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INJECTIVES AND PROJECTIVES IN TERM FINITE VARIETIES OF ALGEBRAS

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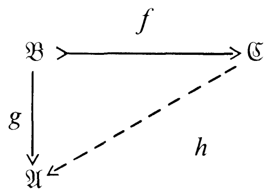
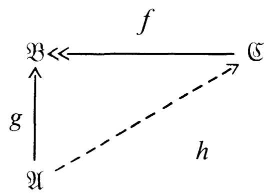
0. Introduction. Let V be a class of similar algebras. An algebra $\mathfrak{A}$ is *V-injective* provided $\mathfrak{A} \in V$ and whenever $\mathfrak{B}, \mathfrak{C} \in V$ and f is a one-to-one homomorphism from $\mathfrak{B}$ into $\mathfrak{C}$ and g is a homomorphism from $\mathfrak{B}$ into $\mathfrak{A}$, then there is a homomorphism h from $\mathfrak{C}$ into $\mathfrak{A}$ such that $h \circ f = g$. So $\mathfrak{A}$ is injective provided all diagrams of the following sort can be completed.



Dually, $\mathfrak{A}$ is *V-projective* provided $\mathfrak{A} \in V$ and whenever $\mathfrak{B}, \mathfrak{C} \in V$ and f is a homomorphism from $\mathfrak{C}$ onto $\mathfrak{B}$ and g is a homomorphism from $\mathfrak{A}$ into $\mathfrak{B}$, then there is a homomorphism h from $\mathfrak{A}$ into $\mathfrak{C}$ such that $f \circ h = g$. So $\mathfrak{A}$ is projective provided all diagrams of the following sort can be completed:



This usage of the words "projective" and "injective" differs somewhat from the usage current in category theory.

Let V be a variety of algebras. It is easy to see that all algebras in V which have only one element are V-injective. Likewise, it is evident that all

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algebras free relative to V are projective. Further, it is well-known (and not difficult) that an algebra is V-projective if and only if it is a retract of a V-free algebra. (Roughly, $\mathfrak{A}$ is a *retract* of $\mathfrak{B}$ when $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ and there is an endomorphism of $\mathfrak{B}$ onto $\mathfrak{A}$ fixing each member of $\mathfrak{A}$.)

Injective and projective modules have been subjected to intense investigation since 1940. See, for instance, [**11**]. Halmos [**9**] deals with injective and projective Boolean algebras. R. Balbes [**1**] characterizes injective and projective distributive lattices, while Day [**4**] demonstrates that all injectives are trivial in any variety of lattices except the variety of distributive lattices. Freese and Nation [**6**] take up projective lattices. Horn and Kimura [**10**] investigate injective and projective semilattices, but see also [**2**]. Gerhard [**7**] is concerned with injective bands, while Nordahl and Scheiblich [**15**] take up projective bands. B. M. Schein [**17**] describes injectives in the class of commutative regular semigroups. See [**5**] and [**19**] for the theory of injective hulls. Calabi [**3**] noted that the projectives in the variety of all semigroups were free, while Grillet [**4**] established that the projectives in the variety of commutative semigroups were free, as well.

After seeing the results in this paper, B. M. Schein observed that some results already appearing in the literature were in error. His remarks are contained in [**18**] where some of our arguments have been sketched.

In this note we are concerned with a large assortment of varieties in which injectives appear to be trivial, while projectives turn out to be free.

A variety V is *term finite* provided $\{\tau : V \models \sigma \approx \tau\}$ is finite for each term σ. The concept of term finite variety is wider than the concept of balanced variety, although these concepts are closely related. These notions have been put to diverse uses in [**12**], [**13**], [**14**], [**16**], and [**20**]. In Theorem D below we prove that a variety of semigroups is term finite if and only if every commutative semigroup belongs to the variety. This entails that there are $2^{\aleph_0}$ term finite varieties of semigroups.

Algebras are denoted by upper case German script: $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, . . . , while their universes are usually denoted by the corresponding Latin capitals: A, B, C, ... Terms are denoted by lower case Greek letters: σ, τ, ϕ, ψ, θ, ... σ denotes the term function on $\mathfrak{A}$ which is the interpretation of σ. $\approx$ is the formal equality symbol. $\omega = \{0, 1, 2, \ldots\}$, $\mathbf{N} = \{1, 2, 3, \ldots\}$ and $\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.

Our principal objective is to establish the first two theorems below.

THEOREM A. *If* $(\mathbf{N}, +)$ *is* V*-free and* $\mathfrak{A} \in V$, *then* $\mathfrak{A}$ *is* V*-injective if and only if* $\mathfrak{A}$ *has only one element.*

THEOREM B. *If V is a term finite variety and $\mathfrak{A} \in V$ then $\mathfrak{A}$ is V-projective if and only if $\mathfrak{A}$ is V-free.*

Theorem A is established in Section 1 with the help of some lemmas that depend on a kind of absorption property. We have been unable to describe the injectives of term finite varieties in general.

Theorem B is proved in Section 2.

A variety V is *balanced* provided the length of σ equals the length of τ whenever $V \models \sigma \approx \tau$.

THEOREM C. *Let V be a variety of groupoids. The following are equivalent*:

(i) $(N, +) \in V$.

(ii) *V is balanced.*

(iii) *All commutative semigroups belong to V.*

This theorem is very easy to prove.

THEOREM D. *Let V be a variety of semigroups. The following are equivalent*:

(i) $(N, +)$ *is V-free.*

(ii) *V is balanced.*

(iii) *V is term-finite.*

Proof. Since V is a variety of semigroups and $(\mathbf{N}, +)$ is a free semigroup, $(\mathbf{N}, +) \in V$ if and only if $(\mathbf{N}, +)$ is V-free. So (i) and (ii) are equivalent by Theorem C. Evidently, (ii) implies (iii). To see the converse, suppose $V \models \sigma \approx \tau$ where the length of σ is n and the length of τ is k with $n = k + r$ and $r > 0$. We can suppose x is the only variable occurring in σ and in τ. Hence $V \models x^{k+r} \approx x^k$. But then $V \models x^{k+mr} = x^k$ for every $m \in \omega$. Therefore V is not term finite.

Among varieties V of groupoids

$$(\mathbf{N}, +) \text{ is } V\text{-free} \Rightarrow V \text{ is balanced} \Rightarrow V \text{ is term-finite.}$$

But none of the implications can be reversed. In fact, the variety based on

$$(x(x(xx)))x \approx (x(xx))x$$

is term finite, see [**14**], but not balanced, while the variety based on

$$x(xx) \approx (xx)x$$

is balanced but $(\mathbf{N}, +)$ is not free relative to this variety.

CONJECTURE. *If V is a term-finite variety, then all the V-injectives are trivial.*

A variety V is said to be *injectively complete* (or to *have enough injectives*) provided every member of V is subalgebra of a V-injective. A weaker but still interesting form of our conjecture asserts that no term-finite variety can be injectively complete.

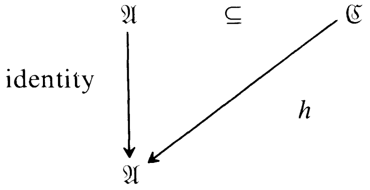
1. V-injectives are trivial if $(\mathbf{N}, +)$ is V-free. We need three lemmas. We state them all under the assumption that $(\mathbf{N}, +)$ is V-free and that $\mathfrak{A}$ is V-injective. Lemma 1.0 is true for any term-finite variety of groupoids, while Lemma 1.2 is true for any balanced variety of groupoids. The V-freeness of $(\mathbf{N}, +)$ plays an essential role in Lemma 1.1.

LEMMA 1.0. *There is a zero in $\mathfrak{A}$. (i.e., an element $o \in A$ such that $ao = oa = a$ for all $a \in A$.)*

Proof. Let ∞ be anything such that $\infty \notin A$. Let $\mathfrak{C}$ be the algebra with universe $A \cup \{\infty\}$ such that for $u, v \in A \cup \{\infty\}$

$$uv = \begin{cases} uv \text{ if } u \neq \infty \neq v \\ \infty \text{ otherwise.} \end{cases}$$

Since V is term-finite, it follows that V is regular; that is, the same variables occur in σ and τ if $V \models \sigma \approx \tau$. Since $\mathfrak{A} \in V$ and V is regular, we conclude that $\mathfrak{C} \in V$. Now consider



where h exists since $\mathfrak{A}$ is injective. Take $o = h(\infty)$. Then

$$o = h(\infty) = h(a\infty) = h(a)h(\infty) = uo \text{ for every } a \in A.$$

Similary $o = oa$.

LEMMA 1.1. *For each $a \in A$ there is $b \in A$ such that $ab = a$ and $bb = b$.*

Proof. Let g be the homomorphism from $(\mathbf{N}, +)$ into $\mathfrak{A}$ such that $g(1) = a$; such a homomorphism exists since $(\mathbf{N}, +)$ is V-free. Obtain h such

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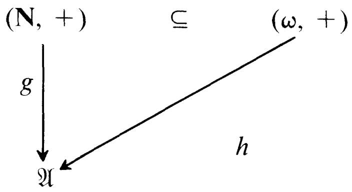
that



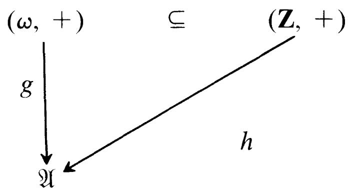
Take $b = h(0)$. Then

$$a = h(1) = h(1 + 0) = h(1)h(0) = ab \quad \text{and}$$

$$b = h(0) = h(0 + 0) = h(0)h(0) = bb.$$

LEMMA 1.2. *$\mathfrak{A}$ has only one idempotent element.*

Proof. Suppose $a \in A$ and $aa = a$. Let $g:\omega \to \{a, o\}$ such that $g(0) = a$ and $g(k) = o$ for $k > 0$. It is easy to see that $\{a, o\}$ is the universe of a subalgebra of $\mathfrak{A}$ and that g is a homomorphism. Obtain h so that



Then

$$a = h(0) = h(-1 + 1) = h(-1)h(1) = h(-1)o = o.$$

Now let $a \in A$ and let $b \in A$ such that b is idempotent and $ab = a$, using Lemma 1.1. By Lemma 1.2, $b = o$, so $a = ab = ao = o$. Hence $\mathfrak{A}$ is trivial and Theorem A is established.

2. Projectives in term finite varieties are free. To prove Theorem B it suffices to establish the lemma below.

LEMMA 2.0. *Retracts of V-free algebras are V-free, for any term finite variety V.*

Proof. Let $\mathfrak{A}$ be a retract of the V-free algebra $\mathfrak{B}$. It does no harm to suppose that $\mathfrak{A}$ is a subalgebra of $\mathfrak{B}$ and that h is a homomorphism from $\mathfrak{B}$ onto $\mathfrak{A}$ such that $h(a) = a$ for all $a \in A$. Let X be a set which freely generates $\mathfrak{B}$. To simplify notation we assume $X = \{x_i : i \geqq 0\}$. We argue that $\mathfrak{A}$ is generated by $\{x : x \in X \text{ and } h(x) = x\}$.

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For each $i \geqq 0$, there is a term τ_i such that

$$h(x_i) = \tau_i^{\mathfrak{B}}(x_0, x_1, \ldots).$$

Define $\tau_{i,n}$ for each natural number n by the following recursion:

$\tau_{i,0}$ is τ_i

$\tau_{i,k+1}$ is obtained from $\tau_{i,k}$ by the simultaneous substitution of τ_j for the variable v_j for each $j \geqq 0$.

Since

$$\tau_i^{\mathfrak{B}}(x_0, x_1, \ldots) = h(x_i) = h(h(x_i)) = \tau_i^{\mathfrak{B}}(\tau_0^{\mathfrak{B}}, \tau_1^{\mathfrak{B}}, \ldots)$$

we can conclude inductively that

$$\mathfrak{B} \models \tau_i \approx \tau_{i,n} \quad \text{for each natural number } n,$$

since $\mathfrak{B}$ is V-freely generated by X. Since V is term finite $\{\tau_{i,n}:n \geqq 0\}$ is finite. Pick n such that $\tau_{i,n}$ is as long as possible and among all such terms of maximum length $\tau_{i,n}$ contains the least number of variables. Suppose v_j occurs in $\tau_{i,n}$. Then τ_j occurs in $\tau_{i,n+1}$. Since $\tau_{i,n+1}$ and $\tau_{i,n}$ must have the same length, τ_j is either a variable or a constant. Since $\tau_{i,n}$ already has the fewest variables, τ_j cannot be a constant. Thus τ_j is a variable; say v_k. This means

$$h(x_j) = \tau_j^{\mathfrak{B}}(x_0, \ldots) = x_k.$$

Therefore $x_k \in A$ and $h(x_k) = x_k$. Hence if v_l is any variable occurring in $\tau_{i,n+1}$, then x_l is fixed by h. Consequently $\tau_{i,n+1}^{\mathfrak{B}}(x_0, \ldots)$ is generated by $\{x:x \in X \text{ and } h(x) = x\}$. But

$$\tau_i^{\mathfrak{B}}(x_0, \ldots) = \tau_{i,n+1}^{\mathfrak{B}}(x_0, \ldots),$$

so

$$\{h(x):x \in X\} = \{\tau_i^{\mathfrak{B}}(x_0, \ldots):i \geqq 0\}$$

is generated by $\{x:x \in X \text{ and } h(x) = x\}$. Since $\mathfrak{A}$ is generated by $\{h(x):x \in X\}$, we conclude that $\mathfrak{A}$ is also generated by $\{x:x \in X \text{ and } h(x) = x\}$.

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