\right)^{(h)},$$

called the big dual exponential map. It has the property that for every classical weight specialization k_0 it is compatible with the classical dual exponential map as follows:

a) *If $k_0 > h-1$, that is, k_0 is a noncritical weight for the slope h, then we have the following commutative diagram with horizontal isomorphisms. Here, we denoted by $\exp_{k_0}^*$ the Kato dual exponential map associated to weight k_0 modular forms.*

$$\begin{array}{ccc}
\left(\mathrm{H}^1\left(G, \mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}_k^o(T_0^\vee)[n](1)\right)^{(h)}\right)\right)_{k_0} & \xrightarrow{\left(\mathrm{Exp}_k^*\right)_{k_0}} & \left(\mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right), \mathbf{W}_{k,\mathrm{dR},\bullet}\right)^{(h)}\right)_{k_0} \\
\downarrow\cong & & \downarrow\cong \\
\mathrm{H}^1\left(G, \mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathrm{Sym}^{k_0}(T_p(E)^\vee)(1)\right)^{(h)}\right) & \xrightarrow{\exp_{k_0}^*} & \mathrm{Fil}^0\mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}, \mathrm{Sym}^{k_0}(\mathrm{H}_E)\right)^{(h)}.
\end{array}$$

b) *If $0 \le k_0 \le h+1$, that is, k_0 is critical with respect to h, we only have a commutative diagram of the form*

$$\begin{array}{ccc}
\left(\mathrm{H}^1\left(G, \mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}_k^o(T_0^\vee)[n](1)\right)^{(h)}\right)\right)_{k_0} & \xrightarrow{\left(\mathrm{Exp}_k^*\right)_{k_0}} & \left(\mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right), \mathbf{W}_{k,\mathrm{dR},\bullet}\right)^{(h)}\right)_{k_0} \\
\downarrow & & \uparrow \\
\mathrm{H}^1\left(G, \mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathrm{Sym}^{k_0}(T_p(E)^\vee)(1)\right)^{(h)}\right) & \xrightarrow{\exp_{k_0}^*} & \mathrm{Fil}^0\mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}, \mathrm{Sym}^{k_0}(\mathrm{H}_E)\right)^{(h)},
\end{array}$$

where the right vertical arrow is induced by restriction.

Proof. Granted Theorem 6.5, we have the following natural B-linear and G-equivariant maps:

$$\begin{aligned}
\mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}_k^o(T_0^\vee)[n](1)\right)^{(h)} &\longrightarrow \mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}_k^o(T_0^\vee)[n](1)\right)^{(h)}\widehat{\otimes}B_{\mathrm{dR}}^+ \xrightarrow{\rho_k} \\
&\xrightarrow{\rho_k} \mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right), \mathbf{W}_{k,\mathrm{dR},\bullet}\right)^{(h)}\widehat{\otimes}\mathrm{Fil}^{-1}B_{\mathrm{dR}}(1),
\end{aligned}$$

whose composition we denote by f_k. Then we define Exp_k^* as the map induced by composing f_k in Galois cohomology with the natural isomorphism:

$$\begin{aligned}
&\mathrm{H}^1\left(G, \mathrm{H}^1\left(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}_k^o(T_0^\vee)[n](1)\right)^{(h)}\right) \\
&\quad\longrightarrow \mathrm{H}^1\left(G, \mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right), \mathbf{W}_{k,\mathrm{dR},\bullet}\right)^{(h)}\widehat{\otimes}\mathrm{Fil}^{-1}B_{\mathrm{dR}}(1)\right) \cong \\
&\cong \left(\mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right), \mathbf{W}_{k,\mathrm{dR},\bullet}\right)^{(h)} \otimes_K \mathrm{H}^1\left(G, \mathrm{Fil}^{-1}B_{\mathrm{dR}}(1)\right) \cong \mathrm{H}^1_{\mathrm{dR}}\left(\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right), \mathbf{W}_{k,\mathrm{dR},\bullet}\right)^{(h)}.
\end{aligned}$$

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The vertical maps on the left-hand side of the diagrams in (a) and (b) are induced by the quotient map $\mathbb{D}_k^o(T_0^\vee)[n]|_{k_0} \to \mathrm{Sym}^{k_0}(T_p(E)^\vee)$. The fact that the induced map on the slope $\leq h$-part of $\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\_\big)$ is an isomorphism upon specialization in weight k_0 follows identifying $\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\_\big)$ with the group cohomology $\mathrm{H}^1\big(\Gamma_0(p^n)\cap\Gamma(N),\_\big)$, arguing as in [5, Prop. 3.18] and using Glenn Stevens' classicality result of modular symbols; see [5, Thm. 3.16 & Thm. 3.17]. The vertical maps on the right-hand side of the diagrams in (a) and (b) are induced by the inclusion $\mathrm{H}^0\big(\mathcal{X},\omega_E^{k_0+2}\big) = \mathrm{Fil}^0\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X},\mathrm{Sym}^{k_0}(\mathrm{H}_E)\big) \subset \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X},\mathrm{Sym}^{k_0}(\mathrm{H}_E)\big)$. The induced map on slope $\leq h$-part in (a) is an isomorphism thanks to Theorem 5.1. □

Remark 6.7. The main reason we twist by 1 the pro-Kummer étale sheaves $\mathbb{D}_k^o(T_0^\vee)[n]$ and $\mathrm{Sym}^{k_0}\big((T_p(E)^\vee\big)$ in the corollary above is because the Galois representations attached to overconvergent eigenforms, respectively classical ones are quotients of pro-Kumer étale cohomology of such sheaves (with the twist, that is).

6.4.1. A refinement of the map $\alpha_{\log}$. Consider the étale cover $j_n\colon \mathcal{I}g_n\big(p/\mathrm{Ha}^{p^s}\big) \to \mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)$ given by choosing a generator of $C_n^\vee$. It is Galois with groups $\Delta_n \cong (\mathbb{Z}/p^n\mathbb{Z})^*$. Let $D_n := (T_p(E)/p^n)/C_n$. Then we get an exact sequence $0 \to D_n^\vee \to T_p(E)^\vee/p^nT_p(E)^\vee \to C_n^\vee \to 0$ with a marked section s of $C_n^\vee$.

Proposition 6.8. *The restriction to $\mathcal{I}g_n\big(p/\mathrm{Ha}^{p^s}\big)$ of the map $\alpha^{\log}_{\mathrm{max},n+1}$ factors via the submodule $\mathrm{H}_E^\# \otimes_{\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}} \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}|_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}$. The induced map*

$$\beta^{\log}_{\mathrm{max},n+1}\colon (T_p(E))^\vee \otimes_{\mathbb{Z}_p} \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}|_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})} \longrightarrow \mathrm{H}_E^\# \otimes_{\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}} \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}|_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}$$

sends the tautological section P of $C_n^\vee$ to the marked section t of $\mathrm{H}_E^\#/p^r\mathrm{H}_E^\#$ and sends $D_n^\vee$ to 0 modulo p^r. In particular, $j_{n,*}(\beta^{\log}_{\mathrm{max},n+1})$ is equivariant for the action of Δ_n.

Proof. Let $\mathcal{J}$ be the kernel of the map $\mathcal{O}\mathbb{A}_{\log} \to \widehat{\mathcal{O}}^+_{\mathcal{X}_{\mathrm{pke}}}$. By construction of $\mathbb{A}_{\mathrm{max},n+1}$, the ideal $\mathcal{J}$ maps to $\frac{\mathcal{I}}{p}\mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1} \subset p^n\mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}$ via the map $\mathcal{O}\mathbb{A}_{\log} \longrightarrow \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}$.

The map $\alpha_{\log}$ modulo $\mathcal{J}$ coincides with the map dlog of equation (2) and thanks to Proposition 4.5, its image coincides with $\omega_E^{\mathrm{mod}} \otimes \widehat{\mathcal{O}}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})} \subset \mathrm{H}_E^\# \otimes \widehat{\mathcal{O}}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})} \subset \mathrm{H}_E \otimes \widehat{\mathcal{O}}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}$. Also, $\mathrm{Ha}^p \cdot \mathrm{H}_E \subset \mathrm{H}_E^\#$. As p/Ha^p is a section of $\mathcal{O}^+_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}$, we conclude that $p\mathrm{H}_E \subset \mathrm{H}_E^\#$. We deduce that the image of $\alpha^{\log}_{\mathrm{max},n+1}$ is contained in

$$\begin{aligned}
&\mathrm{H}_E^\# \otimes_{\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}} \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}|_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})} + \mathcal{J}\,\mathrm{H}_E \otimes_{\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}} \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}|_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})} \subset \\
&\subset \mathrm{H}_E^\# \otimes_{\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}} \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}|_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})} + p\,\mathrm{H}_E \otimes_{\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}} \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}|_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})} \subset \\
&\subset \mathrm{H}_E^\# \otimes_{\mathcal{O}^+_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}} \mathcal{O}\mathbb{A}^{\log}_{\mathrm{max},n+1}|_{\mathcal{I}g_n(p/\mathrm{Ha}^{p^s})}
\end{aligned}$$

as claimed. We also know that $\alpha_{\log}$ maps the tautological section P of $C_n^\vee$ to the marked section t of $\mathrm{H}_E^\#/p^r\mathrm{H}_E^\#$ modulo $p^r\mathcal{O}\mathbb{A}_{\log}+\mathcal{J}$ and sends $D_n^\vee$ to 0 modulo $p^r\mathcal{O}\mathbb{A}_{\log}+\mathcal{J}$. As $\mathcal{J}\subset p^n\mathcal{O}\mathbb{A}_{\max,n+1}$, the conclusion follows.

The equivariance of $j_{n,*}(\beta_{\max,n+1}^{\log})$ follows from the fact that after composing with the inclusion $\mathrm{H}_E^\#\subset j_n^*(\mathrm{H}_E)$, it coincides with $\alpha_{\max,n+1}^{\log}$ restricted to $\mathcal{I}g_n\big(p/\mathrm{Ha}^{p^s}\big)$ and $\alpha_{\max,n+1}^{\log}$ is defined over $\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)$. □

Corollary 6.9. *Applying the formalism of VBMS to $\beta_{\max,n+1}^{\log}$, we get map*

$$\delta\colon \mathbb{W}_k^D\big((T_p(E))^\vee,D_n^\vee,t\big)|_{\mathcal{I}g_n\left(p/\mathrm{Ha}^{p^s}\right)_{\mathrm{pke}}} \longrightarrow \left(\mathbf{W}_{k,\mathrm{dR}}\widehat{\otimes}_{\mathcal{O}^+_{\mathcal{I}g_n\left(p/\mathrm{Ha}^{p^s}\right)}}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}|_{\mathcal{I}g_n\left(p/\mathrm{Ha}^{p^s}\right)}\right)^{\nabla=0}$$

of sheaves over $\mathcal{I}g_n\big(p/\mathrm{Ha}^{p^s}\big)_{\mathrm{pke}}$ such that $j_{n,*}(\delta)$ is Δ_n-equivariant.

Composing with the map defined in Corollary 3.9, we get a map of sheaves on $\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)_{\mathrm{pke}}$:

$$\zeta_k\colon \mathbb{D}_k^o(T_0^\vee)[n]\longrightarrow\left(\mathbf{W}_{k,\mathrm{dR}}\widehat{\otimes}_{\mathcal{O}^+_{\mathcal{X}\left(p/\mathrm{Ha}^{p^s}\right)}}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}[p^{-1}]\right)^{\nabla=0}$$

that is functorial with respect to isogenies $E\to E'$ inducing an isomorphism on canonical subgroups.

Proof. We define δ for a log affinoid perfectoid open W of $\mathcal{I}g_n\big(p/\mathrm{Ha}^{p^s}\big)_{\mathrm{pke}}$ over an open affinoid $U=\mathrm{Spa}(R,R^+)$ of $\mathcal{I}g_n\big(p/\mathrm{Ha}^{p^s}\big)$. We assume that $T_p(E)(W)$ is trivial and that $H_E^\#(U)$ is a free of rank two R^+-module. It is easy to see that we have isomorphisms

$$\begin{aligned}&\mathbb{V}_0^D\big((T_p(E))^\vee,D_n^\vee,t\big)\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W)\\&\quad\cong\mathbb{V}_0^D\big((T_p(E)\otimes_{\mathbb{Z}_p}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W))^\vee,D_n^\vee\otimes_{\mathbb{Z}}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W),t\big)\end{aligned}$$

and that

$$\mathbb{V}_0\big(\mathrm{H}_E^\#,t\big)\widehat{\otimes}_{R^+}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W)\cong\mathbb{V}_0\big(\mathrm{H}_E^\#\widehat{\otimes}_{R^+}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W),t\big).$$

Thanks to Proposition 6.8, the map $\beta_{\max,n+1}^{\log}$ induces a map

$$\begin{aligned}&\mathbb{V}_0\big(\mathrm{H}_E^\#\widehat{\otimes}_{R^+}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W),t\big)\\&\quad\to\mathbb{V}_0^D\big((T_p(E)\otimes_{\mathbb{Z}_p}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W))^\vee,D_n^\vee\otimes_{\mathbb{Z}}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W),t\big).\end{aligned}$$

In conclusion, we get a map

$$\mathbb{V}_0\big(\mathrm{H}_E^\#,t\big)\widehat{\otimes}_{R^+}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W)\longrightarrow\mathbb{V}_0^D\big((T_p(E))^\vee,D_n^\vee,t\big)\widehat{\otimes}_{\mathbb{Z}_p}\mathcal{O}\mathbb{A}_{\max,n+1}^{\log}(W).$$

This induces the claimed map $\delta(W)$. As $j_{n,*}(\beta_{\max,n+1}^{\log})$ is Δ_n-equivariant due to Proposition 6.8, also δ is.

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The formalism of VBMS implies that these maps are functorial with respect to isogenies and base change. The connection on $\mathbf{W}_{k,\mathrm{dR}}$ is defined in [2] using Grothendieck's approach: Passing to the base-change over the first infinitesimal neighbourhood of the diagonal of $\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)$, it is realized as an isomorphism between the pullback via the two projections to $\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)$ using the functoriality of the VBMS formalism. The fact that $\beta^{\log}_{\max,\mathrm{n}+1}$ has image annihilated by ∇ implies that the image of $\delta(W)$ is also annihilated by the connection. □

Recall that $\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}^o_k(T_0^\vee)[n]\big)$ admits slope decompositions for the U_p-operator thanks to Proposition 4.13.

Lemma 6.10. *For every h, there exists m such the map*

$$\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}^o_k(T_0^\vee)[n]\big)^{(h)} \longrightarrow \mathrm{H}^1\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)_{\overline{K},\mathrm{pke}},\mathbf{W}_{k,\mathrm{dR},\bullet}\widehat{\otimes}_{\mathcal{O}^+_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}}\mathcal{O}\mathbb{A}^{\log}_{\max,\mathrm{n}+1}[p^{-1}]\big),$$

induced by ζ_k, factors via

$$\mathrm{H}^1\big(\mathcal{X}(p/\mathrm{Ha}^{p^s})_{\overline{K},\mathrm{pke}},\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}\otimes_{\mathcal{O}^+_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}}\mathcal{O}\mathbb{A}^{\log}_{\max,\mathrm{n}+1}[p^{-1}]\big).$$

Proof. Consider the complex $\mathbf{W}_{k,\mathrm{dR},\bullet}/\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}$. We claim that for $i=0$ and 1

$$\mathrm{H}^i\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)_{\overline{K},\mathrm{pke}},\mathbf{W}_{k,\mathrm{dR},\bullet}/\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}\widehat{\otimes}_{\mathcal{O}^+_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}}\mathcal{O}\mathbb{A}^{\log}_{\max,\mathrm{n}+1}[p^{-1}]\big)$$

admits a slope h-decomposition and the $\le h$-part is zero, for $m\gg 0$. The claim of the lemma then follows upon taking long exact sequences in cohomology associated to the short exact sequences $0\to\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}\to\mathbf{W}_{k,\mathrm{dR},\bullet}\to\mathbf{W}_{k,\mathrm{dR},\bullet}/\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}\to 0$ and using that the slope h-decomposition is an exact operation.

It follows from [2, Lemma 3.33] that the operator U_p on

$$\mathrm{H}^i\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big)_{\overline{K},\mathrm{pke}},\mathbf{W}_{k,\mathrm{dR},\bullet}/\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}\widehat{\otimes}_{\mathcal{O}^+_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}}\mathcal{O}\mathbb{A}^{\log}_{\max,\mathrm{n}+1}\big)$$

is integrally defined and can be written as $p^{h+1}U_p'$ for some operator U_p' for $m\gg 0$. The proof then follows from [6], Lemma 5.8 and the subsequent claim: One shows that for every polynomial P of slope $\le h$, $P(U_p)$ is invertible on this space after inverting p. □

6.4.2. Proof of Theorem 6.5. Consider the map $\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}\otimes_{\mathcal{O}^+_{\mathcal{Y}(p/\mathrm{Ha}^{p^s})}}\mathcal{O}\mathbb{A}_{\max,\mathrm{n}+1}[p^{-1}]\to\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR},\bullet}\otimes_{\mathcal{O}^+_{\mathcal{Y}(p/\mathrm{Ha}^{p^s})}}\mathrm{Fil}^0\mathcal{O}\mathbb{B}_{\mathrm{dR},\log}$ obtained from the morphism $\mathcal{O}\mathbb{A}^{\log}_{\max,n+1}\to\mathcal{O}\mathbb{B}^+_{\mathrm{dR},\log}$ of Lemma 6.2. We recall that the connection ∇' : $\mathbf{W}_{k,\mathrm{dR}}\widehat{\otimes}\mathcal{O}\mathbb{B}_{\mathrm{dR}}\longrightarrow\mathbf{W}_{k+2,\mathrm{dR}}\widehat{\otimes}\mathcal{O}\mathbb{B}_{\mathrm{dR}}$ has the form $\nabla'=\nabla_k\widehat{\otimes}1+1\widehat{\otimes}\nabla_{\mathrm{dR}}$, where ∇_k is the connection on $\mathbf{W}_{k,\mathrm{dR}}$ and ∇_{dR} the one on $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$. Moreover, both ∇_k and ∇_{dR} satisfy the Griffith-transversality property with respect to the respective filtrations of $\mathbf{W}_{k,\mathrm{dR}}$ and respectively $\mathcal{O}\mathbb{B}_{\mathrm{dR}}$, where let us recall that the first sheaf has an increasing filtration $\mathrm{Fil}_\bullet\mathbf{W}_{k,\mathrm{dR}}$ while the second has a decreasing filtration $\mathrm{Fil}^\bullet\mathcal{O}\mathbb{B}_{\mathrm{dR}}$.

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For every $s \geq 1$, we consider the composition, which we still denote ∇':

$$\mathrm{Fil}_m \mathbf{W}_{k,\mathrm{dR}} \otimes \frac{\mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}}{\mathrm{Fil}^s \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}} \xrightarrow{\nabla'} \mathrm{Fil}_{m+1} \mathbf{W}_{k+2,\mathrm{dR}} \otimes \frac{\mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}}{\mathrm{Fil}^s \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}} + \mathrm{Fil}_m \mathbf{W}_{k+2,\mathrm{dR}} \otimes \frac{\mathrm{Fil}^{-1} \mathcal{O}\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^{s-1} \mathcal{O}\mathbb{B}_{\mathrm{dR}}} \longrightarrow \mathrm{Fil}_{m+1} \mathbf{W}_{k+2,\mathrm{dR}} \otimes \frac{\mathrm{Fil}^{-1} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}}{\mathrm{Fil}^{s-1} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}},$$

and we have:

Lemma 6.11. *For every $u \geq 1$, the natural map of complexes*

$$\begin{array}{ccc}
\mathrm{H}^0\Big(\mathcal{X}\big(\frac{p}{\mathrm{Ha}^{p^{r s}}}\big)_{\overline{K}}, \mathrm{Fil}_m \mathbf{W}_{k,\mathrm{dR}}\Big) \widehat{\otimes} \frac{\mathrm{Fil}^0 B_{\mathrm{dR}}}{\mathrm{Fil}^u B_{\mathrm{dR}}} & \xrightarrow{\nabla} & \mathrm{H}^0\Big(\mathcal{X}\big(\frac{p}{\mathrm{Ha}^{p^s}}\big)_{\overline{K}}, \mathrm{Fil}_{m+1} \mathbf{W}_{k+2,\mathrm{dR}}\Big) \widehat{\otimes} \dfrac{\mathrm{Fil}^{-1} B_{\mathrm{dR}}}{\mathrm{Fil}^{u-1} B_{\mathrm{dR}}} \\
\downarrow & & \downarrow \\
\mathrm{H}^0\Big(\mathcal{X}\big(\frac{p}{\mathrm{Ha}^{p^s}}\big)_{\overline{K}}, \mathrm{Fil}_m \mathbf{W}_{k,\mathrm{dR}} \otimes \frac{\mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}}{\mathrm{Fil}^u \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}}\Big) & \xrightarrow{\nabla'} & \mathrm{H}^0\Big(\mathcal{X}\big(\frac{p}{\mathrm{Ha}^{p^s}}\big)_{\overline{K}}, \mathrm{Fil}_{m+1} \mathbf{W}_{k+2,\mathrm{dR}} \otimes \dfrac{\mathrm{Fil}^{-1} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}}{\mathrm{Fil}^{u-1} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}}\Big),
\end{array}$$

where the cohomology is taken with respect to the pro-Kummer étale topology, is an isomorphism.

In the above diagram, B_{dR} denotes Fontaine's classical period ring. Furthermore, this complex represents the cohomology $\mathrm{R}\Gamma\big(\mathcal{X}(p/\mathrm{Ha}^{p^s})_{\overline{K},\mathrm{pke}}, \mathrm{Fil}_m \mathbf{W}_{k,\mathrm{dR},\bullet} \otimes_{\mathcal{O}^+_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}} \frac{\mathrm{Fil}^0 \mathcal{O}\mathbb{B}_{\mathrm{dR}}}{\mathrm{Fil}^u \mathcal{O}\mathbb{B}_{\mathrm{dR}}}\big)$.

Proof. Recall that $\mathrm{Fil}_m \mathbf{W}_{k,\mathrm{dR}}[1/p]$ is a locally free $\mathcal{O}_{\mathcal{X}(p/\mathrm{Ha}^{p^s})}$-module for every m. We prove the result restricting to an affinoid cover $\{\mathcal{U}_i\}_{i\in I}$, where Fil_m and Fil_{m+1} are free. Since $\mathcal{X}(p/\mathrm{Ha}^{p^s})$ is affinoid, the Chech complex for $\mathrm{Fil}_m \mathbf{W}_{k,\mathrm{dR}}[1/p]$ w.r.t. the $\mathcal{U}_i$'s is exact. As $\mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^u B_{\mathrm{dR}}$ is an iterated extension of $\mathbb{C}_p$-vector spaces for every h, the Chech complex remains exact also after taking $\widehat{\otimes}\mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^u B_{\mathrm{dR}}$. From the result for the $\mathcal{U}_i$'s, we then deduce the lemma.

We are left to show the claim for each $\mathcal{U}_i = \mathrm{Spa}(R_i, R_i^+)$. Then [16, Lemma 3.3.2] implies that the group $\mathrm{H}^j\big(\mathcal{U}_{i,\overline{K},\mathrm{pke}}, \mathrm{Fil}^n \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}/\mathrm{Fil}^{n+u} \mathcal{O}\mathbb{B}_{\mathrm{dR,log}}\big) = 0$ for $j \geq 1$ and coincides with $\mathrm{Fil}^n\big(R_i^+ \widehat{\otimes} B_{\mathrm{dR}}\big)/\mathrm{Fil}^{u+n}\big(R_i^+ \widehat{\otimes} B_{\mathrm{dR}}\big)$ for $i = 0$, for $n = 0, -1$. As the latter coincides with $R_i^+ \widehat{\otimes}\big(\mathrm{Fil}^n B_{\mathrm{dR}}/\mathrm{Fil}^{u+n} B_{\mathrm{dR}}\big)$ by [16, Def. 3.1.1], the conclusion follows. □

We deduce from Lemma 6.10 and Lemma 6.11 that for every positive integer u we have a natural map

$$\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}}, \mathbb{D}^o_k(T_0^\vee)[n]\big)^{(h)} \widehat{\otimes} \frac{\mathrm{Fil}^0 B_{\mathrm{dR}}}{\mathrm{Fil}^u B_{\mathrm{dR}}} \to \frac{\mathrm{H}^0\big(\mathcal{X}(p/\mathrm{Ha}^{p^s}), \mathrm{Fil}_{m+1} \mathbf{W}_{k+2,\mathrm{dR}}\big) \widehat{\otimes} \big(\mathrm{Fil}^{-1} B_{\mathrm{dR}}/\mathrm{Fil}^{u-1} B_{\mathrm{dR}}\big)}{\nabla_k\big(\mathrm{H}^0\big(\mathcal{X}(p/\mathrm{Ha}^{p^s}), \mathrm{Fil}_m \mathbf{W}_{k,\mathrm{dR}}\big)\big) \widehat{\otimes} \big(\mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^u B_{\mathrm{dR}}\big)}.$$

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Recall from Proposition 6.4 that $\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathrm{Fil}_m\mathbf{W}_{k,\mathrm{dR}}[1/p]\big)$ admits a slope $\leq h-1$ decomposition and $\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathrm{Fil}_{m+1}\mathbf{W}_{k+2,\mathrm{dR}}[1/p]\big)$ admits a slope $\leq h$ decomposition and the slope $\leq h-1$ part, resp. $\leq h$ part, coincides with $\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k,\mathrm{dR}}[1/p]\big)^{(h-1)}$, resp. $\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k+2,\mathrm{dR}}[1/p]\big)^{(h)}$.

The same then holds after $\widehat{\otimes}\mathrm{Fil}^0 B_{\mathrm{dR}}/\mathrm{Fil}^u B_{\mathrm{dR}}$, respectively $\widehat{\otimes}\mathrm{Fil}^{-1},B_{\mathrm{dR}}/\mathrm{Fil}^{u-1}B_{\mathrm{dR}}$, and for their quotient via ∇ (by the five lemma for slope decompositions cf. [6, Thm. 5.7]). As $\mathrm{Fil}^0 B_{\mathrm{dR}} = B_{\mathrm{dR}}^+$, $\mathrm{Fil}^{-1}B_{\mathrm{dR}} = t^{-1}B_{\mathrm{dR}}^+$ and $\mathrm{Fil}^{n+u}B_{\mathrm{dR}} = t^{n+u}B_{\mathrm{dR}}^+$ for $n\in\{-1,0\}$, we get maps

$$\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}_k^o(T_0^\vee)[n]\big)^{(h)}\widehat{\otimes}\frac{B_{\mathrm{dR}}^+}{t^uB_{\mathrm{dR}}^+}\longrightarrow \mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k,\mathrm{dR},\bullet}[1/p]\big)^{(h)}\widehat{\otimes}\frac{t^{-1}B_{\mathrm{dR}}^+}{t^{u-1}B_{\mathrm{dR}}^+}$$

with

$$\mathrm{H}^1_{\mathrm{dR}}\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k,\mathrm{dR},\bullet}[1/p]\big)^{(h)}\cong\frac{\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k+2,\mathrm{dR}}[1/p]\big)^{(h)}}{\nabla_k\Big(\mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\mathbf{W}_{k,\mathrm{dR}}[1/p]\big)^{(h-1)}\Big)}$$

by [2, Lemma 3.33 & Eq. (6)]. As they are obtained from maps of sheaves on $\mathcal{X}_{\mathrm{pke}}$, the equivariance for the $\mathrm{Gal}(\overline{K}/K)$-action is clear. The compatibility with Hecke operators follows from the fact that ζ_k is compatible with the map induced by isogenies preserving the canonical subgroup that are used to define the Hecke operators T_ℓ, for $\ell \not| pN$, and the Hecke operator U_p. It is compatible with weight specializations as the map ζ_k is. Taking the inverse limits for $u\to\infty$, we get the statement of Theorem 6.5, except for (i) and (ii). Claim (ii) follows from [2, section §3.9]. Using (ii), we get a map

$$\mathrm{H}^1\big(\mathcal{X}_{\overline{K},\mathrm{pke}},\mathbb{D}_k^o(T_0^\vee)[n]\big)^{(h)}\widehat{\otimes}\frac{B_{\mathrm{dR}}^+}{t^uB_{\mathrm{dR}}^+}\longrightarrow \mathrm{H}^0\big(\mathcal{X}\big(p/\mathrm{Ha}^{p^s}\big),\omega_E^k\big)^{(h)}\widehat{\otimes}\frac{t^{-1}B_{\mathrm{dR}}^+}{t^{u-1}B_{\mathrm{dR}}^+},$$

which we'd like to prove is surjective under the hypothesis of i). By devissage it suffices to prove surjectivity for $u=1$. As $t^{-1}B_{\mathrm{dR}}^+/B_{\mathrm{dR}}^+\cong\mathbb{C}_p(-1)$, the surjectivity follows from Theorem 5.1.

7. Appendix: Integral slope decomposition

Let us start by formulating the following general property.

We let R be a p-torsion-free $\mathbb{Z}_p$-algebra and T an R-module equipped with an R-linear operator $v: T\to T$ and let $\alpha\in R$ be an element such that there is $r\in\mathbb{N}$ and $\gamma\in R$ with $\alpha\gamma = p^r$. We denote by $\rho\colon T\longrightarrow T[1/p] := T\otimes_R R[1/p]$ and denote by $T^{\mathrm{tors}} := \mathrm{Ker}(\rho)$, $T^{\mathrm{tf}} := T/T^{\mathrm{tors}} = \mathrm{Im}(\rho)$ and remark that $v-\alpha$ respects the submodule T^{tors} and therefore induces an R-linear map on T^{tf}.

Definition 7.1. We say that the triple (T,v,α) has property $(*)$ if

1) There is $w\in\mathbb{N}$ such that $p^w\big(T^{\mathrm{tors}}\big)^{v=\alpha}=0$.
2) There is a $\eta\in\mathbb{N}$, which depends only on α, such that for every $x\in\big(T^{\mathrm{tf}}\big)^{v=\alpha}$ there is $\tilde{x}\in T^{v=\alpha}$ such that $(\tilde{x})^{\mathrm{tf}} = p^\eta x$, where we denoted $(\tilde{x})^{\mathrm{tf}}$ the image of $\tilde{x}$ in T^{tf}.

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The main result of this appendix is the following. Let B denote a ring, and let $k\colon \mathbb{Z}_p^* \longrightarrow B^*$ be a B-valued weight as in Definition 3.6, which is s-analytic. Let $\mathcal{X}_\infty^{(u)}$ be the adic subspace of the adic modular curve $\mathcal{X} := \mathcal{X}_0(p^n,N)$ for $n \geq u$ as in Proposition 4.5. Let $\mathbb{D}_k(T_0^\vee)[n]$ denote the pro-Kummer étale sheaf of weight k distributions, for $n \geq s$, over $\mathcal{X}_\infty^{(u)}$ and $\mathfrak{D}_{k,\infty}^{o,(m)}[n] := \mathbb{D}_k(T_0^\vee)[n]\widehat{\otimes}\mathcal{O}^+_{(\mathcal{X}_\infty^{(u)})_{\rm pke}}$, where we have denoted $\mathcal{O}^+_{(\mathcal{X}_\infty^{(u)})_{\rm pke}}$ the structure sheaf of the pro-Kummer étale site of $\mathcal{X}_\infty^{(u)}$. We write $R := \mathcal{O}^+_{\mathcal{X}_\infty^{(u)}}(\mathcal{X}_\infty^{(u)})\widehat{\otimes}_{\mathbb{Z}_p}B$ and $T := \mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\rm pke},\mathfrak{D}_{k,\infty}^{o,(m)}[n]\big)$. On $T[1/p]$, we have a $B[1/p]$-linear operator U_p which has finite slope decompositions by Proposition 4.15.

Let $Q(X) \in \big(B\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}\big)[X]$ be the polynomial with the property that $T[1/p]^{(b)}$, for some $b \in \mathbb{N}$ is the subset of elements $x \in T[1/p]$ such that $Q(U_p)x = 0$. Such a polynomial exists as $T[1/p]^{(b)} \cong \mathrm{H}^0\big(\mathcal{X}_\infty^{(u)},\omega_E^{k+2}\big)[1/p]^{(b)}$ by Proposition 4.15, and on $\mathrm{H}^0\big(\mathcal{X}_\infty^{(u)},\omega_E^{k+2}\big)$ the U_p operator is compact and has a Fredholm determinant. Then $\alpha := -Q(0) \in p^a(B\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p})^*$ with $a \leq b\cdot\deg(Q(X))$. We write $Q(X) = P(X) - \alpha$, with $P(X) = XR(X)$ and $P(X),R(X) \in \big(B\widehat{\otimes}\mathcal{O}_{\mathbb{C}_p}\big)[X]$. We denote $v := P(U_p)$ and remark that $x \in T[1/p]^{(b)}$ if and only if $v(x) = \alpha x$. We have

Theorem 7.2. *After localizing B to a new p-adically complete ring which we denote by B' and replacing R by $R' := \mathcal{O}^+_{\mathcal{X}_\infty^{(u)}}(\mathcal{X}_\infty^{(u)})\widehat{\otimes}_{\mathbb{Z}_p}B'$ and T by $T' := T\otimes_R R'$ the triple $(T',v\otimes 1_{R'},\alpha)$ above satisfies property $(*)$ of definition 7.1.*

Before we start on the proof of this theorem, we need a few lemmas. We remind the reader that the sheaf $\mathfrak{D}_{k,\infty}^{o,(m)}[n]$ has a decreasing filtration $\big(\mathfrak{Fil}^\nu\big)_{\nu\geq 0}$ by subsheaves with the property (see Proposition 4.15): For all $\nu \geq 0$ and $i \geq 0$ we have

$$U_p\Big(\mathrm{H}^i\big((\mathcal{X}_\infty^{(u)})_{\rm pke},\mathfrak{Fil}^\nu\big)\Big) \subset p^{\nu+1}\mathrm{H}^i\big((\mathcal{X}_\infty^{(u)})_{\rm pke},\mathfrak{Fil}^\nu\big).$$

Moreover, we have $\mathfrak{Fil}^\nu/\mathfrak{Fil}^{\nu+1} \cong \omega_E^{k-2\nu-2}\widehat{\otimes}\mathcal{O}_{\mathcal{X}_\infty^{(u)}}(\nu+1)$ by equation (5). With these notations we have:

Lemma 7.3. *For every $\nu \in \mathbb{N}$ large enough and $i \geq 0$ the triple $\Big(T_i^\nu := \mathrm{H}^i\big((\mathcal{X}_\infty^{(u)})_{\rm pke},\mathfrak{Fil}^\nu\big),$ $v = P(U_p),\alpha\Big)$ satisfies property $(*)$ of definition 7.1.*

Proof. We have the following commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \big(T_i^\nu\big)^{\rm tors} & \longrightarrow & T_i^\nu & \longrightarrow & \big(T_i^\nu\big)^{,\rm tf} & \longrightarrow & 0 \\
 & & \downarrow v-\alpha & & \downarrow v-\alpha & & \downarrow v-\alpha & & \\
0 & \longrightarrow & \big(T_i^\nu\big)^{\rm tors} & \longrightarrow & T_i^\nu & \longrightarrow & \big(T_i^\nu\big)^{,\rm tf} & \longrightarrow & 0.
\end{array}$$

We remark that the above property about the behavior of U_p with respect to the cohomology of the filtration and the fact that v is the composition of U_p, with an endomorphism of T_i^ν which commutes with U_p, implies that there is ν_0 such that for all $\nu \geq \nu_0$ we have $(v - \alpha\cdot\mathrm{Id}_{T_i^\nu}) = \alpha U_i'$, where $U_i' : T_i^\nu \longrightarrow T_i^\nu$ is an isomorphism. Therefore, $\big(T_i^\nu)\big)^{v=\alpha} = T_i^\nu[\alpha] = \big(T_i^\nu\big)^{\rm tors}[\alpha]$ and $\big(T_i^\nu\big)^{\rm tf}[\alpha] = 0$. Therefore, 1) of property $(*)$ follows:

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$\alpha\Big(\big(T_i^\nu\big)^{\rm tors}\Big)^{v=\alpha} = \alpha(T_i^\nu[\alpha]) = 0$. For property 2), let $x \in \Big(\big(T_i^\nu\big)^{\rm tf}\Big)^{v=\alpha} = 0$. For every $y \in \big(T_i^\nu\big)^{v=\alpha} = T_i^\nu[\alpha]$, we have $\alpha y = 0 = x$. □

Lemma 7.4. *For every $\nu \in \mathbb{N}$, the triple $\Big(T_\nu := \mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\rm pke}, \mathfrak{D}_k^o/\mathfrak{Fil}^\nu\big), u = P(U_p), \alpha\Big)$ satisfies the property $(*)$ of Definition 7.1.*

Proof. In order to prove the lemma, we'll use induction on $\nu \geq 0$. For $\nu = 0$, we have $\mathfrak{D}_k/\mathfrak{Fil}^0 \cong \omega_E^k \widehat{\otimes} \mathcal{O}^+_{\mathcal{X}_\infty^{(u)}}$; therefore, we have:

1) $p^{1/(p-1)}(T_0)^{\rm tors} = p^{1/(p-1)}\mathrm{H}^1\big((\mathcal{X}_\infty^{(u)})_{\rm pke}, \widehat{\mathcal{O}}^+_{\mathcal{X}_\infty^{(u)}}\big)^{\rm tors} \widehat{\otimes} \mathrm{H}^0(\mathcal{X}_\infty^{(u)}, \omega_E^k) = 0$ as computed by Faltings; see [22, Lemma 5.5 & 5.6] when the log structure is trivial and their analogues [15, Lemma 6.1.7 & 6.1.11] in the general case;

and

2) If $x \in (T_0)^{\rm tf}$ is such that $(v-\alpha)x = 0$, let $y \in T_0$ be any lift of x. Then $(v-\alpha)(y) \in T_0^{\rm tors}$; therefore, $z := p^{1/(p-1)}y \in T_0$ satisfies: $(v-\alpha)(z) = 0$ and $z^{\rm tf} = p^{1/(p-1)}x$, where we wrote $z^{\rm tf}$ for the image of z in $T_0^{\rm tf}$. Let us observe that we proved more then 1) of property $(*)$; namely, we showed that there is $r \geq 0$ such that $p^r T_0^{\rm tors} = 0$. We'll prove the same property, call it $(**)$ for all $\nu \geq 0$.

Suppose now that $(**)$ is true for T_ν, $\nu \geq 1$, and let us prove it for $T_{\nu+1}$. We have an exact sequence of pro-Kumer étale sheaves on the site $\mathcal{V} := (\mathcal{X}_\infty^{(u)})_{\rm pke}$:

$$0 \longrightarrow \mathfrak{Fil}^\nu/\mathfrak{Fil}^{nu+1} \longrightarrow \mathfrak{D}_k^o/\mathfrak{Fil}^{\nu+1} \longrightarrow \mathfrak{D}_k^o/\mathfrak{Fil}^\nu \longrightarrow 0,$$

therefore a long exact cohomology sequence:

$$\mathcal{A} := \mathrm{H}^0(\mathcal{V}, \mathfrak{D}_k/\mathfrak{Fil}^\nu) \xrightarrow{\beta} \mathcal{B} := \mathrm{H}^1(\mathcal{V}, \mathfrak{Fil}^\nu/\mathfrak{Fil}^{\nu+1}) \xrightarrow{\gamma}$$
$$\mathcal{C} := \mathrm{H}^1(\mathcal{V}, \mathfrak{D}/\mathfrak{Fil}^{\nu+1}) \xrightarrow{\delta} \mathcal{D} := \mathrm{H}^1(\mathcal{V}, \mathfrak{D}/\mathfrak{Fil}^\nu) \to 0.$$

By the induction hypothesis, there is r such that $p^r\mathcal{A}^{\rm tors} = p^r\mathcal{B}^{\rm tors} = p^r\mathcal{D}^{\rm tors} = 0$. We have the commutative diagram with the middle row exact:

$$(1)\quad \begin{array}{ccccccccc} \mathcal{A}^{\rm tors} & \xrightarrow{\beta} & \mathcal{B}^{\rm tors} & \xrightarrow{\gamma} & \mathcal{C}^{\rm tors} & \xrightarrow{\delta} & \mathcal{D}^{\rm tors} & & \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{A} & \xrightarrow{\beta} & \mathcal{B} & \xrightarrow{\gamma} & \mathcal{C} & \xrightarrow{\delta} & \mathcal{D} & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\ \mathcal{A}^{\rm tf} & \xrightarrow{\beta} & \mathcal{B}^{\rm tf} & \xrightarrow{\gamma} & \mathcal{C}^{\rm tf} & \xrightarrow{\delta} & \mathcal{D}^{\rm tf}. & & \end{array}$$

Let us suppose that there is $s \geq 0$ such that $p^s\Big(\mathcal{B}^{\rm tf}/\beta(\mathcal{A}^{\rm tf})\Big)^{\rm tors} = 0$. Then we claim that $p^{2r+s}\mathcal{C}^{\rm tors} = 0$. To see it, let $c \in \mathcal{C}^{\rm tors}$, then $p^r x = \gamma(y)$, $y \in B$. We denote by $[y]$ the image of y in $\mathcal{B}^{\rm tf}/\beta(\mathcal{A}^{\rm tf})$. As $p^N x = 0$ for some $N \geq 0$ we have that $p^N[y] = 0$, therefore $[y] \in \Big(\mathcal{B}^{\rm tf}/\beta(\mathcal{A}^{\rm tf})\Big)^{\rm tors}$ and so by the above assumption $p^s[y] = 0$. Let $z \in \mathcal{A}$ be such that $\beta(z^{\rm tf}) = p^s y^{\rm tf}$ in $\mathcal{B}^{\rm tf}$. It follows that $\beta(z) - p^s y \in \mathcal{B}^{\rm tors}$ which implies that $p^{r+s}y = p^r\beta(z)$. Therefore, $p^{s+2r}x = 0$.

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Let us now prove the remaining claim, namely that there is $s \geq 0$ such that $p^s\left(\mathcal{B}^{\mathrm{tf}}/\beta(\mathcal{A}^{\mathrm{tf}})\right)^{\mathrm{tors}} = 0$. For this, let us recall [6] that we have a commutative diagram with exact rows

$$\begin{array}{ccccccc}
\mathrm{H}^0(\mathcal{X}_\infty^{(u)}, \omega_E^{k-2\nu}) & \xrightarrow{\tilde{\beta}} & \mathrm{H}^0(\mathcal{X}_\infty^{(u)}, \omega_E^{k-2\nu}) & \longrightarrow & \mathrm{Coker}(\tilde{\beta}) & \longrightarrow & 0 \\
\downarrow i & & \downarrow j & & \downarrow u & & \\
A^{\mathrm{tf}} & \xrightarrow{\beta} & \mathcal{B}^{\mathrm{tf}} & \longrightarrow & \mathcal{B}^{\mathrm{tf}}/\beta(\mathcal{A}^{\mathrm{tf}}) & \longrightarrow & 0,
\end{array}$$

where i and j are injective with cokernels killed by $p^{\nu/(p-1)}$ and $p^{1/(p-1)}$, respectively. Moreover, it follows using the explicit basis of the filtration described in Proposition 4.14 and [6, Prop. 5.2] that $\tilde{\beta} = \left(\prod_{n=0}^{\nu}(u_k - n)\right)\beta'$, with β' an isomorphism. As $p^{1/(p-1)}$ kills $\mathrm{Coker}(j)$, it also kills $\mathrm{Coker}(u)$. Therefore, it is enough to prove the claim for $\mathrm{Coker}(\tilde{\beta})^{\mathrm{tors}}$. We have two possibilities. Either $u_k = n$ for some $0 \leq n \leq \nu$ and then $\mathcal{B}^{\mathrm{tf}}/\beta(\mathcal{A}^{\mathrm{tf}}) = \mathcal{B}^{\mathrm{tf}}$ so that the claim is obvious. Else $\prod_{n=0}^{\nu}(u_k - n) \in (B[1/p])^*$ due to our assumption on B in Definition 3.6, that is, there exists $s \in \mathbb{N}$ and $v \in B$ such that $\prod_{n=0}^{\nu}(u_k - n)\cdot v = p^s$. Then $\tilde{\beta}$ is injective with the cokernel annihilated by p^s and the claim is proven also in this case.

So we have proved that the property $(**)$ holds for triples $(T_\nu, v = P(U_p), \alpha)$ for all $\nu \geq 0$, which implies 1) of property $(*)$.

Let us prove 2) of property $(*)$ for $\mathcal{C}$, supposing that it holds for $\mathcal{D}$. We recall our diagram (1) and let $x \in \mathcal{C}^{\mathrm{tf}}$ be such that $(v-\alpha)(x) = 0$. Then $\delta(x) \in \mathcal{D}^{\mathrm{tf}}$ is such that $(v-\alpha)(\delta(x)) = 0$; therefore, by the induction hypothesis there is $m \geq 0$ and $y \in \mathcal{D}^{v=\alpha}$ such that $y^{\mathrm{tf}} = p^m\delta(x)$. Let $z \in \mathcal{C}$ be such that $\delta(z) = y$ and let $\tilde{x} \in \mathcal{C}$ be such that $\tilde{x}^{\mathrm{tf}} = x$. Then there is $q \in \mathcal{B}$ such that $\gamma(p^m q) - z + p^m\tilde{x} \in \mathcal{C}^{\mathrm{tors}}$ and so $p^m z - p^{m+r}\gamma(q) = p^{m+r}\tilde{x}$. Let $t = p^m z - p^{m+r}\gamma(q) \in \mathcal{C}$. It has the property that $(v-\alpha)(t) = (v-\alpha)p^{m+r}(\tilde{x})$ and so $(v-\alpha)(t)^{\mathrm{tf}} = 0$, that is, $(v-\alpha)(t) \in \mathcal{C}^{\mathrm{tors}}$. Therefore, $(v-\alpha)(p^r t) = 0$ and $(p^r t)^{\mathrm{tf}} = p^{m+2r}x$. □

Proof. (of Theorem 7.2). Let $\nu \in \mathbb{N}$ be large enough so that Lemma 7.3 is satisfied for the triple $\left(\mathrm{H}^1\left((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, \mathfrak{Fil}^\nu\right), v = P(U_p), \alpha\right)$, and consider the exact sequence of sheaves on the pro-Kummer étale site $\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}$:

$$0 \longrightarrow \mathfrak{Fil}^\nu \longrightarrow \mathfrak{D}_k \longrightarrow \mathfrak{D}_k/\mathfrak{Fil}^\nu \longrightarrow 0,$$

where we use the notations introduced before Lemma 7.4. It induces the long exact sequence of pro-Kummer étale cohomology groups which, in order to simplify notations, we write $\mathrm{H}^\bullet(F)$ instead of $\mathrm{H}^\bullet\left((\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}, F\right)$, where F is a sheaf on $(\mathcal{X}_\infty^{(u)})_{\mathrm{pke}}$.

$$\begin{array}{ccccc}
\mathcal{A} = \mathrm{H}^0(\mathfrak{D}_k/\mathfrak{Fil}^\nu) & \xrightarrow{\beta} & \mathcal{B} = \mathrm{H}^1(\mathfrak{Fil}^\nu) & \xrightarrow{\gamma} & \mathcal{C} = \mathrm{H}^1(\mathfrak{D}_k) \\
& \xrightarrow{\delta} & \mathcal{D} = \mathrm{H}^1(\mathfrak{D}_k/\mathfrak{Fil}^\nu) & \xrightarrow{\epsilon} & \mathcal{E} = \mathrm{H}^2(\mathfrak{Fil}^\nu).
\end{array}$$

We'd like to show the v-module $\mathcal{C}$ satisfies the property $(*)$.

1) Let $x \in \mathcal{C}^{v=\alpha}$ such that x is a p-power torsion element. Then $\delta(x) \in \mathcal{D}^{v=\alpha}$ is a p-power torsion element, and let $s = s(\mathcal{D}) \in \mathbb{N}$ be such that $p^s\delta(x) = 0$. Then let $y \in \mathcal{B}$ be such

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that $\gamma(y)=p^s x$. We also have $\gamma\big((v-\alpha)y\big)=(v-\alpha)\gamma(y)=(v-\alpha)(p^s x)=0$. Therefore, let $z\in\mathcal{A}$ be such that $\beta(z)=(v-\alpha)(y)$.

At this point, we recall the following result from [21], Proposition 12, and [13], Propositions A.4.2 and A.4.3: As $\mathcal{A}[1/p]$ is a finitely generated, free $R[1/p]$-module and U_p is completely continuous on it, after localizing B to B' and replacing R by R', there are $d=d(\mathcal{A},\alpha), e=e(\mathcal{A},\alpha)\in\mathbb{N}$ and $e_\alpha=p^d P_\alpha(U_p)$, with $p^d P_\alpha(T)$ a series with integral coefficients such that for every $z\in\mathcal{A}$ we denote by $z_\alpha:=e_\alpha z\in\mathcal{A}$ and by $z^\perp:=p^d z-z_\alpha\in\mathcal{A}$. Then $(v-\alpha)(z_\alpha)=0$. Moreover there is a $w_\alpha\in\mathcal{A}$ with $(v-\alpha)(w_\alpha)=p^e z_\alpha^\perp$.

Let now e,d be as above, then $p^{d+e}z=p^e z_\alpha+(v-\alpha)(w_\alpha)$. Therefore, we have:

$$(v-\alpha)\big(p^{d+e}y-\beta(w_\alpha)\big)=p^e\beta(z_\alpha),\ \text{and so we have}\ (v-\alpha)^2\big(p^{d+e}y-\beta(w_\alpha)\big)=0.$$

If we set $m:=p^{d+e}y-\beta(w_\alpha)$, we have $(v-\alpha)^2(m)=0$ and $\gamma(m)=p^{d+e}\gamma(y)=p^{d+e+s}x$. As $(v-\alpha)^2 m=0$, we have $\alpha^2 m=0$; therefore, $\alpha^2 p^{d+e+s}x=0$. This concludes 1) of property $(*)$ for $\mathcal{C}$.

2) Let $x\in\big(\mathcal{C}^{\mathrm{tf}}\big)^{v=\alpha}$. Then there is $r:=r(\mathcal{D},\alpha)$ and $y\in\mathcal{D}^{v=\alpha}$ such that $y^{\mathrm{tf}}=p^r\delta(x)$, where we denoted y^{tf} the image of y in $\mathcal{D}^{\mathrm{tf}}$. The image $\epsilon(y)\in\mathcal{E}^{v=\alpha}$ is annihilated by α, and therefore, there is $z\in\mathcal{C}$ such that $\delta(z)=\alpha y$. There is $w\in\mathcal{D}$ such that $(v-\alpha)(z)=\gamma(w)$ and let $q\in\mathcal{B}$ be such that $\alpha w=(v-\alpha)(q)$. Therefore, $(v-\alpha)(\alpha z-\gamma(q))=0$. Let $\tilde{x}:=\alpha z-\gamma(q)\in\mathcal{C}^{v=\alpha}$. Then $\delta(\tilde{x})=\delta(\alpha z)=\alpha^2 y$. The image $\alpha^2 y^{\mathrm{tf}}$ of $\alpha^2 y$ in $(\mathcal{D}[1/p])^{v=\alpha}$ is $\alpha^2 p^r\delta(x)$. But δ induces an isomorphism $\big(\mathcal{C}[1/p]\big)^{v=\alpha}\overset{\delta}{\cong}\big(\mathcal{D}[1/p]\big)^{v=\alpha}$ (see [6]). Therefore, the image $(\tilde{x})^{\mathrm{tf}}$ of $\tilde{x}$ in $\big(\mathcal{C}[1/p]\big)^{v=\alpha}$ is $\alpha^2 p^r x$ which proves the claim. □

References

[1] F. Andreatta and A. Iovita, 'Semistable sheaves and comparison isomorphisms in the semistable case', *Rend. Semin. Mat. Univ. Padova* **128** (2012), 131–285.

[2] F. Andreatta and A. Iovita, 'Triple product p-adic L-functions associated to a triple of finite slope p-adic families of modular forms', *Duke Math. J.* **170** (2021), 1989–2083.

[3] F. Andreatta, A. Iovita and V. Pilloni, 'p-adic families of Siegel modular cuspforms', *Ann. of Math.* **181** (2015), 623–697.

[4] F. Andreatta, A. Iovita and V. Pilloni, 'p-adic variation of automorphic sheaves', in *Proceedings of the International Congress of Mathematicians—Rio de Janeiro 2018, Invited Lectures,* vol. II, (World Sci. Publ., Hackensack, NJ, 2018), 249–276.

[5] F. Andreatta, A. Iovita and G. Stevens, 'Overconvergent Eichler–Shimura isomorphisms', *Journal of the Institute of Mathematics of Jussieu* **14** (2015), 221–274.

[6] F. Andreatta, A. Iovita and G. Stevens, 'A 0.5 overconvergent Eichler–Shimura isomorphism', *Annales mathématiques du Québec* **40** (2016), 121–148.

[7] D. Barrera and S. Gao, 'Overconvergent Eichler–Shimura isomorphisms for quaternionic modular forms over $\mathbb{Q}$', *International Journal of Number Theory* **13** (2017), 2487–2504.

[8] B. Bhatt, M. Morrow and P. Scholze, 'Integral p-adic Hodge theory', Preprint, 2019, arXiv:1602.03148v3.

[9] B. Bhatt and P. Scholze, '*Prisms and prismatic cohomology*', Preprint, 2019.

[10] A. Caraiani and P. Scholze, 'On the generic part of the cohomology of compact unitary Shimura varieties', *Ann. of Math.* **186** (2017), 649–766.

[11] K. Cesnavicius and T. Koshikawa, 'The A_{inf}-cohomology in the semi-stable case', *Comp. Math.* **155**, (11) (2019), 2039–2128.

[12] P. Chojecki, D. Hansen and C. Johansson, 'Overconvergent modular forms and perfectoid Shimura curves', *Documenta Math.* **22** (2017), 191–262.

[13] R. Coleman, 'p-adic Banach spaces and families of modular forms', *Invent. Math.* **127** (1997), 417–479.

[14] P. Colmez, G. Dospinescu and W. Niziol, '*Intregral étale cohomology of Drinfeld symmetric spaces*', Preprint, 2019.

[15] H. Diao, K.-W. Lan, R. Liu and X. Zhu, 'Logarithmic adic spaces: some foundational results', Preprint, 2019, arXiv:1912.09836.

[16] H. Diao, K.-W. Lan, R. Liu and X. Zhu, '*Logarithmic Riemann–Hilbert correspondences for rigid varieties*', Preprint, 2019, arXiv:1803.05786. To appear in J. Amer. Math. Soc.

[17] R. Huber, 'Étale cohomology of rigid analytic varieties and adic spaces', *Aspects of Mathematics* **30** (2014).

[18] L. Illusie, 'An overview of the work of K. Fujiwara, K. Kato and C. Nakamura on logarithmic étale cohomology', *Astérisque* **279** (2002), 271–322.

[19] U. Jannsen, 'Continuous étale cohomology', *Math. Ann.* **280** (1988), 207–245.

[20] V. Pilloni and B. Stroh, 'Cohomologie cohérente et représentations Galoisiennes', *Annales mathématiques du Québec* **40** (2016), 167–202.

[21] J.-P. Serre, 'Endomorphismes complètement continus des espaces de Banach p-adiques', *Inst. Hautes Études Sci. Publ. Math.* **12** (1962), 69–85.

[22] P. Scholze, 'p-adic Hodge thery for rigid analytic varieties', *Forum Math. Π* **1** (2013), 77.

[23] P. Scholze, 'On torsion in the cohomology of locally symmetric varieties', *Ann. of Math.* **182** (2015), 945–1066.

[24] P. Scholze and J. Weinstein, 'Moduli of p-divisible groups', *Camb. J. Math.* **1** (2013), 145–237.

[25] F. Tan and J. Tong 'Crystalline comparison isomorphisms in p-adic Hodge theory: The absolutely unramified case', *Algebra & Number Theory* **13** (2019), 1509–1581.

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