ON A FACTORISATION OF POSITIVE DEFINITE MATRICES

Kulendra N. Majindar

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All our matrices are square with real elements. The Schur product of two $n \times n$ matrices $B = (b_{ij})$ and $C = (c_{ij})$ $(i, j, = 1, 2, \ldots, n)$, is an $n \times n$ matrix $A = (a_{ij})$ with $a_{ij} = b_{ij} c_{ij}$, $(i, j = 1, 2, \ldots, n)$.

A result due to Schur [1] states that if B and C are symmetric positive definite matrices then so is their Schur product A. A question now arises. Can any symmetric positive definite matrix be expressed as a Schur product of two symmetric positive definite matrices? The answer is in the affirmative as we show in the following theorem.

THEOREM. A real symmetric positive definite matrix is a Schur product of two real symmetric positive definite matrices.

Proof. Let us first prove the theorem for a real symmetric positive definite matrix $A_1 = (a'_{ij})$ where $a_{ii} = 1$, $(i = 1, 2, \ldots, n)$. For each i, $(i = 1, 2, \ldots, n)$, let the i characteristic roots of the leading $i \times i$ principal submatrix of A_1 (i.e. the submatrix occupying the upper left hand corner of A_1) be $\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{ii}$ with $\lambda_{1i} \geq \lambda_{2i} \geq \ldots \geq \lambda_{ii} > 0$. Write $\lambda = \min_i \lambda_{ii}$. Choose α to be a positive number satisfying $1 - \lambda < \alpha < 1$. If B is an $n \times n$ matrix with 1 in its main diagonal and α elsewhere, and $C = (c_{ij})$ where $c_{ij} = a'_{ij}/\alpha$, $i \neq j$, $c_{ii} = 1$, $(i, j = 1, 2, \ldots, n)$, then the Schur

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product of B_1 and C is A_1.

Now we have to show that the matrices B_1 and C are positive definite. Indeed the i-th leading principal minor is easily seen (see [2]) to be equal to $(1-\alpha)^{i-1}(1+i\alpha-\alpha)$, $(i = 1, 2, \ldots, n)$. So B_1 is positive definite. With regard to C, let us consider the following polynomial of degree i in x:

$$P_i(x) = \begin{vmatrix} x & a'_{12} & a'_{13} & \ldots & a'_{1i} \\ a'_{21} & x & a'_{23} & \ldots & a'_{2i} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a'_{i1} & a'_{i2} & a'_{i3} & \ldots & x \end{vmatrix}.$$

According to the definition of $\lambda_{1i}, \lambda_{2i}, \ldots, \lambda_{ii}$, we have $P_i(1-\lambda_{1i}) = P_i(1-\lambda_{2i}) = \ldots = P_i(1-\lambda_{ii}) = 0$, and $P_i(x) > 0$ whenever $x > 1 - \lambda_{ii}$. The leading principal minor of order i of C is equal to $\alpha^{-i} P_i(\alpha)$. As $\alpha > 0$ and $\alpha > 1 - \lambda_{ii}$, we see that this leading principal minor is positive. This holds for $i = 1, 2, \ldots, n$. Thus C is a symmetric positive definite matrix. Hence we have got a desired factorisation of A.

Let now $A = (a_{ij})$, $(i, j = 1, 2, \ldots, n)$, be a symmetric positive definite matrix in which not all the main diagonal elements are 1. We know that $a_{ii} > 0$ for $i = 1, 2, \ldots, n$.

Put $a'_{ij} = a_{ij}/(a_{ii}a_{jj})^{\frac{1}{2}}$, $(i, j = 1, 2, \ldots, n)$. Then $A_1 = (a'_{ij})$ is a symmetric positive definite matrix, because the leading i-th principal minor of A_1 is equal to the product of

$(a_{11}a_{22}a_{33}\ldots a_{ii})^{-1}$ and the leading i-th principal minor of A, and so is positive, $(i = 1, 2, \ldots, n)$. Further, we define $B = (b_{ij})$

where $b_{ii} = a_{ii}$ and $b_{ij} = \alpha(a_{ii}a_{jj})^{\frac{1}{2}}$ if $i \neq j$.

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Then B is symmetric and positive definite, since its i-th principal minor is equal to the product of $(a_{11}a_{22}\cdots a_{ii})$ and the leading i-th principal minor of B_1, i.e. equal to $a_{11}\cdots a_{ii}(1-\alpha)^{i-1}(1+i\alpha-\alpha)$.

We now see that A is the Schur product of the symmetric positive definite matrices B and C, with α and a'_{ij} as defined above. This completes the proof.

We notice that each real symmetric positive definite matrix can be exhibited as a Schur product of two real symmetric positive definite matrices in infinitely many ways. Combining the above result with Schur's result, we can state that a real symmetric matrix is positive definite if and only if it is the Schur product of two real symmetric positive definite matrices.

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2. F.E. Hohn, Elementary Matrix Algebra, MacMillan Company, 1958, p. 50.

Loyola College, Montreal

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