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# Ramification conjecture and Hirzebruch's property of line arrangements

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Abstract

The ramification of a polyhedral space is defined as the metric completion of the universal cover of its regular locus. We consider mainly polyhedral spaces of two origins: quotients of Euclidean space by a discrete group of isometries and polyhedral metrics on $\mathbb{C}\mathrm{P}^2$ with singularities at a collection of complex lines. In the former case we conjecture that quotient spaces always have a CAT[0] ramification and prove this in several cases. In the latter case we prove that the ramification is CAT[0] if the metric on $\mathbb{C}\mathrm{P}^2$ is non-negatively curved. We deduce that complex line arrangements in $\mathbb{C}\mathrm{P}^2$ studied by Hirzebruch have aspherical complement.

## 1. Introduction

The main objects of this article are Euclidean polyhedral spaces and their *ramifications*. The ramification of a polyhedral space is the metric completion of the universal cover of its regular locus. We are interested in the situation when the ramification is CAT[0].

Two classes of polyhedral spaces that will play the most important role are quotients of $\mathbb{R}^m$ by discrete isometric actions, and polyhedral Kähler manifolds; that is, polyhedral manifolds with a complex structure.

*Quotients of $\mathbb{R}^m$ and ramification conjecture.* We start with the case of $\mathbb{R}^m$ quotients where the ramification space admits an alternative description in terms of arrangements of planes of (real) codimension 2; we will call such planes *hyperlines*.

Consider a discrete isometric and orientation-preserving action $\Gamma \curvearrowright \mathbb{R}^m$. Denote by $\mathcal{L}_\Gamma$ the arrangement of all hyperlines which are fixed by at least one non-identical element in $\Gamma$. Define the ramification of $\Gamma \curvearrowright \mathbb{R}^m$ (briefly $\mathrm{Ram}_\Gamma$) as the universal cover of $\mathbb{R}^m$ branching infinitely along each hyperline in $\mathcal{L}_\Gamma$.

More precisely, if $\tilde{W}_\Gamma$ denotes the universal cover of

$$W_\Gamma = \mathbb{R}^m \Big\backslash \Big( \bigcup_{\ell \in \mathcal{L}_\Gamma} \ell \Big)$$

equipped with the length metric induced from $\mathbb{R}^m$, then $\mathrm{Ram}_\Gamma$ is the metric completion of $\tilde{W}_\Gamma$.

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One of the main motivations of this paper is the following conjecture.

Ramification conjecture 1.1. Let $\Gamma \curvearrowright \mathbb{R}^m$ be a properly discontinuous isometric orientation-preserving action. Then:

(a) $\mathrm{Ram}_\Gamma$ is a CAT[0] space.
(b) the natural inclusion $\tilde{W}_\Gamma \hookrightarrow \mathrm{Ram}_\Gamma$ is a homotopy equivalence.

Assume for an action $\Gamma \curvearrowright \mathbb{R}^m$ that the ramification conjecture holds. Then since CAT[0] spaces are contractible, $\tilde{W}_\Gamma$ is also contractible, and so $W_\Gamma$ is aspherical.

The ramification conjecture generalizes a conjecture of Allcock [All13, Conjecture 1.4] on finite *reflection groups* (recall that a reflection group is a discrete group generated by a set of reflections of a Euclidean space). Allcock considered the case of the action $\Gamma \curvearrowright \mathbb{C}^m$ of a finite reflection group $\Gamma$ that complexifies the orientation-reversing action of $\Gamma$ on $\mathbb{R}^m$ generated by reflections. Allcock's conjecture is related to an earlier conjecture of Charney and Davis (see [CD95, Conjecture 3]), which in turn is motivated by a conjecture of Arnold, Pham and Thom on complex hyperplane arrangements.

In the following theorem we collect the partial cases of the ramification conjecture which we can prove.

Theorem 1.2. *The ramification conjecture holds in the following cases.*

($\mathrm{R}^+$) *If the action $\Gamma \curvearrowright \mathbb{R}^m$ is the orientation-preserving index-two subgroup of a reflection group.*
($\mathbb{Z}_2$) *If $\Gamma$ is isomorphic to $\mathbb{Z}_2^k$.*
($\mathbb{R}^3$) *If $m \leqslant 3$.*
($\mathbb{C}^2$) *If $m = 4$, and the action $\Gamma \curvearrowright \mathbb{R}^4$ preserves a complex structure on $\mathbb{R}^4$.*

The most involved case is ($\mathbb{C}^2$); it is proved in § 9 and relies on Theorem 8.1, which is the main technical result of this paper.

The proofs of other cases are simpler. The case ($\mathrm{R}^+$) follows from the more general Proposition 4.1. In § 7, we give two proofs of the case ($\mathbb{R}^3$); one is based on Theorem 6.3 and Zalgaller's theorem 3.5 and the other on the case ($\mathrm{R}^+$).

Corollary 1.3. *Let $S_3 \curvearrowright \mathbb{C}^3$ be the action of a symmetric group by permuting coordinates of $\mathbb{C}^3$. Then $\mathrm{Ram}_{S_3}$ is a* CAT[0] *space.*

The above corollary is deduced from the ($\mathbb{C}^2$)-case of Theorem 1.2 since the action $S_3 \curvearrowright \mathbb{C}^3$ splits as a sum of an action on $\mathbb{C}^2$ and a trivial action on $\mathbb{C}^1$. This corollary also follows from a result of Charney and Davis in [CD93].

*Polyhedral manifolds and Hirzebruch's question.* Our study of ramifications of polyhedral manifolds sheds some light on a question of Hirzebruch on complex line arrangements in $\mathbb{C}\mathrm{P}^2$ asked in [Hir85]. To state this question, recall the notion of complex reflection groups and arrangements.

A *finite complex reflection* group is a group $\Gamma$ acting on $\mathbb{C}^m$ by complex linear transformations generated by elements that fix a complex hyperplane in $\mathbb{C}^m$. The arrangement of complex hyperplanes[1] $\mathcal{L}_\Gamma$ in $\mathbb{C}^m$ and its projectivization in $\mathbb{C}\mathrm{P}^{m-1}$ are called *complex reflection arrangements*.

[1] That is, the set of hyperplanes fixed by at least one non-trivial element of $\Gamma$.



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*Hirzebruch's question* 1.4 [Hir85]. Let $\mathcal{L}$ be a complex line arrangement in $\mathbb{C}\mathrm{P}^2$ consisting of $3 \cdot n$ lines such that each line of $\mathcal{L}$ intersects others at exactly $n+1$ points. Is it true that $\mathcal{L}$ is a complex reflection arrangement?

The above property will be called *Hirzebruch's property*. Hirzebruch noticed that all complex reflection line arrangements in $\mathbb{C}\mathrm{P}^2$ satisfy this property. These line arrangements consist of two infinite series and five exceptional examples. The infinite series are called $A_m^0$ $(m \geqslant 3)$ and $A_m^3$ $(m \geqslant 2)$ and correspond to reflection groups $G(m,m,3)$ and $G(m,p,3)$ $(p < m)$ from the Shephard–Todd classification. The five exceptional examples correspond to reflection groups $G_{23}, G_{24}, G_{25}, G_{26}$ and $G_{27}$.

Hirzebruch's question is still open, but we are able to prove the following.

Theorem 1.5. *All line arrangements satisfying Hirzebruch's property have aspherical complements.*

Note that if the answer to Hirzebruch's question were positive, this theorem would follow from the work of Bessis [Bes15]. Bessis finished the proof of the old conjecture stating that complements to finite complex reflection arrangements are aspherical. Namely, he proved this statement for the cases of groups $G_{24}$, $G_{27}$, $G_{29}$, $G_{31}$, $G_{33}$ and $G_{34}$. As an immediate corollary of our theorem, we get a new geometric proof of Bessis's theorem for the cases of groups $G_{24}$ and $G_{27}$.

Theorem 1.5 has a generalization to a larger class of arrangements, described in Corollary 11.3. Note that on the one hand line arrangements with aspherical complements are quite rare; on the other hand no idea exists at the present of how to classify them.

*About the proof of Theorem 1.5.* It follows from [Pan09, Corollary 7.8] that for any arrangement satisfying Hirzebruch's property except the union of three lines, there is a non-negatively curved polyhedral metric on $\mathbb{C}\mathrm{P}^2$ with singularities at this arrangement. Hence to prove the theorem it is enough to show that the ramification of this polyhedral metric satisfies conditions (a) and (b) of Conjecture 1.1. Let us sketch how this is done.

Consider a three-dimensional pseudomanifold $\Sigma$ with a piecewise-spherical metric. Define the *singular locus* $\Sigma^\star$ of $\Sigma$ as the set of points in $\Sigma$ which do not admit a neighbourhood isometric to an open domain in the unit 3-sphere.

Then the *ramification* of $\Sigma$ is defined as the completion of the universal cover $\tilde{\Sigma}^\circ$ of the *regular locus* $\Sigma^\circ = \Sigma \backslash \Sigma^\star$. The obtained space will be denoted as $\operatorname{Ram}\Sigma$.

In Theorem 8.1, we characterize spherical polyhedral 3-manifolds $\Sigma$ admitting an isometric $\mathbb{R}^1$-action with geodesic orbits such that $\operatorname{Ram}\Sigma$ is CAT[1]. The key condition in Theorem 8.1 is that all points in $\Sigma$ lie sufficiently close to the singular locus.

The existence of an $\mathbb{R}^1$-action as above on $\Sigma$ is equivalent to the existence of a complex structure on the Euclidean cone over $\Sigma$; see Theorem 3.9. The latter permits us to apply Theorem 8.1 in the proof of Theorem 1.5 since any non-negatively curved polyhedral metric on $\mathbb{C}\mathrm{P}^2$ has complex holonomy. It follows then that the ramification of $\mathbb{C}\mathrm{P}^2$ is locally CAT[0] and by an analogue of the Cartan–Hadamard theorem it is globally CAT[0]; see Proposition 3.7.

## 2. More questions and observations

*Ramification of a polyhedral space.* A Euclidean polyhedral space with non-negative curvature in the sense of Alexandrov has to be a pseudomanifold, possibly with a non-empty boundary.

In fact, a stronger statement holds, a Euclidean polyhedral space $\mathcal{P}$ has curvature bounded from below in the sense of Alexandrov if and only if its regular locus $\mathcal{P}^\circ$ is connected and convex



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in $\mathcal{P}$; that is, any minimizing geodesic between points in $\mathcal{P}^\circ$ lies completely in $\mathcal{P}^\circ$ (compare [Mil69, Theorem 5]).

Recall that the *ramification* of $\mathcal{P}$ is defined as the completion of the universal cover $\tilde{\mathcal{P}}^\circ$ of the *regular locus* $\mathcal{P}^\circ$. The next question is intended to generalize the ramification conjecture to a wider setting that is not related to group actions.

*Question* 2.1. Let $\mathcal{P}$ be a Euclidean polyhedral space. Suppose that $\mathcal{P}$ has non-negative curvature in the sense of Alexandrov. What additional conditions should be imposed on $\mathcal{P}$ to guarantee that $\operatorname{Ram}\mathcal{P}$ is a CAT[0] space and the inclusion $\tilde{\mathcal{P}}^\circ \hookrightarrow \operatorname{Ram}\mathcal{P}$ is a homotopy equivalence?

For a while we thought that no additional condition on $\mathcal{P}$ should be imposed; that is, $\operatorname{Ram}\mathcal{P}$ is always a CAT[0] space. But then we found a counterexample in dimension 4 and higher; see Theorem 10.1.

Nevertheless, Theorem 6.3 joined with Zalgaller's theorem 3.5 imply that no additional condition is needed if $\dim\mathcal{P} \leqslant 3$. Theorem 11.1 also gives an affirmative answer in a particular four-dimensional case. The latter theorem is used to prove Theorem 1.5; it also proves [Pan09, Conjecture 8.2].

We do not know what conditions should be imposed in general if $\dim\mathcal{P} \geqslant 4$ but would like to formulate a conjecture in one interesting non-trivial case.

Conjecture 2.2. Let $\mathcal{P}$ be a Euclidean polyhedral space with non-negative curvature in the sense of Alexandrov. Suppose that $\mathcal{P}$ is homeomorphic to $\mathbb{C}\mathrm{P}^m$ and its singularities form a complex hyperplane arrangement on $\mathbb{C}\mathrm{P}^m$. Then $\operatorname{Ram}\mathcal{P}$ is CAT[0] and the inclusion $\tilde{\mathcal{P}}^\circ \hookrightarrow \operatorname{Ram}\mathcal{P}$ is a homotopy equivalence.

This conjecture holds for $m = 2$ by Theorem 11.1. Existence of higher-dimensional examples of such polyhedral metrics on $\mathbb{C}\mathrm{P}^m$ can be deduced from [CHL05].

*Two-convexity of the regular locus.* The same argument as in [PP14] shows that the regular locus $\mathcal{P}^\circ$ of a polyhedral space is two-convex, that is, it satisfies the following property.

*Assume that $\Delta$ is a flat tetrahedron. Then any locally isometric geodesic immersion in $\mathcal{P}^\circ$ of three faces of $\Delta$ which agrees on three common edges can be extended to a locally isometric immersion $\Delta \looparrowright \mathcal{P}^\circ$.*

From the main result of Alexander *et al.* in [AB93], it follows that every simply connected two-convex flat manifold with a smooth boundary is CAT[0]. Therefore, if one could approximate $(\operatorname{Ram}\mathcal{P})^\circ$ by flat two-convex manifolds with smooth boundary, the Alexander–Berg–Bishop theorem would imply that $\operatorname{Ram}\mathcal{P} \in \mathrm{CAT}[0]$.

This looks as a nice plan to approach the problem, but it turns out that such a smoothing does not exist even for the action $\mathbb{Z}_2^2 \curvearrowright \mathbb{C}^2$ which changes the signs of the coordinates; see the discussion after [PP14, Proposition 5.3] or *Two convexity* in [Pet09] for more details.

*Ramification around a subset.* Given a subset $A$ in a metric space $X$, define $\operatorname{Ram}_A X$ as the completion of the universal cover of $X\backslash A$. Then results of Charney and Davis in [CD93] imply the following.

(i) Let $x$, $y$ and $z$ be distinct points in $\mathbb{S}^2$. Then

$$\operatorname{Ram}_{\{x,y,z\}}\mathbb{S}^2 \in \mathrm{CAT}[1]$$

if and only if the triangle $[xyz]$ has perimeter $2\cdot\pi$. In particular, the points $x$, $y$ and $z$ lie on a great circle of $\mathbb{S}^2$.



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(ii) Let $X$, $Y$ and $Z$ be disjoint great circles in $\mathbb{S}^3$. Then

$$\operatorname{Ram}_{X\cup Y\cup Z}\mathbb{S}^3 \in \mathrm{CAT}[1]$$

if and only if $X$, $Y$ and $Z$ are fibers of the Hopf fibration $\mathbb{S}^3 \to \mathbb{S}^2$ and their images $x, y, z \in \mathbb{S}^2$ satisfy condition (i).[2]

The following two observations give a link between the above results and Question 2.1.

It turns out that if $\mathcal{P}_n$ is a sequence of two-dimensional spherical polyhedral spaces with exactly three singular points that approach $\mathbb{S}^2$ in the sense of Gromov–Hausdorff, then the limit position of singular points on $\mathbb{S}^2$ satisfies (i).

With a bit more work one can show that a similar statement holds in the three-dimensional case. More precisely, let $\mathcal{P}_n$ be a sequence of three-dimensional spherical polyhedral spaces with the singular locus formed by exactly three circles. If $\mathcal{P}_n$ approaches $\mathbb{S}^3$ in the sense of Gromov–Hausdorff, then the limit position of the singular locus satisfies (ii).

We finish the discussion with one more conjecture.

Conjecture 2.3. Let $\mathcal{H}$ be a complex hyperplane arrangement in $\mathbb{C}^m$. Then $\operatorname{Ram}_{\mathcal{H}}\mathbb{C}^m$ is CAT[0] if and only if the following condition holds.

Let $\ell$ be any complex hyperline[3] that belongs to more than one complex hyperplane of $\mathcal{H}$. Then for any complex hyperplane $h \subset \mathbb{C}^m$ containing $\ell$ there is a hyperplane $h' \in \mathcal{H}$ containing $\ell$ such that the angle between $h'$ and $h$ is at most $\pi/4$.

Note that all complex reflection hyperplane arrangements satisfy the conditions of this conjecture. The two-dimensional version of this conjecture is Corollary 8.5, and the 'only if' part follows from this corollary. If this conjecture holds, then, using the orbi-space version of the Cartan–Hadamard theorem 3.7 and Allcock's lemma 3.6 in the same way as in the proof of Theorem 1.2, one shows that the inclusion $(\operatorname{Ram}_{\mathcal{H}}\mathbb{C}^m)^\circ \hookrightarrow (\operatorname{Ram}_{\mathcal{H}}\mathbb{C}^m)$ is a homotopy equivalence. Hence, this conjecture gives an alternative geometric approach to Bessis's result [Bes15] on asphericity of complements to complex reflection arrangements.

## 3. Preliminaries

*Three types of ramifications.* Recall that we consider three types of ramifications which are closely related: for group actions, for polyhedral spaces and for subsets.

- ◇ Given a subset $A$ in a metric space $X$, we define $\operatorname{Ram}_A X$ as the completion of the universal cover of $X\backslash A$. We assume here that $X\backslash A$ is connected.
- ◇ Given a polyhedral space $\mathcal{P}$ (Euclidean, spherical or hyperbolic), the ramification $\operatorname{Ram}\mathcal{P}$ is defined as $\operatorname{Ram}_A\mathcal{P}$, where $A$ is the singular locus of $\mathcal{P}$.
- ◇ Given an isometric and orientation-preserving action $\Gamma \curvearrowright \mathbb{R}^m$, the ramification $\operatorname{Ram}_\Gamma\mathcal{P}$ can be defined as $\operatorname{Ram}(\mathbb{R}^m/\Gamma)$; this definition makes sense since $\mathbb{R}^m/\Gamma$ is a polyhedral space.

*Curvature bounds for polyhedral spaces. A Euclidean polyhedral space* is a simplicial complex equipped with an intrinsic metric such that each simplex is isometric to a simplex in a Euclidean space.

*A spherical polyhedral space* is a simplicial complex equipped with an intrinsic metric such that each simplex is isometric to a simplex in a unit sphere.

[2] More precisely, the quotient metric on the base $\mathbb{S}^2$ has curvature 4, so $[xyz]$ should have perimeter $\pi$.

[3] That is, an affine subspace of complex codimension 2.



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The link of any simplex in a polyhedral space (Euclidean or spherical) equipped with the angle metric forms a spherical polyhedral space.

The following two propositions give a more combinatorial description of polyhedral spaces with curvature bounded from below or above.

Proposition 3.1. *An $m$-dimensional Euclidean (spherical) polyhedral space $\mathcal{P}$ has curvature $\geqslant 0$ (correspondingly $\geqslant 1$) in the sense of Alexandrov if and only if each of the following conditions holds.*

(i) *The link of any $(m-1)$-simplex is isometric to the one-point space $\mathfrak{p}$ or $\mathbb{S}^0$; that is, the two-point space with distance $\pi$ between the distinct points.*

(ii) *The link of any $(m-2)$-simplex is isometric to a closed segment of length $\leqslant \pi$ or a circle with length $\leqslant 2\cdot\pi$.*

(iii) *The link of any $k$-simplex with $k \leqslant m-2$ is connected.*

Corollary 3.2. *The simplicial complex of any polyhedral space $\mathcal{P}$ with a lower curvature bound is a pseudomanifold.*

The following proposition follows from the Cartan–Hadamard theorem and its analogue is proved by Bowditch in [Bow95]; see also [AKP], where both theorems are proved nicely.

Proposition 3.3. *A polyhedral space $\mathcal{P}$ is a* CAT[0] *space if and only if $\mathcal{P}$ is simply connected and the link of each vertex is a* CAT[1] *space.*

*A spherical polyhedral space $\mathcal{P}$ is a* CAT[1] *space if and only if the link of each vertex of $\mathcal{P}$ is a* CAT[1] *space and any closed curve of length $< 2\cdot\pi$ in $\mathcal{P}$ is null-homotopic in the class of curves of length $< 2\cdot\pi$.*

We say that a polyhedral space has *finite shapes* if the number of isometry types of simplices that compose it is finite. The following proposition is proved in [BH99, II. 4.17].

Proposition 3.4. *Let $\mathcal{P}$ be a Euclidean (spherical) polyhedral space with finite shapes and suppose that $\mathcal{P}$ has curvature $\leqslant 0$ (or $\leqslant 1$ correspondingly). If $\mathcal{P}$ is not a* CAT[0] *(correspondingly, not a* CAT[1]*) space, then $\mathcal{P}$ contains an isometrically embedded circle (correspondingly, a circle with length smaller than $2\cdot\pi$).*

*Spherical polyhedral metrics on $\mathbb{S}^2$.* The following theorem appears as an intermediate statement in Zalgaller's proof of rigidity of spherical polygons; see [Zal56].

Zalgaller's theorem 3.5. *Let $\Sigma$ be a spherical polyhedral space homeomorphic to the 2-sphere and with curvature $\geqslant 1$ in the sense of Alexandrov. Assume that there is a point $z \in \Sigma$ such that all singular points lie at the distance $> \pi/2$ from $z$. Then $\Sigma$ is isometric to the standard sphere.*

*A sketch of Zalgaller's proof.* We apply an induction on the number $n$ of singular points. The base case $n = 1$ is trivial. To do the induction step, choose two singular points $p, q \in \Sigma$, cut $\Sigma$ along a geodesic $[pq]$ and patch the hole so that the obtained new polyhedron $\Sigma'$ has curvature $\geqslant 1$. The patch is obtained by doubling[4] a convex spherical triangle across two sides. For a unique

[4] Given a metric length space $X$ with a closed subset $A \subset X$, the *doubling* of $X$ across $A$ is obtained by gluing two copies of $X$ along $A$.



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choice of triangle, the points $p$ and $q$ become regular in $\Sigma'$ and exactly one new singular point appears in the patch.[5] In this way, the case with $n$ singular points is reduced to the case with $n-1$ singular points. □

*A test for homotopy equivalence.* The following lemma is a slight modification of [All13, Lemma 6.2]; the proofs of these lemmas are almost identical.

Allcock's lemma 3.6. *Let $\mathcal{S}$ be an $m$-dimensional pure[6] simplicial complex.*

*Let $K$ be a subcomplex in $\mathcal{S}$ of codimension $\geqslant 1$; set $W = \mathcal{S}\backslash K$. Assume that the link in $\mathcal{S}$ of any simplex in $K$ is contractible. Then the inclusion map $W \hookrightarrow \mathcal{S}$ is a homotopy equivalence.*

*Proof.* Denote by $K_n$ the $n$-skeleton of $K$; set $W_n = \mathcal{S}\backslash K_n$ and set $K_{-1} = \emptyset$.

For each $n \in \{0, 1, \dots, m-1\}$, we will construct a homotopy

$$F_n\colon [0,1] \times W_{n-1} \to W_{n-1}$$

of the identity map $\mathrm{id}_{W_{n-1}}$ into a map with the target in $W_n$.

Note that $W_{m-1} = W$ and $W_{-1} = \mathcal{S}$. Therefore, joining all the homotopies $F_n$, we construct a homotopy of the identity map on $\mathcal{S}$ into a map with the target in $W$. Therefore, the lemma follows once we construct $F_n$ for all $n$.

*Existence of $F_n$.* Note that each open $n$-dimensional simplex $\Delta$ in $\mathcal{S}$ admits a closed neighbourhood $N_\Delta$ in $W_{n-1}$ which is homeomorphic to

$$\Delta \times (\mathfrak{p} \star \operatorname{Link}\Delta),$$

where $\operatorname{Link}\Delta$ denotes the link of $\Delta$, $\mathfrak{p}$ denotes a one-point complex and $\star$ denotes the join. Moreover, we can assume that $\Delta$ lies in $N_\Delta = \Delta \times (\mathfrak{p} * \operatorname{Link}\Delta)$ as $\Delta \times \mathfrak{p}$ and $N_\Delta \cap N_{\Delta'} = \emptyset$ for any two open $(n-1)$-dimensional simplices $\Delta$ and $\Delta'$ in $\mathcal{S}$.

Note that if $\operatorname{Link}\Delta$ is contractible, then $\operatorname{Link}\Delta$ is a strict deformation retract of $\mathfrak{p} \star \operatorname{Link}\Delta$. It follows that for any $(n-1)$-dimensional simplex $\Delta$ in $K$, the relative boundary $\partial_{W_{n-1}} N_\Delta$ is a deformation retract of $N_\Delta$. Clearly, $\partial_{W_{n-1}} N_\Delta \subset W_n$. Hence, the existence of $F_n$ follows. □

*An orbi-space version of the Cartan–Hadamard theorem.*

Proposition 3.7. *Let $\mathcal{P}$ be a polyhedral pseudomanifold. Suppose that for any point $x \in \mathcal{P}$ the ramification of the cone at $x$ is* CAT[0]. *Then:*

(i) $\operatorname{Ram}\mathcal{P}$ *is* CAT[0];

(ii) *for any $y \in \operatorname{Ram}\mathcal{P}$ that projects to $x \in \mathcal{P}$, the cone at $y$ is isometric to the ramification of the cone at $x$.*

The proposition can be proved along the same lines as the Cartan–Hadamard theorem; see for example [AKP].

A closely related statement was rigorously proved by Haefliger in [Hae90]; he showed that if the charts of an orbi-space are CAT[0] then its universal orbi-cover is CAT[0]. Haefliger's definition of orbi-space restricts only to finite isotropy groups, but the above proposition requires only minor modifications of Haefliger's proof.

[5] This patch construction was introduced by Alexandrov; the earliest reference we found is [Ale48, VI, § 7].

[6] That is, each simplex in $\mathcal{S}$ forms a face in an $m$-dimensional simplex.



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*Polyhedral Kähler manifolds.* Let us recall some definitions and results from [Pan09]. We will restrict our consideration to the case of non-negatively curved polyhedra.

Definition 3.8. Let $\mathcal{P}$ be an orientable non-negatively curved Euclidean polyhedral manifold of dimension $2\cdot n$. We say that $\mathcal{P}$ is *polyhedral Kähler* if the holonomy of the metric on $\mathcal{P}^\circ$ belongs to $\mathrm{U}(n)\subset \mathrm{SO}(2\cdot n)$.

In the case when $\mathcal{P}$ is a metric cone piecewise linearly isomorphic to $\mathbb{R}^{2\cdot n}$, we call it a *polyhedral Kähler cone*.

Recall that from a result of Cheeger (see [Che86] and [Pan09, Proposition 2.3]), it follows that the metric of an orientable simply connected non-negatively curved polyhedral compact 4-manifold not homeomorphic to $\mathbb{S}^4$ has unitary holonomy. Moreover in the case when the (unitary) holonomy is irreducible, the manifold has to be diffeomorphic to $\mathbb{C}\mathrm{P}^2$. Metric singularities form a collection of complex curves on $\mathbb{C}\mathrm{P}^2$; see [Pan09] for the details.

The following theorem summarizes some results on non-negatively curved four-dimensional polyhedral Kähler cones proven in [Pan09, Theorems 1.5, 1.7 and 1.8].

Theorem 3.9. *Let $\mathcal{C}^4$ be a non-negatively curved polyhedral Kähler cone and let $\Sigma$ be the unit sphere of this cone centred at its tip.*

(a) *There is a canonical isometric $\mathbb{R}$-action on $\mathcal{C}$ such that its orbits on $\Sigma$ are geodesics. This action is generated by the vector field $J(r(\partial/\partial r))$ in the non-singular part of $\mathcal{C}$, where $J$ is the complex structure on $\mathcal{C}$ and $r(\partial/\partial r)$ is the radial vector field on $\mathcal{C}$.*

(b) *If the metric singularities of the cone are topologically equivalent to a collection of $n\geqslant 3$ complex lines in $\mathbb{C}^2$, then the action of $\mathbb{R}$ on $\Sigma$ factors through $\mathbb{S}^1$ and the map $\Sigma\to\Sigma/\mathbb{S}^1$ is the Hopf fibration.*

(c) *If the metric singularities of the cone are topologically equivalent to a union of two complex lines, then the cone splits as a metric product of two two-dimensional cones.*

*Reshetnyak gluing theorem.* Let us recall the formulation of the Reshetnyak gluing theorem, which will be used in the proof of Proposition 4.1.

Theorem 3.10. *Suppose that $\mathcal{U}_1$, $\mathcal{U}_2$ are $\mathrm{CAT}[\kappa]$ spaces[7] with closed convex subsets $A_i\subset\mathcal{U}_i$ which admit an isometry $\iota: A_1\to A_2$. Let us define a new space $W$ by gluing $\mathcal{U}_1$ and $\mathcal{U}_2$ along the isometry $\iota$; that is, consider the new space*

$$\mathcal{W}=\mathcal{U}_1\sqcup_\sim\mathcal{U}_2,$$

*where the equivalence relation $\sim$ is defined by $a\sim\iota(a)$ with the induced length metric. Then the following holds.*

*The space $\mathcal{W}$ is $\mathrm{CAT}[\kappa]$. Moreover, both canonical mappings $\tau_i\,\mathcal{U}_i\to\mathcal{W}$ are distance preserving, and the images $\tau_i(\mathcal{U}_i)$ are convex subsets in $\mathcal{W}$.*

The following corollary is proved by repeated application of Reshetnyak's theorem.

Corollary 3.11. *Let $S$ be a finite tree. Assume that a convex Euclidean (or spherical) polyhedron $Q_\nu$ corresponds to each node $\nu$ in $S$ and for each edge $[\nu\mu]$ in $S$ there is an isometry $\iota_{\mu\nu}$ from a facet[8] $F\subset Q_\nu$ to a facet $F'\subset Q_\mu$.*

[7] We always assume that $\mathrm{CAT}[\kappa]$ spaces are complete.

[8] A facet is a face of codimension 1.



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*Then the space obtained by gluing all the polyhedra $Q_\nu$ along the isometries $\iota_{\mu\nu}$ forms a* CAT[0] *space (correspondingly a* CAT[1] *space).*

*Flag complexes.*

Definition 3.12. A simplicial complex $\mathcal{S}$ is *flag* if whenever $\{v_0, \dots, v_k\}$ is a set of distinct vertices which are pairwise joined by edges, then $\{v_0, \dots, v_k\}$ spans a $k$-simplex in $\mathcal{S}$.

Note that every flag complex is determined by its 1-skeleton.

Spherical polyhedral CAT[1] spaces glued from right-angled simplices admit the following combinatorial characterization discovered by Gromov [Gro87, p. 122].

Theorem 3.13. *A piecewise-spherical simplicial complex made of right-angled simplices is a* CAT[1] *space if and only if it is a flag complex.*

## 4. On the reflection groups

If the singular locus of a polyhedral space $\mathcal{P}$ coincides with its $(m-2)$-skeleton, then $\mathcal{P}^\circ$ has the homotopy type of a graph (its vertices correspond to the centres of $m$-simplices of $\mathcal{P}$). We will show that in this case the ramification conjecture can be proven by applying the Reshetnyak gluing theorem recursively.

We will prove the following stronger statement.

Proposition 4.1. *Assume that $\mathcal{P}$ is an $m$-dimensional polyhedral space which admits a subdivision into closed sets $\{Q_i\}$ such that each $Q_i$ with the induced metric is isometric to a convex $m$-dimensional polyhedron and each face of dimension $m-2$ of each polyhedron $Q_i$ belongs to the singular locus of $\mathcal{P}$. Then* $\operatorname{Ram}\mathcal{P} \in$ CAT[0].

Note that Theorem 1.2($\mathrm{R}^+$) follows directly from the above proposition. Also the condition in the above proposition holds if $\mathcal{P}$ is isometric to the boundary of a convex polyhedron in Euclidean space and, in particular, by Alexandrov's theorem, it holds if $\mathcal{P}$ is homeomorphic to $\mathbb{S}^2$.

*Proof of Proposition 4.1.* In the subdivision of $\mathcal{P}$ into $Q_i$, colour all the facets in different colours. Consider the graph $\Gamma$ with a node for each $Q_i$, where two nodes are connected by an edge if the corresponding polyhedra have a common facet. Colour each edge of $\Gamma$ in the colour of the corresponding facet.

Denote by $\tilde{\Gamma}$ the universal cover of $\Gamma$. Note that $\tilde{\Gamma}$ has to be a tree.

For each node $\nu$ of $\tilde{\Gamma}$, prepare a copy of $Q_i$ which corresponds to the projection of $\nu$ in $\Gamma$.

Note that the space $\operatorname{Ram}\mathcal{P}$ can be obtained by gluing the prepared copies. Two copies should be glued along two facets of the same colour $z$ if the nodes corresponding to these copies are connected in $\tilde{\Gamma}$ by an edge of colour $z$.

Given a finite subtree $S$ of $\tilde{\Gamma}$, consider the subset $Q_S \subset \operatorname{Ram}\mathcal{P}$ formed by all the copies of $Q_i$ corresponding to the nodes of $S$.

Note that $Q_S$ is a convex subset of $\operatorname{Ram}\mathcal{P}$. Indeed, if a path between points of $Q_S$ escapes from $Q_S$, it has to cross the boundary $\partial Q_S$ at the same facet twice, say at the points $x$ and $y$ in a facet $F \subset \partial Q_S$. Further note that the natural projection $\operatorname{Ram}\mathcal{P} \to \mathcal{P}$ is a short map which is distance preserving on $F$. Therefore, there is a unique geodesic from $x$ to $y$ and it lies in $F$. In particular, a geodesic with ends in $Q_S$ cannot escape from $Q_S$; in other words, $Q_S$ is convex.



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Finally, by Corollary 3.11, the subspace $Q_S$ is CAT[0] for any finite subtree $S$. Clearly, for every triangle $\triangle$ in $\operatorname{Ram}\mathcal{P}$ there is a finite subtree $S$ such that $Q_S \supset \triangle$. Therefore, the CAT[0] comparison holds for any geodesic triangle in $\operatorname{Ram}\mathcal{P}$. □

## 5. Case ($\mathbb{Z}_2$)

In this section we reduce the case ($\mathbb{Z}_2$) of Theorem 1.2 to the case ($\mathrm{R}^+$).

*Proof of Theorem 1.2; case* ($\mathbb{Z}_2$). Every orientation-preserving action of a group $\mathbb{Z}_2^k$ on $\mathbb{R}^m$ arises as the action of a subgroup of the group $\mathbb{Z}_2^m$ generated by reflections in coordinate hyperplanes. By the definition of ramification, we can assume that the action of $\mathbb{Z}_2^k$ is generated by reflections in hyperlines.

Let us write $i \sim j$ if $i = j$ or the reflection in the hyperline $x_i = x_j = 0$ belongs to $\Gamma$. Note that $\sim$ is an equivalence relation.

It follows that $\mathbb{R}^m/\Gamma$ splits as a direct product of the subspaces corresponding to the coordinate subspaces of $\mathbb{R}^m$ for each equivalence relation.

Finally, for each of the factors in this splitting, the statement holds by Theorem 1.2($\mathrm{R}^+$). □

## 6. Two-dimensional spaces

Definition 6.1. An $m$-dimensional spherical polyhedral space $\Sigma$ is called *$\alpha$-extendable* if for any $\varepsilon > 0$, every isometric immersion into $\Sigma$ of a ball of radius $\alpha + \epsilon$ from $\mathbb{S}^m$ extends to an isometric immersion of the whole $\mathbb{S}^m$.

In other words, $\Sigma$ is $\alpha$-extendable if either the distance from any point $x \in \Sigma$ to its singular locus $\Sigma^\star$ is at most $\alpha$ or $\Sigma$ is a space form.

Theorem 6.2. *Let $\Sigma$ be a two-dimensional spherical polyhedral manifold. Then* $\operatorname{Ram}\Sigma$ *is* CAT[1] *if and only if $\Sigma$ is $\pi/2$-extendable.*

*Proof.* Note that if $\Sigma^\star = \emptyset$, then $\Sigma$ is a spherical space form. So we assume that $\Sigma^\star \neq \emptyset$.

Let us show that in the case when $\Sigma$ is $\pi/2$-extendable one can decompose $\Sigma$ into a collection of convex spherical polygons with vertices in $\Sigma^\star$. The proof is almost identical to the proof of [Thu98, Proposition 3.1], so we just recall the construction.

In the case where $\Sigma^\star$ consists of two points, $\Sigma$ can be decomposed into a collection of two-gons. It remains to consider the case when $\Sigma^\star$ has at least three distinct points.

Consider the Voronoi decomposition of $\Sigma$ with respect to the points in $\Sigma^\star$. The vertices of this decomposition consist of points $x$ that have the following property. If $D$ is the maximal open ball in $\Sigma^\circ = \Sigma\backslash\Sigma^\star$ with the centre at $x$, then the radius of $D$ is at most $\pi/2$ and the convex hull of points in $\partial D \cap \Sigma^\star$ contains $x$. Note that such a convex hull is a convex spherical polygon $P(x)$ and $\Sigma$ is decomposed into the union of $P(x)$ for various vertices $x$.

Consider finally the Euclidean cone over $\Sigma$ with the induced decomposition into cones over spherical polygons. Applying Proposition 4.1 to the cone, we see that its ramification is CAT[0]. So $\operatorname{Ram}\Sigma \in$ CAT[1] by Proposition 3.3. □

The next result follows directly from Theorem 6.2 and Proposition 3.3.

Theorem 6.3. *Let $\mathcal{Y}$ be a three-dimensional polyhedral cone. Then* $\operatorname{Ram}\mathcal{Y}$ *is* CAT[0] *if and only if $\mathcal{Y}$ satisfies one of the following conditions.*



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(i) *The singular locus $\mathcal{Y}^\star$ is formed by the tip or it is empty.*
(ii) *For any direction $v \in \mathcal{Y}$, there is a direction $w \in \mathcal{Y}^\star$ such that $\measuredangle(v, w) \leqslant \pi/2$.*

Indeed, the link $\Sigma$ of $\mathcal{Y}$ is a space form if and only if $\mathcal{Y}^\star$ is formed by the tip or it is empty. If $\mathcal{Y}^\star$ contains more than one point, the condition 2 of this theorem means literally that $\Sigma$ is $\pi/2$-extendable.

## 7. Case ($\mathbb{R}^3$)

Here we present two proofs of Theorem 1.2 case ($\mathbb{R}^3$); the first one is based on Theorem 6.3 and the second on Theorem 1.2 case ($\mathrm{R}^+$).

In both of these proofs we assume that $\Gamma$ is finite. The case when $\Gamma$ is infinite can be done in the same way as the $\mathbb{C}^2$ case; see § 9.

*Proof 1.* Since $\Gamma$ is finite, without loss of generality we may assume that $\Gamma$ fixes the origin.

By Zalgaller's theorem 3.5, the link of the origin in the quotient $\mathbb{R}^3/\Gamma$ is $\pi/2$-extendable. Applying Theorem 6.3, we get the result. □

*Proof 2.* By Theorem 1.2, it is sufficient to prove that $\Gamma$ is an index-two subgroup in a group $\Gamma_1$ generated by reflections in planes.

If $\Gamma$ fixes a line in $\mathbb{R}^3$, then it is a cyclic group and it is an index-two subgroup of a dihedral group.

Otherwise $\mathbb{S}^2/\Gamma$ is an orbifold with three orbi-points glued from two copies of a Coxeter spherical triangle. Such an orbifold has an involution $\sigma$ such that $(\mathbb{S}^2/\Gamma)/\sigma$ is a Coxeter triangle $\Delta$. So $\Gamma_1$ is the group generated by reflections in the sides of $\Delta$. □

## 8. 3-spaces with a geodesic actions

The following theorem is the main technical result.

Theorem 8.1. *Let $\Sigma$ be a three-dimensional spherical polyhedral manifold. Assume that $\Sigma$ admits an isometric action of $\mathbb{R}$ with geodesic orbits.*

*Then* $\operatorname{Ram}\Sigma$ *is* $\mathrm{CAT}[1]$ *if and only if $\Sigma$ is $\pi/4$-extendable or* $\operatorname{Ram}\Sigma$ *is the completion of the universal cover of $\mathbb{S}^3\backslash\mathbb{S}^1$.*

*Example.* We will further apply this theorem to unit spheres of polyhedral cones that are quotients of $\mathbb{C}^2$ by a finite group of unitary isometries. The action of $\mathbb{R}$ in this case comes from the action on $\mathbb{C}^2$ by multiplication by complex units.

The proof of Theorem 8.1 relies on several lemmas. The following lemma is the spherical analogue of the theorem proved by Pestov and Ionin in [PI59]; a different proof via curve-shortening flow was given by Pankrashkin in [Pan15]; see also *The moon in the puddle* in [Pet09].

Drop lemma 8.2. *Let $D$ be a disk with a metric of curvature* 1*, whose boundary consists of several smooth arcs of curvature at most $\kappa$ that meet at angles larger than $\pi$ at all points except at most one. Then:*

(a) *$D$ contains an isometric copy of a disk whose boundary has curvature $\kappa$;*
(b) *if the length of $\partial D$ is less than the length of the circle with curvature $\kappa$ on the unit sphere, then $D$ contains an isometric copy of a unit half-sphere.*



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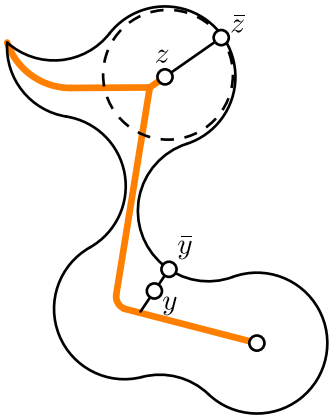
*Proof.* (a) Recall that the *cut locus* of $D$ with respect to its boundary $\partial D$ is defined as the closure of the set of all points $x \in D$ such that the restriction of the distance function $\operatorname{dist}_x|_{\partial D}$ attains its global minimum at two or more points of $\partial D$. The cut locus will be denoted as $\operatorname{CutLoc} D$.

After a small perturbation of $\partial D$, we may assume that $\operatorname{CutLoc} D$ is a graph embedded in $D$ with a finite number of edges.




Note that $\operatorname{CutLoc} D$ is a deformation retract of $D$. The retraction can be obtained by moving each point $y \in D \setminus \operatorname{CutLoc} D$ towards $\operatorname{CutLoc} D$ along the geodesic containing $y$ and the point $\bar{y} \in \partial D$ closest to $y$. In particular, $\operatorname{CutLoc} D$ is a tree.

Since $\operatorname{CutLoc} D$ is a tree, it has at least two vertices of valence one. Among all points of $\partial D$ only the non-smooth point of $\partial D$ with angle less than $\pi$ belongs to $\operatorname{CutLoc} D$. So there is at least one point $z$ of $\operatorname{CutLoc} D$ of valence one contained in the interior of $D$. The point $z$ has to be a focal point of $\partial D$; this means that the disk of radius $\operatorname{dist}_{\partial D} z$ centred at $z$ touches $\partial D$ with multiplicity at least two at some point $\bar{z}$. At $\bar{z}$ the curvature of the boundary of the disk centred at $z$ equals the curvature of $\partial D$ and so it is at most $\kappa$. So this disk contains a disk with boundary of curvature $\kappa$.

(b) By (a), we can assume that $\kappa > 0$. Consider a locally isometric immersion of $D$ into the unit sphere, $\varphi : D \looparrowright \mathbb{S}^2$. Since the length of $\partial D$ is less than $2 \cdot \pi$, by Crofton's formula, $\partial D$ does not intersect one of the equators. Therefore, the curve $\varphi(\partial D)$ is contained in a half-sphere, say $\mathbb{S}^2_+$.

Note that it is sufficient to show that $\varphi(D)$ contains the complement of $\mathbb{S}^2_+$. Suppose the contrary; note that in this case $\varphi(D) \subset \mathbb{S}^2_+$. Applying (a), we get that $\varphi(D)$ contains a disk bounded by a circle, say $\sigma_\kappa$, of curvature $\kappa$. Note that $\partial[\varphi(D)]$ cuts $\sigma_\kappa$ from its antipodal circle; therefore,

$$\operatorname{length} \partial[\varphi(D)] \geqslant \operatorname{length} \sigma_\kappa.$$

Note that

$$\operatorname{length} \partial D \geqslant \operatorname{length} \partial[\varphi(D)].$$

On the other hand, by the assumptions,

$$\operatorname{length} \partial D < \operatorname{length} \sigma_\kappa,$$

which is a contradiction. □

Lemma 8.3. *Assume that $\Sigma$ is a spherical polyhedral 3-manifold with an isometric $\mathbb{R}$-action, whose orbits are geodesic. Then the quotient $\Lambda = (\operatorname{Ram} \Sigma)/\mathbb{R}$ is a spherical polyhedral surface of curvature 4, and there are two possibilities.*



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(a) *If $\Lambda$ is not contractible, then it is isometric to the sphere of curvature 4, further denoted as $\frac{1}{2}\cdot\mathbb{S}^2$. In this case,* $\operatorname{Ram}\Sigma$ *is isometric to the unit $\mathbb{S}^3$ or to $\operatorname{Ram}_{\mathbb{S}^1}\mathbb{S}^3$, where $\mathbb{S}^1$ is a closed geodesic in $\mathbb{S}^3$.*

(b) *If $\Lambda$ is contractible, then a point $x\in\operatorname{Ram}\Sigma$ is singular if and only if so is its projection $\bar{x}\in\Lambda$. Moreover, the angle around each singular point $\bar{x}\in\Lambda$ is infinite.*

*Proof.* We will consider two cases.

*Case 1.* Assume that the action $\mathbb{R}\curvearrowright\Sigma$ is not periodic; that is, it does not factor through an $\mathbb{S}^1$-action. Then the group of isometries of $\Sigma$ contains a torus $\mathbb{T}^2$.

From [Pan09, Proposition 3.9], one can deduce that the Euclidean cone over $\operatorname{Ram}\Sigma$ is isometric to the ramification of $\mathbb{R}^4$ in one 2-plane or in a pair of two orthogonal 2-planes. It follows that $\operatorname{Ram}\Sigma$ is either

$$\operatorname{Ram}_{\mathbb{S}^1}\mathbb{S}^3 \quad\text{or}\quad \operatorname{Ram}_{\mathbb{S}^1_a\cup\mathbb{S}^1_b}\mathbb{S}^3,$$

where $\mathbb{S}^1_a$ and $\mathbb{S}^1_b$ are two opposite Hopf circles. In both cases, the $\mathbb{R}$-action is lifted from the Hopf $\mathbb{S}^1$-action on $\mathbb{S}^3$.

If $\operatorname{Ram}\Sigma=\operatorname{Ram}_{\mathbb{S}^1}\mathbb{S}^3$, then $\Lambda=\frac{1}{2}\cdot\mathbb{S}^2$ and therefore (a) holds.

If $\operatorname{Ram}\Sigma=\operatorname{Ram}_{\mathbb{S}^1_a\cup\mathbb{S}^1_b}\mathbb{S}^3$, then $\Lambda=\operatorname{Ram}_{\{a,b\}}(\frac{1}{2}\cdot\mathbb{S}^2)$, where $a$ and $b$ are two poles of the sphere; therefore, (b) holds.

*Case 2.* Assume that the $\mathbb{R}$-action is periodic. Let $s$ be the number of orbits in the singular locus $\Sigma^\star$ and let $m$ be the number of multiple orbits in the regular locus $\Sigma^\circ$.

Note that the space $\Sigma^\circ/\mathbb{S}^1$ is an orbifold with constant curvature 4; it has $m$ orbi-points. Passing to the completion of $\Sigma^\circ/\mathbb{S}^1$, we get $\Sigma/\mathbb{S}^1$. In this way we add $s$ points to $\Sigma^\circ/\mathbb{S}^1$, which we will call the *punctures*; this is a finite set of points formed by the projection of the singular locus $\Sigma^\star$ in the quotient space $\Sigma/\mathbb{S}^1$.

Now we will consider a few subcases.

Assume that $s=0$; in other words, $\Sigma^\star=\emptyset$. Then $\operatorname{Ram}\Sigma$ is isometric to $\mathbb{S}^3$ and the $\mathbb{R}$-action factors through the standard Hopf action; that is, the first part of (a) holds.

Assume that either $s\geqslant 2$ or $s\geqslant 1$ and $m\geqslant 2$. Then the orbifold fundamental group of $\Sigma^\circ/\mathbb{S}^1$ is infinite, the universal orbi-cover is a disk and it branches infinitely over every puncture of $\Sigma/\mathbb{S}^1$. The completion of the cover is contractible; that is, (b) holds.

It remains to consider the subcase $s=1$ and $m=1$. In this subcase the universal orbi-cover of $\Sigma^\circ/\mathbb{S}^1$ is a once-punctured $\mathbb{S}^2$ of curvature 4 and $\operatorname{Ram}\Sigma=\operatorname{Ram}_{\mathbb{S}^1}\mathbb{S}^3$; that is, (a) holds. □

*Proof of Theorem 8.1.* Suppose first that $\Lambda=(\operatorname{Ram}\Sigma)/\mathbb{R}$ is not contractible. By Lemma 8.3, $\Lambda$ is isometric to $\mathbb{S}^2$, and the ramification $\operatorname{Ram}\Sigma$ is isometric either to $\mathbb{S}^3$ or $\operatorname{Ram}_{\mathbb{S}^1}\mathbb{S}^3$. Both of these spaces are CAT[1]; so the theorem follows.

From now on we consider the case when $\Lambda$ is contractible and will prove in this case that $\operatorname{Ram}\Sigma\in\mathrm{CAT}[1]$ if and only if $\Sigma$ is $\pi/4$-extendable.

*If part.* From Lemma 8.3, it follows that $\operatorname{Ram}\Sigma$ branches infinitely over singular circles of $\Sigma$. So $\operatorname{Ram}\Sigma$ is locally CAT[1] and we only need to show that any closed geodesic $\gamma$ in $\operatorname{Ram}\Sigma$ has length at least $2\cdot\pi$ (see Proposition 3.4).

Let $\gamma$ be a closed geodesic in $\operatorname{Ram}\Sigma$; denote by $\bar{\gamma}$ its projection in $\Lambda$. The curve $\bar{\gamma}$ is composed of arcs of constant curvature, say $\kappa$, joining singularities of $\Lambda$. Moreover, for each singular point $p$ of $\Lambda$ that belongs to $\bar{\gamma}$, the angle between the arcs of $\bar{\gamma}$ at $p$ is at least $\pi$.

Both of the above statements are easy to check; the first one is also proved in [Pan11, Lemma 3.1]. The following lemma follows directly from [Pan11, Proposition 3.6(2))].



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Lemma 8.4. *Assume that $\Sigma$ and $\Lambda$ are as in the formulation of Lemma 8.3. Then for every geodesic $\gamma$ in $\operatorname{Ram}\Sigma$ its projection $\bar\gamma$ in $\Lambda$ has a point of self-intersections.*

Summarizing all the above, we can choose two subloops in $\bar\gamma$, say $\bar\gamma_1$ and $\bar\gamma_2$, which bound disks on $\operatorname{Ram}\Sigma/\mathbb{R}$ and both of these disks satisfy the conditions of Lemma 8.2 for some $\kappa$. Clearly, we can choose $\bar\gamma_1$ and $\bar\gamma_2$ so that $\bar\gamma_1\cap\bar\gamma_2$ is at most a finite set.

By our assumptions, the disks bounded by $\bar\gamma_i$ cannot contain points at distance more than $\pi/4$ from their boundary, otherwise $\Sigma$ would not be $\pi/4$-extendable. So we deduce from Lemma 8.2(b) that

$$\operatorname{length}\bar\gamma_i\geqslant\ell(\kappa),\tag{$*$}$$

where $\ell(\kappa)=(2\cdot\pi)/\sqrt{\kappa^2+4}$ is the length of a circle of curvature $\kappa$ on the sphere of radius $\frac{1}{2}$.

Let $\alpha$ be an arc of $\gamma$ and $\bar\alpha$ be its projection in $\Lambda$. Note that

$$\operatorname{length}\alpha=\frac{\pi}{\ell(\kappa)}\cdot\operatorname{length}\bar\alpha.$$

Together with ($*$), this implies that $\operatorname{length}\gamma\geqslant 2\cdot\pi$.

*Only if part.* Suppose now that $\Sigma$ contains an immersed copy of a ball with radius $\pi/4+\varepsilon$. Consider a lift of this ball to $\operatorname{Ram}\Sigma$ and denote it by $B$.

Set as before $\Lambda=(\operatorname{Ram}\Sigma)/\mathbb{R}$. The projection of $B$ in $\Lambda$ is a disk, say $D$, of radius $\pi/4+\varepsilon$ and curvature 4, isometrically *immersed* in $\Lambda$. Since $\Lambda$ is contractible, $D$ has to be *embedded* in $\Lambda$.

Consider a closed geodesic $\bar\gamma\subset\Lambda\backslash D$ which is obtained from $\partial D$ by a curve-shortening process. Such a geodesic has to contain at least two singular points; let $x$ be one of such points. Choose now a lift of $\bar\gamma$ to a horizontal geodesic path $\gamma$ on $\operatorname{Ram}\Sigma$ with two (possibly distinct) ends at the $\mathbb{R}$-orbit over $x$.

Finally, consider a deck transformation $\iota$ of $\operatorname{Ram}\Sigma$ that fixes the $\mathbb{R}$-orbit over $x$ and rotates $\operatorname{Ram}\Sigma$ around it by an angle larger than $\pi$. The union of $\gamma$ with $\iota\circ\gamma$ forms a closed geodesic in $\operatorname{Ram}\Sigma$ of length less than $2\cdot\pi$. □

The following statement is proved by the same methods as in the theorem.

*Corollary from the proof* 8.5. Let $n\geqslant 2$ be an integer and $X$ be a union of $n$ fibers of the Hopf fibration on the unit $\mathbb{S}^3$. Then $\operatorname{Ram}_X\mathbb{S}^3$ is CAT[1] if and only if there is no point on $\mathbb{S}^3$ at distance more than $\pi/4$ from $X$.

## 9. Case ($\mathbb{C}^2$)

*Proof of Theorem 1.2; case* ($\mathbb{C}^2$). Let us show that $\operatorname{Ram}_\Gamma\mathbb{C}^2$ is CAT[0].

First assume that $\Gamma$ is finite. Without loss of generality, we can assume that the origin is fixed by $\Gamma$. Let $L$ be the union of all the lines in $\mathbb{C}^2$ fixed by some non-identity elements of $\Gamma$. Note that $\operatorname{Ram}_\Gamma=\operatorname{Ram}_L\mathbb{C}^2$; here $\operatorname{Ram}_A X$ denotes the completion of the universal cover of $X\backslash A$.

If $L=\varnothing$ or $L$ is a single line, the statement is clear.

Set $\Theta=\mathbb{S}^3\cap L$; this is a union of Hopf circles. If the circles in $\Theta$ satisfy the conditions of Corollary 8.5, then $\operatorname{Ram}_\Theta\mathbb{S}^3$ is CAT[1]. Therefore,

$$\operatorname{Ram}_\Gamma=\operatorname{Ram}_L\mathbb{C}^2=\operatorname{Cone}(\operatorname{Ram}_\Theta\mathbb{S}^3)\in\mathrm{CAT}[0].$$

Suppose now that the conditions of Corollary 8.5 are not satisfied. Denote by $\Xi$ the projection of $\Theta$ in $\frac{1}{2}\cdot\mathbb{S}^2=\mathbb{S}^3/\mathbb{S}^1$; note that $\Xi$ is a finite set of points. In this case there is an open



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half-sphere containing all points $\Xi$. Denote by $P$ the convex hull of $\Xi$. Note that $\Xi$ and therefore $P$ are $\Gamma$-invariant sets. Therefore, the action on $\mathbb{S}^2$ is cyclic. The latter means that $L$ consists of one line.

If $\Gamma$ is infinite, we can apply the above argument to each isotropy group of $\Gamma$. We get that $\mathrm{Ram}_{\Gamma_x}$ is CAT[0] for the isotropy group $\Gamma_x$ at any point $x \in \mathbb{C}^2$. Then it remains to apply Proposition 3.7(i).

Now let us show that the inclusion $W_\Gamma \hookrightarrow \mathrm{Ram}_\Gamma$ is a homotopy equivalence. Fix a singular point $y$ in $\mathrm{Ram}_\Gamma$ and let $x$ be its projection to $\mathbb{C}^2/\Gamma$. By Proposition 3.7(ii), the link at $y$ is the same as the link of the ramification of the cone at $x$. The latter space is the ramification of $\mathbb{S}^3$ in a non-empty collection of Hopf circles, which is clearly contractible. It remains to apply Allcock's lemma 3.6. □

## 10. The counterexample

In this section we use the technique introduced above to show that the answer to the Question 2.1 is negative without additional assumptions on $\mathcal{P}$.

Theorem 10.1. *There is a positively curved spherical polyhedral space $\mathcal{P}$ such that* $\mathrm{Ram}\,\mathcal{P}$ *is not* CAT[1]. *Moreover, one can assume that $\mathcal{P}$ is homeomorphic to $\mathbb{S}^3$ and it admits an isometric $\mathbb{S}^1$-action with geodesic orbits.*

*Proof.* Consider a triangle $\Delta$ on the sphere of curvature 4 with one angle $\pi/n$ and the other two $(\pi \cdot (n+1))/(2 \cdot n) + \varepsilon$; here $n$ is a positive integer and $\varepsilon > 0$. Note that two sides of $\Delta$ are longer than $\pi/4$.

Denote by $\Lambda$ the doubling of $\Delta$. The space $\Lambda$ is a spherical polyhedral space with curvature 4; it has three singular points which correspond to the vertices of $\Delta$. Label the point with angle $(2 \cdot \pi)/n$ by $x$.

According to [Pan09, Theorem 1.8], there is a unique up to isometry polyhedral spherical space $\mathcal{P}$ with an isometric action $\mathbb{S}^1 \curvearrowright \mathcal{P}$ such that $\mathbb{S}^1$-orbits are geodesic, $\Lambda$ is isometric to the quotient space $\mathcal{P}/\mathbb{S}^1$ and the point $x$ corresponds to the orbit of multiplicity $n$, while the rest of the orbits are simple.

Note that the points in $\mathcal{P}$ on the $\mathbb{S}^1$-fiber over $x$ are regular. The distance from this fiber to the singularities of $\mathbb{S}^3$ is more than $\pi/4$; that is, $\mathcal{P}$ is not $\pi/4$-extendable. By Theorem 8.1, we conclude that $\mathrm{Ram}\,\mathcal{P}$ is not CAT[1]. □

## 11. Line arrangements

The following theorem is the main result of this section.

Theorem 11.1. *Let $\mathcal{P}$ be a non-negatively curved polyhedral space homeomorphic to $\mathbb{C}\mathrm{P}^2$ whose singularities form a complex line arrangement on $\mathbb{C}\mathrm{P}^2$. Then* $\mathrm{Ram}\,\mathcal{P}$ *is a* CAT[0] *space and the inclusion* $(\mathrm{Ram}\,\mathcal{P})^\circ \hookrightarrow \mathrm{Ram}\,\mathcal{P}$ *is a homotopy equivalence.*

It follows that all complex line arrangements in $\mathbb{C}\mathrm{P}^2$ appearing as singularities of non-negatively curved polyhedral metrics have aspherical complements. The class of such arrangements is characterized in Theorem 11.2; this class includes all the arrangements from Theorem 1.5.



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*Proof.* According to [Che86] and [Pan09], the metric on $\mathcal{P}$ is polyhedral Kähler.

First let us show that $\operatorname{Ram}\mathcal{P}$ is CAT[0]. By Theorem 3.7, it is sufficient to show that the ramification of the cone of each singular point $x$ in $\mathcal{P}$ is CAT[0].

If there are exactly two lines meeting at $x$, then the cone of $x$ is a direct product by Theorem 3.9(c), and the statement is clear.

If more than two lines meet at $x$, consider the link $\Sigma$ of the cone at $x$. According to Theorem 3.9, there is a free $\mathbb{S}^1$-action on $\Sigma$ inducing on it the structure of the Hopf fibration. The quotient $\Sigma/\mathbb{S}^1$ is a 2-sphere with spherical polyhedral metric of curvature 4 and the conical angle is at most $2\cdot\pi$ around any point. It follows from Zalgaller's theorem that $\Sigma$ is $\pi/4$-extendable. So by Theorem 8.1 $\operatorname{Ram}\Sigma$ is CAT[1].

It remains to show that $\operatorname{Ram}\mathcal{P}^\circ \hookrightarrow \operatorname{Ram}\mathcal{P}$ is a homotopy equivalence. The latter follows from Allcock's lemma 3.6 in the same way as at the end of the proof of Theorem 1.2, case $\mathbb{C}^2$. □

*Proof of Theorem 1.5.* By Theorem 11.1, it suffices to know that there is a non-negatively curved polyhedral metric on $\mathbb{C}\mathrm{P}^2$ with singularities at the line arrangement. It is shown in [Pan09] that for any arrangement of $3\cdot n$ lines that satisfies Hirzebruch's property and such that no $2\cdot n$ lines of the arrangement pass through one point, such a metric exists.

We are left with the case when at least $2\cdot n$ lines of the arrangement pass through one point, say $p$. Take any other line that does not pass through $p$. This line has at least $2\cdot n$ distinct intersections with other lines of the arrangement. So $n+1\geqslant 2\cdot n$, and we conclude that the arrangement is composed of three generic lines; hence, its complement is aspherical. □

*General line arrangements.* Let $(\ell_1,\dots,\ell_n)$ be a line arrangement in $\mathbb{C}\mathrm{P}^2$. The number of lines $\ell_i$ passing through a given point $x\in\mathbb{C}\mathrm{P}^2$ will be called the *multiplicity* of $x$, briefly $\operatorname{mult}_x$.

Let us associate to the arrangement a symmetric $n\times n$ matrix $(b_{ij})$. For $i\neq j$, put $b_{ij}=1$ if the point $x_{ij}=\ell_i\cap\ell_j$ has multiplicity 2 and $b_{ij}=0$ if its multiplicity is 3 or higher. The number $b_{jj}+1$ equals the number of points on $\ell_j$ with the multiplicity 3 and higher.

The next theorem follows from [Pan09, Theorem 1.12 and Lemma 7.9]; it reduces the existence of a non-negatively curved polyhedral Kähler metric on $\mathbb{C}\mathrm{P}^2$ with singularities at a given line arrangement to the existence of a solution of a certain system of linear equalities and inequalities.

Theorem 11.2. *Let $(\ell_1,\dots,\ell_n)$ be a line arrangement in $\mathbb{C}\mathrm{P}^2$ and $(b_{ij})$ be its matrix. There exists a non-negatively curved polyhedral Kähler metric on $\mathbb{C}\mathrm{P}^2$ with the singular locus formed by the lines $\ell_i$ if and only there are real numbers $(z_1,\dots,z_n)$ such that:*

(i) *for each $k$, we have*
$$0<z_k<1;$$

(ii) *for each $j$, we have*
$$\sum_k b_{jk}\cdot z_k=1$$
*and*
$$\sum_k z_k=3;$$

(iii) *for each point $x\in\mathbb{C}\mathrm{P}^2$ with multiplicity at least 3, we have*
$$\alpha_x=1-\frac{1}{2}\cdot\sum_{\{k\,|\,x\in\ell_k\}} z_k>0.$$



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Let us explain the geometric meaning of the above conditions. If $(z_1, \dots, z_n)$ satisfy the conditions, then there is a polyhedral Kähler metric on $\mathbb{C}\mathrm{P}^2$ with the conical angle around $\ell_i$ equal to $2 \cdot \pi \cdot (1 - z_i)$. The inequalities (i) say that conical angles are positive and less than $2\pi$.

Each of the $n$ equalities (ii) is the Gauss–Bonnet formula for the flat metric with conical singularities at a line of the arrangement; the additional equality expresses the fact that the canonical bundle of $\mathbb{C}\mathrm{P}^2$ is $O(-3)$.

The link $\Sigma_x$ at $x$ with the described metric is isometric to a 3-sphere with an $\mathbb{S}^1$-invariant metric. A straightforward calculation shows that the length of an $\mathbb{S}^1$-fiber in $\Sigma_x$ is $2 \cdot \pi \cdot \alpha_x$, where $\alpha_x$ is as in (iii). Equivalently, $\pi \cdot \alpha_x$ is the area of the quotient space $\Sigma_x/\mathbb{S}^1$.

The construction of the metric in this theorem relies on a parabolic version of Kobayshi–Hitchin correspondence established by Mochizuki [Moc06]. Surprisingly, the system of $n$ linear equations in (ii) is equivalent to the following quadratic equation. (The equation implies the system by [Pan09, Lemma 7.9] and the converse implication is a direct computation.)

$$\sum_{\{x|\mathrm{mult}_x>2\}} (\alpha_x - 1)^2 - \sum_{j=1}^{n} z_j^2 \cdot b_{jj} = \frac{3}{2}.$$

This equation is the border case of a parabolic Bogomolov–Miayoka inequality. Geometrically it expresses the second Chern class of $\mathbb{C}\mathrm{P}^2$ as a sum of contributions of singularities of the metric.

The following corollary generalizes Theorem 1.5.

Corollary 11.3. *Any line arrangement $(\ell_1, \dots, \ell_n)$ in $\mathbb{C}\mathrm{P}^2$ for which one can find positive $z_j$ satisfying equalities and inequalities of Theorem 11.2 has an aspherical complement.*

The arrangements of lines as in Theorem 1.5 satisfy the conditions in Theorem 11.2 with $z_i = 1/n$ at all $3 \cdot n$ lines of the arrangement. This is proved by an algebraic computation; see [Pan09, Corollary 7.8]. The restriction that at most $2 \cdot n - 1$ lines pass through one point follows from (iii). Therefore, the corollary above is a generalization of Theorem 11.2.

*Proof.* By Theorem 11.2, there is a non-negatively curved polyhedral metric on $\mathbb{C}\mathrm{P}^2$ with singularities along $(\ell_1, \dots, \ell_n)$ and so one can apply Theorem 11.1. □

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D. Panov dmitri.panov@kcl.ac.uk
Department of Mathematics, King's College London, Strand,
London, WC2R 2LS, UK

A. Petrunin petrunin@math.psu.edu
Penn State University Mathematics Department, University Park,
State College, PA 16802 USA



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