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CRITICAL BRANCHING AS A PURE DEATH PROCESS COMING DOWN FROM INFINITY

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Abstract

We consider the critical Galton–Watson process with overlapping generations stemming from a single founder. Assuming that both the variance of the offspring number and the average generation length are finite, we establish the convergence of the finite-dimensional distributions, conditioned on non-extinction at a remote time of observation. The limiting process is identified as a pure death process coming down from infinity. This result brings a new perspective on Vatutin's dichotomy, claiming that in the critical regime of age-dependent reproduction, an extant population either contains a large number of short-living individuals or consists of few long-living individuals.

Keywords: Galton–Watson process with overlapping generations; Bellman–Harris process; Sevastyanov process; Crump–Mode–Jagers process; convergence of finite-dimensional distributions; Vatutin's dichotomy

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1. Introduction

Consider a self-replicating system evolving in the discrete-time setting according to the following rules:

Rule 1: The system is founded by a single individual, the founder, born at time 0.

Rule 2: The founder dies at a random age L and gives a random number N of births at random ages τ_j satisfying $1 \le \tau_1 \le \ldots \le \tau_N \le L$.

Rule 3: Each new individual lives independently from others according to the same life law as the founder.

An individual that was born at time t_1 and dies at time t_2 is considered to be alive during the time interval $[t_1, t_2 - 1]$. Letting $Z(t)$ stand for the number of individuals alive at time t, we study the random dynamics of the sequence

$$Z(0) = 1, Z(1), Z(2), \ldots,$$

which is a natural extension of the well-known Galton–Watson process, or *GW process* for short; see [13]. The process $Z(\cdot)$ is the discrete-time version of what is usually called the

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Crump–Mode–Jagers process or the general branching process; see [5]. To emphasise the discrete-time setting, we call it a GW process with overlapping generations, or *GWO process* for short.

Put $b := \frac{1}{2}\mathrm{var}(N)$. This paper deals with the GWO processes satisfying

$$\mathrm{E}(N) = 1, \quad 0 < b < \infty. \tag{1}$$

The condition $\mathrm{E}(N) = 1$ says that the reproduction regime is critical, implying $\mathrm{E}(Z(t)) \equiv 1$ and making extinction inevitable, provided $b > 0$. According to [1, Chapter I.9], given (1), the survival probability

$$Q(t) := \mathrm{P}(Z(t) > 0)$$

of a GW process satisfies the asymptotic formula $tQ(t) \to b^{-1}$ as $t \to \infty$ (this was first proven in [6] under a third moment assumption). A direct extension of this classical result for the GWO processes,

$$tQ(ta) \to b^{-1}, \quad t \to \infty, \quad a := \mathrm{E}(\tau_1 + \ldots + \tau_N),$$

was obtained in [3, 4] under the conditions (1), $a < \infty$,

$$t^2\mathrm{P}(L > t) \to 0, \quad t \to \infty, \tag{2}$$

plus an additional condition. (Notice that by our definition, $a \geq 1$, and $a = 1$ if and only if $L \equiv 1$, that is, when the GWO process in question is a GW process.) Treating a as the *mean generation length* (see [5, 8]), we may conclude that the asymptotic behaviour of the critical GWO process with *short-living individuals* (see the condition (2)) is similar to that of the critical GW process, provided time is counted generation-wise.

New asymptotic patterns for the critical GWO processes are found under the assumption

$$t^2\mathrm{P}(L > t) \to d, \quad 0 \leq d < \infty, \quad t \to \infty, \tag{3}$$

which, compared to (2), allows the existence of *long-living individuals* given $d > 0$. The condition (3) was first introduced in the pioneering paper [12] dealing with the *Bellman–Harris processes*. In the current discrete-time setting, the Bellman–Harris process is a GWO process subject to two restrictions: (a) $\mathrm{P}(\tau_1 = \ldots = \tau_N = L) = 1$, so that all births occur at the moment of an individual's death, and (b) the random variables L and N are independent. For the Bellman–Harris process, the conditions (1) and (3) imply $a = \mathrm{E}(L)$, $a < \infty$, and according to [12, Theorem 3], we get

$$tQ(t) \to h, \quad t \to \infty, \qquad h := \frac{a + \sqrt{a^2 + 4bd}}{2b}. \tag{4}$$

As was shown in [11, Corollary B] (see also [7, Lemma 3.2] for an adaptation to the discrete-time setting), the relation (4) holds even for the GWO processes satisfying the conditions (1), (3), and $a < \infty$.

The main result of this paper, Theorem 1 of Section 2, considers a critical GWO process under the above-mentioned set of assumptions (1), (3), $a < \infty$, and establishes the convergence of the finite-dimensional distributions conditioned on survival at a remote time of observation. A remarkable feature of this result is that its limit process is fully described by a single parameter $c := 4bda^{-2}$, regardless of complicated mutual dependencies between the random variables τ_j, N, L.

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Our proof of Theorem 1, requiring an intricate asymptotic analysis of multi-dimensional probability generating functions, is split into two sections for the sake of readability. Section 3 presents a new proof of (4) inspired by the proof of [12]. The crucial aspect of this approach, compared to the proof of [7, Lemma 3.2], is that certain essential steps do not rely on the monotonicity of the function $Q(t)$. In Section 4, the technique of Section 3 is further developed to finish the proof of Theorem 1.

We conclude this section by mentioning the illuminating family of GWO processes called the *Sevastyanov processes* [9]. The Sevastyanov process is a generalised version of the Bellman–Harris process, with possibly dependent L and N. In the critical case, the mean generation length of the Sevastyanov process, $a = \mathrm{E}(LN)$, can be represented as

$$a = \mathrm{cov}(L, N) + \mathrm{E}(L).$$

Thus, if L and N are positively correlated, the average generation length a exceeds the average life length $\mathrm{E}(L)$.

Turning to a specific example of the Sevastyanov process, take

$$\mathrm{P}(L = t) = p_1 t^{-3} (\ln \ln t)^{-1}, \quad \mathrm{P}(N = 0 | L = t) = 1 - p_2, \quad \mathrm{P}(N = n_t | L = t) = p_2, \ t \geq 2,$$

where $n_t := \lfloor t (\ln t)^{-1} \rfloor$ and (p_1, p_2) are such that

$$\sum_{t=2}^{\infty} \mathrm{P}(L = t) = p_1 \sum_{t=2}^{\infty} t^{-3} (\ln \ln t)^{-1} = 1, \quad \mathrm{E}(N) = p_1 p_2 \sum_{t=2}^{\infty} n_t t^{-3} (\ln \ln t)^{-1} = 1.$$

In this case, for some positive constant c_1,

$$\mathrm{E}\big(N^2\big) = p_1 p_2 \sum_{t=1}^{\infty} n_t^2 t^{-3} (\ln \ln t)^{-1} < c_1 \int_2^{\infty} \frac{d(\ln t)}{(\ln t)^2 \ln \ln t} < \infty,$$

implying that the condition (1) is satisfied. Clearly, the condition (3) holds with $d = 0$. At the same time,

$$a = \mathrm{E}(NL) = p_1 p_2 \sum_{t=1}^{\infty} n_t t^{-2} (\ln \ln t)^{-1} > c_2 \int_2^{\infty} \frac{d(\ln t)}{(\ln t)(\ln \ln t)} = \infty,$$

where c_2 is a positive constant. This example demonstrates that for the GWO process, unlike for the Bellman–Harris process, the conditions (1) and (3) do not automatically imply the condition $a < \infty$.

2. The main result

Theorem 1. *For a GWO process satisfying* (1), (3) *and $a < \infty$, there holds a weak convergence of the finite-dimensional distributions*

$$(Z(ty), 0 < y < \infty | Z(t) > 0) \xrightarrow{\text{fdd}} (\eta(y), 0 < y < \infty), \quad t \to \infty.$$

The limiting process is a continuous-time pure death process $(\eta(y), 0 \leq y < \infty)$, whose evolution law is determined by a single compound parameter $c = 4bda^{-2}$, as specified next.

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The finite-dimensional distributions of the limiting process $\eta(\cdot)$ are given below in terms of the k-dimensional probability generating functions $\mathrm{E}\big(z_1^{\eta(y_1)}\cdots z_k^{\eta(y_k)}\big)$, $k\geq 1$, assuming

$$0=y_0<y_1<\ldots<y_j<1\leq y_{j+1}<\ldots<y_k<y_{k+1}=\infty,\qquad 0\leq j\leq k,\quad 0\leq z_1,\ldots,z_k<1. \tag{5}$$

Here the index j highlights the pivotal value 1 corresponding to the time of observation t of the underlying GWO process.

As will be shown in Section 4.2, if $j=0$, then

$$\mathrm{E}\big(z_1^{\eta(y_1)}\cdots z_k^{\eta(y_k)}\big)=1-\frac{1+\sqrt{1+\sum_{i=1}^{k}z_1\cdots z_{i-1}(1-z_i)\Gamma_i}}{\big(1+\sqrt{1+c}\big)y_1},\qquad \Gamma_i:=c(y_1/y_i)^2,$$

and if $j\geq 1$,

$$\mathrm{E}\big(z_1^{\eta(y_1)}\cdots z_k^{\eta(y_k)}\big)=\frac{\sqrt{1+\sum_{i=1}^{j}z_1\cdots z_{i-1}(1-z_i)\Gamma_i+cz_1\cdots z_jy_1^2}-\sqrt{1+\sum_{i=1}^{k}z_1\cdots z_{i-1}(1-z_i)\Gamma_i}}{\big(1+\sqrt{1+c}\big)y_1}.$$

In particular, for $k=1$, we have

$$\mathrm{E}\big(z^{\eta(y)}\big)=\frac{\sqrt{1+c(1-z)+czy^2}-\sqrt{1+c(1-z)}}{\big(1+\sqrt{1+c}\big)y},\qquad 0<y<1,$$
$$\mathrm{E}\big(z^{\eta(y)}\big)=1-\frac{1+\sqrt{1+c(1-z)}}{\big(1+\sqrt{1+c}\big)y},\qquad y\geq 1.$$

It follows that $\mathrm{P}(\eta(y)\geq 0)=1$ for $y>0$, and moreover, putting here first $z=1$ and then $z=0$ yields

$$\mathrm{P}(\eta(y)<\infty)=\frac{\sqrt{1+cy^2}-1}{\big(1+\sqrt{1+c}\big)y}\cdot 1_{\{0<y<1\}}+\bigg(1-\frac{2}{\big(1+\sqrt{1+c}\big)y}\bigg)\cdot 1_{\{y\geq 1\}},$$
$$\mathrm{P}(\eta(y)=0)=\frac{y-1}{y}\cdot 1_{\{y\geq 1\}},$$

implying that $\mathrm{P}(\eta(y)=\infty)>0$ for all $y>0$. In fact, letting $y\to 0$, we may set $\mathrm{P}(\eta(0)=\infty)=1$.

To demonstrate that the process $\eta(\cdot)$ is indeed a pure death process, consider the function

$$\mathrm{E}\big(z_1^{\eta(y_1)-\eta(y_2)}\cdots z_{k-1}^{\eta(y_{k-1})-\eta(y_k)}z_k^{\eta(y_k)}\big)$$

determined by

$$\mathrm{E}\big(z_1^{\eta(y_1)-\eta(y_2)}\cdots z_{k-1}^{\eta(y_{k-1})-\eta(y_k)}z_k^{\eta(y_k)}\big)=\mathrm{E}\big(z_1^{\eta(y_1)}(z_2/z_1)^{\eta(y_2)}\cdots(z_k/z_{k-1})^{\eta(y_k)}\big).$$

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This function is given by two expressions:

$$\begin{aligned} &\frac{\big(1+\sqrt{1+c}\big)y_1 - 1 - \sqrt{1+\sum_{i=1}^{k}(1-z_i)\gamma_i}}{\big(1+\sqrt{1+c}\big)y_1}, \quad \text{for } j=0,\\ &\frac{\sqrt{1+\sum_{i=1}^{j-1}(1-z_i)\gamma_i + (1-z_j)\Gamma_j + cz_j y_1^2} - \sqrt{1+\sum_{i=1}^{k}(1-z_i)\gamma_i}}{\big(1+\sqrt{1+c}\big)y_1}, \quad \text{for } j\geq 1,\end{aligned}$$

where $\gamma_i := \Gamma_i - \Gamma_{i+1}$ and $\Gamma_{k+1}=0$. Setting $k=2$, $z_1=z$, and $z_2=1$, we deduce that the function

$$\mathrm{E}\big(z^{\eta(y_1)-\eta(y_2)}; \eta(y_1)<\infty\big), \quad 0<y_1<y_2, \quad 0\leq z\leq 1, \tag{6}$$

is given by one of the following three expressions, depending on whether $j=2$, $j=1$, or $j=0$:

$$\begin{aligned} &\frac{\sqrt{1+cy_1^2+c(1-z)\big(1-(y_1/y_2)^2\big)} - \sqrt{1+c(1-z)\big(1-(y_1/y_2)^2\big)}}{\big(1+\sqrt{1+c}\big)y_1}, \quad y_2<1,\\ &\frac{\sqrt{1+cy_1^2+c(1-z)\big(1-y_1^2\big)} - \sqrt{1+c(1-z)\big(1-(y_1/y_2)^2\big)}}{\big(1+\sqrt{1+c}\big)y_1}, \quad y_1<1\leq y_2,\\ &1-\frac{1+\sqrt{1+c(1-z)\big(1-(y_1/y_2)^2\big)}}{\big(1+\sqrt{1+c}\big)y_1}, \quad 1\leq y_1.\end{aligned}$$

Since the generating function (6) is finite at $z=0$, we conclude that

$$\mathrm{P}(\eta(y_1)<\eta(y_2); \eta(y_1)<\infty)=0, \quad 0<y_1<y_2.$$

This implies

$$\mathrm{P}(\eta(y_2)\leq\eta(y_1))=1, \quad 0<y_1<y_2,$$

meaning that unless the process $\eta(\cdot)$ is sitting at the infinity state, it evolves by negative integer-valued jumps until it gets absorbed at zero.

Consider now the conditional probability generating function

$$\mathrm{E}\big(z^{\eta(y_1)-\eta(y_2)} | \eta(y_1)<\infty\big), \quad 0<y_1<y_2, \quad 0\leq z\leq 1. \tag{7}$$

In accordance with the three expressions given above for (6), the generating function (7) is specified by the following three expressions:

$$\begin{aligned} &\frac{\sqrt{1+cy_1^2+c(1-z)\big(1-(y_1/y_2)^2\big)} - \sqrt{1+c(1-z)\big(1-(y_1/y_2)^2\big)}}{\sqrt{1+cy_1^2}-1}, \quad y_2<1,\\ &\frac{\sqrt{1+cy_1^2+c(1-z)\big(1-y_1^2\big)} - \sqrt{1+c(1-z)\big(1-(y_1/y_2)^2\big)}}{\sqrt{1+cy_1^2}-1}, \quad y_1<1\leq y_2,\\ &1-\frac{\sqrt{1+c(1-z)\big(1-(y_1/y_2)^2\big)}-1}{\big(1+\sqrt{1+c}\big)y_1-2}, \quad 1\leq y_1.\end{aligned}$$

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In particular, setting $z=0$ here, we obtain

$$\mathrm{P}(\eta(y_1)-\eta(y_2)=0|\eta(y_1)<\infty)=\begin{cases}\frac{\sqrt{1+c\left(1+y_1^2-(y_1/y_2)^2\right)}-\sqrt{1+c\left(1-(y_1/y_2)^2\right)}}{\sqrt{1+cy_1^2}-1} & \text{for } 0<y_1<y_2<1,\\ \frac{\sqrt{1+c}-\sqrt{1+c\left(1-(y_1/y_2)^2\right)}}{\sqrt{1+cy_1^2}-1} & \text{for } 0<y_1<1\le y_2,\\ 1-\frac{\sqrt{1+c\left(1-(y_1/y_2)^2\right)}-1}{\left(1+\sqrt{1+c}\right)y_1-2} & \text{for } 1\le y_1<y_2.\end{cases}$$

Notice that given $0<y_1\le 1$,

$$\mathrm{P}(\eta(y_1)-\eta(y_2)=0|\eta(y_1)<\infty)\to 0,\qquad y_2\to\infty,$$

which is expected because of $\eta(y_1)\ge\eta(1)\ge 1$ and $\eta(y_2)\to 0$ as $y_2\to\infty$.

The random times

$$T=\sup\{u:\eta(u)=\infty\},\qquad T_0=\inf\{u:\eta(u)=0\}$$

are major characteristics of a trajectory of the limit pure death process. Since

$$\mathrm{P}(T\le y)=\mathrm{E}\left(z^{\eta(y)}\right)\Big|_{z=1},\qquad \mathrm{P}(T_0\le y)=\mathrm{E}\left(z^{\eta(y)}\right)\Big|_{z=0},$$

in accordance with the above-mentioned formulas for $\mathrm{E}\left(z^{\eta(y)}\right)$, we get the following marginal distributions:

$$\mathrm{P}(T\le y)=\frac{\sqrt{1+cy^2}-1}{\left(1+\sqrt{1+c}\right)y}\cdot 1_{\{0\le y<1\}}+\left(1-\frac{2}{\left(1+\sqrt{1+c}\right)y}\right)\cdot 1_{\{y\ge 1\}},$$
$$\mathrm{P}(T_0\le y)=\frac{y-1}{y}\cdot 1_{\{y\ge 1\}}.$$

The distribution of T_0 is free from the parameter c and has the Pareto probability density function

$$f_0(y)=y^{-2}1_{\{y>1\}}.$$

In the special case (2), that is, when (3) holds with $d=0$, we have $c=0$ and $\mathrm{P}(T=T_0)=1$. If $d>0$, then $T\le T_0$, and the distribution of T has the following probability density function:

$$f(y)=\begin{cases}\frac{1}{\left(1+\sqrt{1+c}\right)y^2}\left(1-\frac{1}{\sqrt{1+cy^2}}\right) & \text{for } 0\le y<1,\\ \frac{2}{\left(1+\sqrt{1+c}\right)y^2} & \text{for } y\ge 1,\end{cases}$$

which has a positive jump at $y=1$ of size $f(1)-f(1-)=(1+c)^{-1/2}$; see Figure 1. Observe that $\frac{f(1-)}{f(1)}\to\frac{1}{2}$ as $c\to\infty$.

Intuitively, the limiting pure death process counts the long-living individuals in the GWO process, that is, those individuals whose life length is of order t. These long-living individuals may have descendants, however none of them would live long enough to be detected by the

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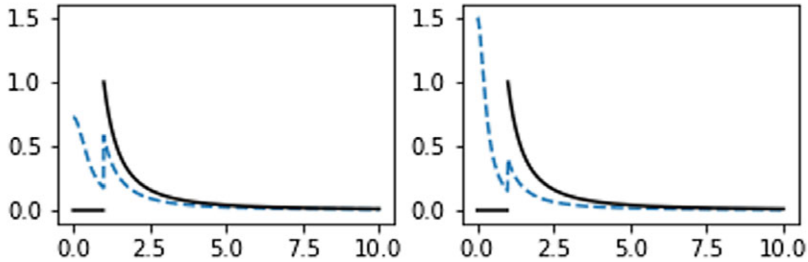

FIGURE 1. The dashed line is the probability density function of T; the solid line is the probability density function of T_0. The left panel illustrates the case $c = 5$, and the right panel illustrates the case $c = 15$.

finite-dimensional distributions at the relevant time scale, see Lemma 2 below. Theorem 1 suggests a new perspective on Vatutin's dichotomy (see [12]), claiming that the long-term survival of a critical age-dependent branching process is due to either a large number of short-living individuals or a small number of long-living individuals. In terms of the random times $T \le T_0$, Vatutin's dichotomy discriminates between two possibilities: if $T > 1$, then $\eta(1) = \infty$, meaning that the GWO process has survived thanks to a large number of individuals, while if $T \le 1 < T_0$, then $1 \le \eta(1) < \infty$, meaning that the GWO process has survived thanks to a small number of individuals.

3. Proof that $tQ(t) \to h$

This section deals with the survival probability of the critical GWO process

$$Q(t) = 1 - P(t), \quad P(t) := \mathrm{P}(Z(t) = 0).$$

By its definition, the GWO process can be represented as the sum

$$Z(t) = 1_{\{L>t\}} + \sum\nolimits_{j=1}^{N} Z_j\left(t - \tau_j\right), \quad t = 0, 1, \ldots, \tag{8}$$

involving N independent daughter processes $Z_j(\cdot)$ generated by the founder individual at the birth times τ_j, $j = 1, \ldots, N$ (here it is assumed that $Z_j(t) = 0$ for all negative t). The branching property (8) implies the relation

$$1_{\{Z(t)=0\}} = 1_{\{L\le t\}} \prod\nolimits_{j=1}^{N} 1_{\left\{Z_j\left(t-\tau_j\right)=0\right\}},$$

which says that the GWO process goes extinct by the time t if, on one hand, the founder is dead at time t and, on the other hand, all daughter processes are extinct by the time t. After taking expectations of both sides, we can write

$$P(t) = \mathrm{E}\left(\prod\nolimits_{j=1}^{N} P\left(t - \tau_j\right); L \le t\right). \tag{9}$$

As shown next, this nonlinear equation for $P(\cdot)$ implies the asymptotic formula (4) under the conditions (1), (3), and $a < \infty$.

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3.1. Outline of the proof of (4)

We start by stating four lemmas and two propositions. Let

$$\Phi(z) := \mathrm{E}\big((1-z)^N - 1 + Nz\big), \tag{10}$$

$$W(t) := \left(1 - ht^{-1}\right)^N + Nht^{-1} - \sum\nolimits_{j=1}^N Q\left(t-\tau_j\right) - \prod\nolimits_{j=1}^N P\left(t-\tau_j\right), \tag{11}$$

$$D(u, t) := \mathrm{E}\Big(1 - \prod\nolimits_{j=1}^N P\left(t-\tau_j\right); u < L \le t\Big) + \mathrm{E}\big(\left(1-ht^{-1}\right)^N - 1 + Nht^{-1}; L > u\big), \tag{12}$$

$$\mathrm{E}_u(X) := \mathrm{E}(X; L \le u), \tag{13}$$

where $0 \le z \le 1$, $u > 0$, $t \ge h$, and X is an arbitrary random variable.

Lemma 1. *Given (10), (11), (12), and (13), assume that* $0 < u \le t$ *and* $t \ge h$. *Then*

$$\Phi\left(ht^{-1}\right) = \mathrm{P}(L > t) + \mathrm{E}_u\left(\sum\nolimits_{j=1}^N Q\left(t-\tau_j\right)\right) - Q(t) + \mathrm{E}_u(W(t)) + D(u, t).$$

Lemma 2. *If (1) and (3) hold, then* $\mathrm{E}(N; L > ty) = o\left(t^{-1}\right)$ *as* $t \to \infty$ *for any fixed* $y > 0$.

Lemma 3. *If (1), (3), and* $a < \infty$ *hold, then for any fixed* $0 < y < 1$,

$$\mathrm{E}_{ty}\left(\sum\nolimits_{j=1}^N \left(\frac{1}{t-\tau_j} - \frac{1}{t}\right)\right) \sim at^{-2}, \quad t \to \infty.$$

Lemma 4. *Let* $k \ge 1$. *If* $0 \le f_j, g_j \le 1$ *for* $j = 1, \ldots, k$, *then*

$$\prod\nolimits_{j=1}^k (1-g_j) - \prod\nolimits_{j=1}^k (1-f_j) = \sum\nolimits_{j=1}^k (f_j - g_j) r_j,$$

where $0 \le r_j \le 1$ *and*

$$1 - r_j = \sum\nolimits_{i=1}^{j-1} g_i + \sum\nolimits_{i=j+1}^k f_i - R_j,$$

for some $R_j \ge 0$. *If moreover* $f_j \le q$ *and* $g_j \le q$ *for some* $q > 0$, *then*

$$1 - r_j \le (k-1)q, \qquad R_j \le kq, \qquad R_j \le k^2 q^2.$$

Proposition 1. *If (1), (3), and* $a < \infty$ *hold, then* $\limsup_{t\to\infty} tQ(t) < \infty$.

Proposition 2. *If (1), (3), and* $a < \infty$ *hold, then* $\liminf_{t\to\infty} tQ(t) > 0$.

According to these two propositions, there exists a triplet of positive numbers (q_1, q_2, t_0) such that

$$q_1 \le tQ(t) \le q_2, \quad t \ge t_0, \quad 0 < q_1 < h < q_2 < \infty. \tag{14}$$

The claim $tQ(t) \to h$ is derived using (14) by accurately removing asymptotically negligible terms from the relation for $Q(\cdot)$ stated in Lemma 1, after setting $u = ty$ with a fixed $0 < y < 1$, and then choosing a sufficiently small y. In particular, as an intermediate step, we will show that

$$Q(t) = \mathrm{E}_{ty}\left(\sum\nolimits_{j=1}^N Q\left(t-\tau_j\right)\right) + \mathrm{E}_{ty}(W(t)) - aht^{-2} + o\left(t^{-2}\right), \quad t \to \infty. \tag{15}$$

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Then, restating our goal as $\phi(t)\to 0$ in terms of the function $\phi(t)$, defined by

$$Q(t)=\frac{h+\phi(t)}{t},\qquad t\geq 1, \tag{16}$$

we rewrite (15) as

$$\frac{h+\phi(t)}{t}=\mathrm{E}_{ty}\Big(\sum\nolimits_{j=1}^{N}\frac{h+\phi\left(t-\tau_j\right)}{t-\tau_j}\Big)+\mathrm{E}_{ty}(W(t))-aht^{-2}+o\big(t^{-2}\big),\qquad t\to\infty. \tag{17}$$

It turns out that the three terms involving h, outside $W(t)$, effectively cancel each other, yielding

$$\frac{\phi(t)}{t}=\mathrm{E}_{ty}\Big(\sum\nolimits_{j=1}^{N}\frac{\phi\left(t-\tau_j\right)}{t-\tau_j}+W(t)\Big)+o\big(t^{-2}\big),\qquad t\to\infty. \tag{18}$$

Treating $W(t)$ in terms of Lemma 4 yields

$$\phi(t)=\mathrm{E}_{ty}\Big(\sum\nolimits_{j=1}^{N}\phi\left(t-\tau_j\right)r_j(t)\frac{t}{t-\tau_j}\Big)+o\big(t^{-1}\big), \tag{19}$$

where $r_j(t)$ is a counterpart of r_j in Lemma 4. To derive from here the desired convergence $\phi(t)\to 0$, we will adapt a clever trick from Chapter 9.1 of [10], which was further developed in [12] for the Bellman–Harris process, with possibly infinite var(N). Define a non-negative function $m(t)$ by

$$m(t):=\ |\phi(t)|\ \ln t,\qquad t\geq 2. \tag{20}$$

Multiplying (19) by $\ln t$ and using the triangle inequality, we obtain

$$m(t)\leq \mathrm{E}_{ty}\Big(\sum\nolimits_{j=1}^{N}m\left(t-\tau_j\right)r_j(t)\frac{t\ln t}{\left(t-\tau_j\right)\ln\left(t-\tau_j\right)}\Big)+v(t), \tag{21}$$

where $v(t)\geq 0$ and $v(t)=o(t^{-1}\ln t)$ as $t\to\infty$. It will be shown that this leads to $m(t)=o(\ln t)$, thereby concluding the proof of (4).

3.2. Proof of lemmas and propositions

Proof of Lemma 1. For $0<u\leq t$, the relations (9) and (13) give

$$P(t)=\mathrm{E}_u\Big(\prod\nolimits_{j=1}^{N}P\left(t-\tau_j\right)\Big)+\mathrm{E}\Big(\prod\nolimits_{j=1}^{N}P\left(t-\tau_j\right);u<L\leq t\Big). \tag{22}$$

On the other hand, for $t\geq h$,

$$\Phi\big(ht^{-1}\big)\overset{(10)}{=}\mathrm{E}_u\Big(\big(1-ht^{-1}\big)^N-1+Nht^{-1}\Big)+\mathrm{E}\Big(\big(1-ht^{-1}\big)^N-1+Nht^{-1};L>u\Big).$$

Adding the latter relation to

$$1=\mathrm{P}(L\leq u)+\mathrm{P}(L>t)+\mathrm{P}(u<L\leq t)$$

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and subtracting (22) from the sum, we get

$$\Phi\left(ht^{-1}\right) + Q(t) = \mathrm{E}_u\left(\left(1 - ht^{-1}\right)^N + Nht^{-1} - \prod_{j=1}^N P\left(t - \tau_j\right)\right) + \mathrm{P}(L > t) + D(u, t),$$

with $D(u, t)$ defined by (12). After a rearrangement, we obtain the statement of the lemma. □

Proof of Lemma 2. For any fixed $\epsilon > 0$,

$$\begin{aligned}\mathrm{E}(N; L > t) &= \mathrm{E}(N; N \le t\epsilon, L > t) + \mathrm{E}\big(N; 1 < N(t\epsilon)^{-1}, L > t\big)\\ &\le t\epsilon \mathrm{P}(L > t) + (t\epsilon)^{-1}\mathrm{E}\big(N^2; L > t\big).\end{aligned}$$

Thus, by (1) and (3),

$$\limsup_{t\to\infty} (t\mathrm{E}(N; L > t)) \le d\epsilon,$$

and the assertion follows as $\epsilon \to 0$. □

Proof of Lemma 3. For $t = 1, 2, \ldots$ and $y > 0$, put

$$B_t(y) := t^2\, \mathrm{E}_{ty}\left(\sum_{j=1}^N \left(\frac{1}{t - \tau_j} - \frac{1}{t}\right)\right) - a.$$

For any $0 < u < ty$, using

$$a = \mathrm{E}_u(\tau_1 + \ldots + \tau_N) + A_u, \quad A_u := \mathrm{E}(\tau_1 + \ldots + \tau_N; L > u),$$

we get

$$\begin{aligned}B_t(y) &= \mathrm{E}_u\left(\sum_{j=1}^N \frac{t}{t - \tau_j}\tau_j\right) + \mathrm{E}\left(\sum_{j=1}^N \frac{t}{t - \tau_j}\tau_j\,; u < L \le ty\right)\\ &\quad - \mathrm{E}_u(\tau_1 + \ldots + \tau_N) - A_u\\ &= \mathrm{E}\left(\sum_{j=1}^N \frac{\tau_j}{1 - \tau_j/t}; u < L \le ty\right) + \mathrm{E}_u\left(\sum_{j=1}^N \frac{\tau_j^2}{t - \tau_j}\right) - A_u.\end{aligned}$$

For the first term on the right-hand side, we have $\tau_j \le L \le ty$, so that

$$\mathrm{E}\left(\sum_{j=1}^N \frac{\tau_j}{1 - \tau_j/t}; u < L \le ty\right) \le (1 - y)^{-1}A_u.$$

For the second term, $\tau_j \le L \le u$ and therefore

$$\mathrm{E}_u\left(\sum_{j=1}^N \frac{\tau_j^2}{t - \tau_j}\right) \le \frac{u^2}{t - u}\mathrm{E}_u(N) \le \frac{u^2}{t - u}.$$

This yields

$$-A_u \le B_t(y) \le (1 - y)^{-1}A_u + \frac{u^2}{t - u}, \quad 0 < u < ty < t,$$

implying

$$-A_u \le \liminf_{t\to\infty} B_t(y) \le \limsup_{t\to\infty} B_t(y) \le (1-y)^{-1}A_u.$$

Since $A_u \to 0$ as $u \to \infty$, we conclude that $B_t(y) \to 0$ as $t \to \infty$. □

Proof of Lemma 4. Let

$$r_j := \big(1-g_1\big)\dots\big(1-g_{j-1}\big)\big(1-f_{j+1}\big)\dots\big(1-f_k\big), \quad 1 \le j \le k.$$

Then $0 \le r_j \le 1$, and the first stated equality is obtained by telescopic summation of

$$\begin{aligned}
\big(1-g_1\big)\prod\nolimits_{j=2}^k\big(1-f_j\big) - \prod\nolimits_{j=1}^k\big(1-f_j\big) &= (f_1-g_1)r_1,\\
\big(1-g_1\big)\big(1-g_2\big)\prod\nolimits_{j=3}^k\big(1-f_j\big) - \big(1-g_1\big)\prod\nolimits_{j=2}^k\big(1-f_j\big) &= (f_2-g_2)r_2, \dots,\\
\prod\nolimits_{j=1}^k\big(1-g_j\big) - \prod\nolimits_{j=1}^{k-1}\big(1-g_j\big)\big(1-f_k\big) &= (f_k-g_k)r_k.
\end{aligned}$$

The second stated equality is obtained with

$$\begin{aligned}
R_j := &\sum_{i=j+1}^{k} f_i\big(1-\big(1-f_{j+1}\big)\dots\big(1-f_{i-1}\big)\big)\\
&+\sum_{i=1}^{j-1} g_i\big(1-\big(1-g_1\big)\dots(1-g_{i-1})\big(1-f_{j+1}\big)\dots\big(1-f_k\big)\big),
\end{aligned}$$

by performing telescopic summation of

$$\begin{aligned}
1-\big(1-f_{j+1}\big) &= f_{j+1},\\
\big(1-f_{j+1}\big)-\big(1-f_{j+1}\big)\big(1-f_{j+2}\big) &= f_{j+2}\big(1-f_{j+1}\big), \dots,\\
\prod\nolimits_{i=j+1}^{k-1}(1-f_i) - \prod\nolimits_{i=j+1}^{k}(1-f_i) &= f_k\prod\nolimits_{i=j+1}^{k-1}(1-f_i),\\
\prod\nolimits_{i=j+1}^{k}(1-f_i) - (1-g_1)\prod\nolimits_{i=j+1}^{k}(1-f_i) &= g_1\prod\nolimits_{i=j+1}^{k}(1-f_i), \dots,\\
\prod\nolimits_{i=1}^{j-2}(1-g_i)\prod\nolimits_{i=j+1}^{k}(1-f_i) - \prod\nolimits_{i=1}^{j-1}(1-g_i)\prod\nolimits_{i=j+1}^{k}(1-f_i) &= g_{j-1}\prod\nolimits_{i=1}^{j-2}(1-g_i)\prod\nolimits_{i=j+1}^{k}(1-f_i).
\end{aligned}$$

By the above definition of R_j, we have $R_j \ge 0$. Furthermore, given $f_j \le q$ and $g_j \le q$, we get

$$R_j \le \sum\nolimits_{i=1}^{j-1} g_i + \sum\nolimits_{i=j+1}^{k} f_i \le (k-1)q.$$

It remains to observe that

$$1-r_j \le 1-(1-q)^{k-1} \le (k-1)q,$$

and from the definition of R_j,

$$R_j \le q\sum\nolimits_{i=1}^{k-j-1}(1-(1-q)^i) + q\sum\nolimits_{i=1}^{j-1}\big(1-(1-q)^{k-j+i-1}\big) \le q^2\sum\nolimits_{i=1}^{k-2} i \le k^2q^2.$$

□

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Proof of Proposition 1. By the definition of $\Phi(\cdot)$, we have

$$\Phi(Q(t)) + P(t) = \mathrm{E}_u\big(P(t)^N\big) + \mathrm{P}(L > u) - \mathrm{E}\big(1 - P(t)^N; L > u\big),$$

for any $0 < u < t$. This and (22) yield

$$\Phi(Q(t)) = \mathrm{E}_u\bigg(P(t)^N - \prod\nolimits_{j=1}^N P\left(t-\tau_j\right)\bigg) + \mathrm{P}(L > u) - \mathrm{E}\big(1 - P(t)^N; L > u\big) - \mathrm{E}\bigg(\prod\nolimits_{j=1}^N P\left(t-\tau_j\right); u < L \le t\bigg). \tag{23}$$

We therefore obtain the upper bound

$$\Phi(Q(t)) \le \mathrm{E}_u\bigg(P(t)^N - \prod\nolimits_{j=1}^N P\left(t-\tau_j\right)\bigg) + \mathrm{P}(L > u),$$

which together with Lemma 4 and the monotonicity of $Q(\cdot)$ implies

$$\Phi(Q(t)) \le \mathrm{E}_u\bigg(\sum\nolimits_{j=1}^N \left(Q\left(t-\tau_j\right) - Q(t)\right)\bigg) + \mathrm{P}(L > u). \tag{24}$$

Borrowing an idea from [11], suppose to the contrary that

$$t_n := \min\{t{:}tQ(t) \ge n\}$$

is finite for any natural n. It follows that

$$Q(t_n) \ge \frac{n}{t_n}, \qquad Q(t_n - u) < \frac{n}{t_n - u}, \qquad 1 \le u \le t_n - 1.$$

Putting $t = t_n$ into (24) and using the monotonicity of $\Phi(\cdot)$, we find

$$\Phi\big(nt_n^{-1}\big) \le \Phi(Q(t_n)) \le \mathrm{E}_u\bigg(\sum\nolimits_{j=1}^N \left(\frac{n}{t_n - \tau_j} - \frac{n}{t_n}\right)\bigg) + \mathrm{P}(L > u).$$

Setting $u = t_n/2$ here and applying Lemma 3 together with (3), we arrive at the relation

$$\Phi\big(nt_n^{-1}\big) = O\big(nt_n^{-2}\big), \quad n \to \infty.$$

Observe that under the condition (1), the L'Hospital rule gives

$$\Phi(z) \sim bz^2, \quad z \to 0. \tag{25}$$

The resulting contradiction, $n^2 t_n^{-2} = O\big(nt_n^{-2}\big)$ as $n \to \infty$, finishes the proof of the proposition. □

Proof of Proposition 2. The relation (23) implies

$$\Phi(Q(t)) \ge \mathrm{E}_u\bigg(P(t)^N - \prod\nolimits_{j=1}^N P\left(t-\tau_j\right)\bigg) - \mathrm{E}\big(1 - P(t)^N; L > u\big).$$

By Lemma 4,

$$P(t)^N - \prod\nolimits_{j=1}^N P\left(t-\tau_j\right) = \sum_{j=1}^N \left(Q\left(t-\tau_j\right) - Q(t)\right) r_j^*(t),$$

where $0 \le r_j^*(t) \le 1$ is a counterpart of the term r_j in Lemma 4. By the monotonicity of $P(\cdot)$, we have, again referring to Lemma 4,

$$1 - r_j^*(t) \le (N-1)Q(t-L).$$

Thus, for $0 < y < 1$,

$$\Phi(Q(t)) \ge \mathrm{E}_{ty}\left(\sum_{j=1}^{N} \left(Q\left(t-\tau_j\right) - Q(t)\right) r_j^*(t)\right) - \mathrm{E}\left(1 - P(t)^N; L > ty\right). \tag{26}$$

The assertion $\liminf_{t\to\infty} tQ(t) > 0$ is proven by contradiction. Assume that $\liminf_{t\to\infty} tQ(t) = 0$, so that

$$t_n := \min\left\{t{:}tQ(t) \le n^{-1}\right\}$$

is finite for any natural n. Plugging $t = t_n$ into (26) and using

$$Q(t_n) \le \frac{1}{nt_n}, \quad Q(t_n - u) - Q(t_n) \ge \frac{1}{n(t_n - u)} - \frac{1}{nt_n}, \quad 1 \le u \le t_n - 1,$$

we get

$$\Phi\left(\frac{1}{nt_n}\right) \ge n^{-1}\mathrm{E}_{t_ny}\left(\sum\nolimits_{j=1}^{N} \left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n}\right) r_j^*(t_n)\right) - \frac{1}{nt_n}\mathrm{E}(N; L > t_ny).$$

Given $L \le ty$, we have

$$1 - r_j^*(t) \le NQ(t(1-y)) \le N\frac{q_2}{t(1-y)},$$

where the second inequality is based on the already proven part of (14). Therefore,

$$\mathrm{E}_{t_ny}\left(\sum\nolimits_{j=1}^{N} \left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n}\right)\left(1 - r_j^*(t_n)\right)\right) \le \frac{q_2 y}{t_n^2(1-y)^2}\mathrm{E}\left(N^2\right),$$

and we derive

$$nt_n^2\Phi\left(\frac{1}{nt_n}\right) \ge t_n^2\mathrm{E}_{t_ny}\left(\sum\nolimits_{j=1}^{N} \left(\frac{1}{t_n - \tau_j} - \frac{1}{t_n}\right)\right) - \frac{\mathrm{E}\left(N^2\right)q_2 y}{(1-y)^2} - t_n\mathrm{E}(N; L > t_ny).$$

Sending $n \to \infty$ and applying (25), Lemma 2, and Lemma 3, we arrive at the inequality

$$0 \ge a - yq_2\mathrm{E}\left(N^2\right)(1-y)^{-2}, \quad 0 < y < 1,$$

which is false for sufficiently small y. □

3.3. Proof of (18) and (19)

Fix an arbitrary $0 < y < 1$. Lemma 1 with $u = ty$ gives

$$\Phi\left(ht^{-1}\right) = \mathrm{P}(L > t) + \mathrm{E}_{ty}\left(\sum\nolimits_{j=1}^{N} Q\left(t - \tau_j\right)\right) - Q(t) + \mathrm{E}_{ty}(W(t)) + D(ty, t). \tag{27}$$

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Let us show that

$$D(ty,t)=o\big(t^{-2}\big),\quad t\to\infty. \tag{28}$$

Using Lemma 2 and (14), we find that for an arbitrarily small $\epsilon>0$,

$$\mathrm{E}\Big(1-\prod_{j=1}^{N}P\left(t-\tau_j\right);ty<L\le t(1-\epsilon)\Big)=o\big(t^{-2}\big),\quad t\to\infty.$$

On the other hand,

$$\mathrm{E}\Big(1-\prod_{j=1}^{N}P\left(t-\tau_j\right);t(1-\epsilon)<L\le t\Big)\le \mathrm{P}(t(1-\epsilon)<L\le t),$$

so that in view of (3),

$$\mathrm{E}\Big(1-\prod_{j=1}^{N}P\left(t-\tau_j\right);ty<L\le t\Big)=o\big(t^{-2}\big),\quad t\to\infty.$$

This, (12), and Lemma 2 imply (28).

Observe that

$$bh^2=ah+d. \tag{29}$$

Combining (27), (28), and

$$\mathrm{P}(L>t)-\Phi\big(ht^{-1}\big)\overset{(3)(25)}{=}dt^{-2}-bh^2t^{-2}+o\big(t^{-2}\big)\overset{(29)}{=}-aht^{-2}+o\big(t^{-2}\big),\quad t\to\infty,$$

we derive (15), which in turn gives (17). The latter implies (18) since by Lemmas 2 and 4,

$$\mathrm{E}_{ty}\Big(\sum_{j=1}^{N}\frac{h}{t-\tau_j}\Big)-\frac{h}{t}=\mathrm{E}_{ty}\Big(\sum_{j=1}^{N}\Big(\frac{h}{t-\tau_j}-\frac{h}{t}\Big)\Big)-ht^{-1}\mathrm{E}(N;L>ty)=aht^{-2}+o\big(t^{-2}\big).$$

Turning to the proof of (19), observe that the random variable

$$W(t)=\big(1-ht^{-1}\big)^N-\prod_{j=1}^{N}\Big(1-\frac{h+\phi\left(t-\tau_j\right)}{t-\tau_j}\Big)+\sum_{j=1}^{N}\Big(\frac{h}{t}-\frac{h+\phi\left(t-\tau_j\right)}{t-\tau_j}\Big)$$

can be represented in terms of Lemma 4 as

$$\begin{aligned}W(t)&=\prod_{j=1}^{N}(1-f_j(t))-\prod_{j=1}^{N}(1-g_j(t))+\sum_{j=1}^{N}(f_j(t)-g_j(t))\\&=\sum_{j=1}^{N}(1-r_j(t))(f_j(t)-g_j(t)),\end{aligned}$$

by assigning

$$f_j(t):=\ ht^{-1},\quad g_j(t):=\frac{h+\phi\left(t-\tau_j\right)}{t-\tau_j}. \tag{30}$$

Here $0\le r_j(t)\le 1$, and for sufficiently large t,

$$1-r_j(t)\overset{(14)}{\le}Nq_2t^{-1}. \tag{31}$$

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After plugging into (18) the expression

$$W(t)=\sum_{j=1}^{N}\left(\frac{h}{t}-\frac{h}{t-\tau_j}\right)(1-r_j(t))-\sum_{j=1}^{N}\frac{\phi\left(t-\tau_j\right)}{t-\tau_j}(1-r_j(t)),$$

we get

$$\frac{\phi(t)}{t}=\mathrm{E}_{ty}\Big(\sum_{j=1}^{N}\frac{\phi\left(t-\tau_j\right)}{t-\tau_j}r_j(t)\Big)+\mathrm{E}_{ty}\Big(\sum_{j=1}^{N}\Big(\frac{h}{t-\tau_j}-\frac{h}{t}\Big)(1-r_j(t))\Big)+o\big(t^{-2}\big),\quad t\to\infty.$$

The latter expectation is non-negative, and for an arbitrary $\epsilon>0$, it has the following upper bound:

$$\mathrm{E}_{ty}\Big(\sum_{j=1}^{N}\Big(\frac{h}{t-\tau_j}-\frac{h}{t}\Big)(1-r_j(t))\Big)\overset{(31)}{\leq} q_2\epsilon\mathrm{E}_{ty}\Big(\sum_{j=1}^{N}\Big(\frac{h}{t-\tau_j}-\frac{h}{t}\Big)\Big)+\frac{q_2h}{(1-y)t^2}\mathrm{E}\big(N^2;N>t\epsilon\big).$$

Thus, in view of Lemma 3,

$$\frac{\phi(t)}{t}=\mathrm{E}_{ty}\Big(\sum_{j=1}^{N}\frac{\phi\left(t-\tau_j\right)}{t-\tau_j}r_j(t)\Big)+o\big(t^{-2}\big),\quad t\to\infty.$$

Multiplying this relation by t, we arrive at (19).

3.4. Proof of $\phi(t)\to 0$

Recall (20). If the non-decreasing function

$$M(t):=\max_{1\leq j\leq t}m(j)$$

is bounded from above, then $\phi(t)=O\big(\frac{1}{\ln t}\big)$, proving that $\phi(t)\to 0$ as $t\to\infty$. If $M(t)\to\infty$ as $t\to\infty$, then there is an integer-valued sequence $0<t_1<t_2<\ldots$, such that the sequence $M_n:=M(t_n)$ is strictly increasing and converges to infinity. In this case,

$$m(t)\leq M_{n-1}<M_n,\quad 1\leq t<t_n,\quad m(t_n)=M_n,\quad n\geq 1. \tag{32}$$

Since $|\phi(t)|\leq\frac{M_n}{\ln t_n}$ for $t_n\leq t<t_{n+1}$, to finish the proof of $\phi(t)\to 0$, it remains to verify that

$$M_n=o(\ln t_n),\quad n\to\infty. \tag{33}$$

Fix an arbitrary $y\in(0,\,1)$. Putting $t=t_n$ in (21) and using (32), we find

$$M_n\leq M_n\mathrm{E}_{t_ny}\Big(\sum_{j=1}^{N}r_j(t_n)\frac{t_n\ln t_n}{\left(t_n-\tau_j\right)\ln\left(t_n-\tau_j\right)}\Big)+\big(t_n^{-1}\ln t_n\big)o_n.$$

Here and elsewhere, o_n stands for a non-negative sequence such that $o_n\to 0$ as $n\to\infty$. In different formulas, the sign o_n represents different such sequences. Since

$$0\leq\frac{t\ln t}{(t-u)\ln(t-u)}-1\leq\frac{u(1+\ln t)}{(t-u)\ln(t-u)},\quad 0\leq u<t-1,$$

and $r_j(t_n)\in[0,\,1]$, it follows that

$$M_n-M_n\mathrm{E}_{t_ny}\Big(\sum_{j=1}^{N}r_j(t_n)\Big)\leq M_n\mathrm{E}_{t_ny}\Big(\sum_{j=1}^{N}\frac{\tau_j(1+\ln t_n)}{t_n(1-y)\ln(t_n(1-y))}\Big)+\big(t_n^{-1}\ln t_n\big)o_n.$$

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Recalling that $a = \mathrm{E}(\sum_{j=1}^{N} \tau_j)$, observe that

$$\mathrm{E}_{t_n y}\bigg(\sum\nolimits_{j=1}^{N} \frac{\tau_j(1+\ln t_n)}{t_n(1-y)\ln(t_n(1-y))}\bigg) \le \frac{a(1+\ln t_n)}{t_n(1-y)\ln(t_n(1-y))} = \big(a(1-y)^{-1}+o_n\big)t_n^{-1}.$$

Combining the last two relations, we conclude

$$M_n\mathrm{E}_{t_n y}\bigg(\sum\nolimits_{j=1}^{N} (1-r_j(t_n))\bigg) \le a(1-y)^{-1}t_n^{-1}M_n + t_n^{-1}(M_n+\ln t_n)o_n. \tag{34}$$

Now it is time to unpack the term $r_j(t)$. By Lemma 4 with (30),

$$1-r_j(t) = \sum_{i=1}^{j-1} \frac{h+\phi(t-\tau_i)}{t-\tau_i} + (N-j)\frac{h}{t} - R_j(t),$$

where, provided $\tau_j \le ty$,

$$0 \le R_j(t) \le Nq_2t^{-1}(1-y)^{-1}, \quad R_j(t) \le N^2q_2^2t^{-2}(1-y)^{-2}, \quad t > t^*,$$

for a sufficiently large t^*. This allows us to rewrite (34) in the form

$$\begin{aligned} &M_n\mathrm{E}_{t_n y}\bigg(\sum\nolimits_{j=1}^{N}\bigg(\sum_{i=1}^{j-1}\frac{h+\phi(t_n-\tau_i)}{t_n-\tau_i}+(N-j)\frac{h}{t_n}\bigg)\bigg) \\ &\qquad \le M_n\mathrm{E}_{t_n y}\bigg(\sum\nolimits_{j=1}^{N} R_j(t_n)\bigg) + a(1-y)^{-1}t_n^{-1}M_n + t_n^{-1}(M_n+\ln t_n)o_n. \end{aligned}$$

To estimate the last expectation, observe that if $\tau_j \le ty$, then for any $\epsilon > 0$,

$$R_j(t) \le Nq_2t^{-1}(1-y)^{-1}1_{\{N>t\epsilon\}} + N^2q_2^2t^{-2}(1-y)^{-2}1_{\{N\le t\epsilon\}}, \quad t > t^*,$$

implying that for sufficiently large n,

$$\mathrm{E}_{t_n y}\bigg(\sum\nolimits_{j=1}^{N} R_j(t_n)\bigg) \le q_2t_n^{-1}(1-y)^{-1}\mathrm{E}\big(N^2; N > t_n\epsilon\big) + q_2^2\epsilon t_n^{-1}(1-y)^{-2}\mathrm{E}\big(N^2\big),$$

so that

$$\begin{aligned} &M_n\mathrm{E}_{t_n y}\bigg(\sum\nolimits_{j=1}^{N}\bigg(\sum\nolimits_{i=1}^{j-1}\frac{h+\phi(t_n-\tau_i)}{t_n-\tau_i}+(N-j)\frac{h}{t_n}\bigg)\bigg) \\ &\qquad \le a(1-y)^{-1}t_n^{-1}M_n + t_n^{-1}(M_n+\ln t_n)o_n. \end{aligned}$$

Since

$$\sum\nolimits_{j=1}^{N}\sum\nolimits_{i=1}^{j-1}\bigg(\frac{h}{t_n-\tau_i}-\frac{h}{t_n}\bigg) \ge 0,$$

we obtain

$$\begin{aligned} &M_n\mathrm{E}_{t_n y}\bigg(\sum\nolimits_{j=1}^{N}\bigg(\sum_{i=1}^{j-1}\frac{\phi(t_n-\tau_i)}{t_n-\tau_i}+(N-1)\frac{h}{t_n}\bigg)\bigg) \\ &\qquad \le a(1-y)^{-1}t_n^{-1}M_n + t_n^{-1}(M_n+\ln t_n)o_n. \end{aligned}$$

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By (16) and (14), we have $\phi(t) \geq q_1 - h$ for $t \geq t_0$. Thus, for $\tau_j \leq L \leq t_n y$ and sufficiently large n,

$$\frac{\phi(t_n - \tau_i)}{t_n - \tau_i} \geq \frac{q_1 - h}{t_n(1-y)}.$$

This gives

$$\sum\nolimits_{j=1}^{N} \left(\sum_{i=1}^{j-1} \frac{\phi(t_n - \tau_i)}{t_n - \tau_i} + (N-1)\frac{h}{t_n} \right) \geq \left(h + \frac{q_1 - h}{2(1-y)} \right) t_n^{-1} N(N-1),$$

which, after multiplying by $t_n M_n$ and taking expectations, yields

$$\left(h + \frac{q_1 - h}{2(1-y)} \right) M_n \mathrm{E}_{t_n y}(N(N-1)) \leq a(1-y)^{-1} M_n + (M_n + \ln t_n) o_n.$$

Finally, since

$$\mathrm{E}_{t_n y}(N(N-1)) \to 2b, \quad n \to \infty,$$

we derive that for any $0 < \epsilon < y < 1$, there is a finite n_ϵ such that for all $n > n_\epsilon$,

$$M_n \big(2bh(1-y) + bq_1 - bh - a - \epsilon \big) \leq \epsilon \ln t_n.$$

By (29), we have $bh \geq a$, and therefore

$$2bh(1-y) + bq_1 - bh - a - \epsilon \geq bq_1 - 2bhy - y.$$

Thus, choosing $y = y_0$ such that $bq_1 - 2bhy_0 - y_0 = \frac{bq_1}{2}$, we see that

$$\limsup_{n\to\infty} \frac{M_n}{\ln t_n} \leq \frac{2\epsilon}{bq_1},$$

which implies (33) as $\epsilon \to 0$, concluding the proof of $\phi(t) \to 0$.

4. Proof of Theorem 1

We will use the following notational conventions for the k-dimensional probability generating function

$$\mathrm{E}\Big(z_1^{Z(t_1)} \cdots z_k^{Z(t_k)} \Big) = \sum_{i_1=0}^{\infty} \ldots \sum_{i_k=0}^{\infty} \mathrm{P}(Z(t_1) = i_1, \ldots, Z(t_k) = i_k) z_1^{i_1} \cdots z_k^{i_k},$$

with $0 < t_1 \leq \ldots \leq t_k$ and $z_1, \ldots, z_k \in [0, 1]$. We define

$$P_k\big(\bar{t}, \bar{z}\big) := P_k(t_1, \ldots, t_n; z_1, \ldots, z_k) := \mathrm{E}\Big(z_1^{Z(t_1)} \cdots z_k^{Z(t_k)} \Big)$$

and write, for $t \geq 0$,

$$P_k\big(t + \bar{t}, \bar{z}\big) := P_k(t + t_1, \ldots, t + t_k; z_1, \ldots, z_k).$$

Moreover, for $0 < y_1 < \ldots < y_k$, we write

$$P_k(t\bar{y}, \bar{z}) := P_k(ty_1, \ldots, ty_k; z_1, \ldots, z_k),$$

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and assuming $0 < y_1 < \ldots < y_k < 1$,

$$P_k^*(t, \bar{y}, \bar{z}) := \mathrm{E}\Big(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)}; Z(t) = 0\Big) = P_{k+1}(ty_1, \ldots, ty_k, t; z_1, \ldots, z_k, 0).$$

These conventions will be similarly applied to the functions

$$Q_k(\bar{t}, \bar{z}) := 1 - P_k(\bar{t}, \bar{z}), \quad Q_k^*(t, \bar{y}, \bar{z}) := 1 - P_k^*(t, \bar{y}, \bar{z}). \tag{35}$$

Our special interest is in the function

$$Q_k(t) := Q_k(t + \bar{t}, \bar{z}), \quad 0 = t_1 < \ldots < t_k, \quad z_1, \ldots, z_k \in [0, 1), \tag{36}$$

to be viewed as a counterpart of the function $Q(t)$ treated by Theorem 2. Recalling the compound parameters

$$h = \frac{a + \sqrt{a^2 + 4bd}}{2b}$$

and $c = 4bda^{-2}$, put

$$h_k := h\frac{1 + \sqrt{1 + cg_k}}{1 + \sqrt{1 + c}}, \quad g_k := g_k(\bar{y}, \bar{z}) := \sum_{i=1}^{k} z_1 \cdots z_{i-1}(1 - z_i)y_i^{-2}. \tag{37}$$

The key step of the proof of Theorem 1 is to show that for any given $1 = y_1 < y_2 < \ldots < y_k$,

$$tQ_k(t) \to h_k, \quad t_i := t(y_i - 1), \quad i = 1, \ldots, k, \quad t \to \infty. \tag{38}$$

This is done following the steps of our proof of $tQ(t) \to h$ given in Section 3.

Unlike $Q(t)$, the function $Q_k(t)$ is not monotone over t. However, monotonicity of $Q(t)$ was used in the proof of Theorem 2 only for the proof of (14). The corresponding statement

$$0 < q_1 \le tQ_k(t) \le q_2 < \infty, \quad t \ge t_0,$$

follows from the bounds $(1 - z_1)Q(t) \le Q_k(t) \le Q(t)$, which hold by the monotonicity of the underlying generating functions over $z_1, \ldots, z_n$. Indeed,

$$Q_k(t) \le Q_k(t, t + t_2, \ldots, t + t_k; 0, \ldots, 0) = Q(t),$$

and on the other hand,

$$Q_k(t) = Q_k(t, t + t_2, \ldots, t + t_k; z_1, \ldots, z_k) = \mathrm{E}\Big(1 - z_1^{Z(t)} z_2^{Z(t+t_2)} \cdots z_k^{Z(t+t_k)}\Big) \ge \mathrm{E}\Big(1 - z_1^{Z(t)}\Big),$$

where

$$\mathrm{E}\Big(1 - z_1^{Z(t)}\Big) \ge \mathrm{E}\Big(1 - z_1^{Z(t)}; Z(t) \ge 1\Big) \ge (1 - z_1)Q(t).$$

4.1. Proof of $tQ_k(t) \to h_k$

The branching property (8) of the GWO process gives

$$\prod_{i=1}^{k} z_i^{Z(t_i)} = \prod_{i=1}^{k} z_i^{1_{\{L>t_i\}}} \prod_{j=1}^{N} z_i^{Z_j(t_i - \tau_j)}.$$

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Given $0 < t_1 < \ldots < t_k < t_{k+1} = \infty$, we use

$$\prod_{i=1}^{k} z_i^{1_{\{L>t_i\}}} = 1_{\{L\le t_1\}} + \sum_{i=1}^{k} z_1 \cdots z_i 1_{\{t_i<L\le t_{i+1}\}}$$

to deduce the following counterpart of (9):

$$P_k(\bar{t}, \bar{z}) = \mathrm{E}_{t_1}\left(\prod_{j=1}^{N} P_k(\bar{t} - \tau_j, \bar{z})\right) + \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{E}\left(\prod_{j=1}^{N} P_k(\bar{t} - \tau_j, \bar{z}); t_i < L \le t_{i+1}\right).$$

This implies

$$\begin{aligned} P_k(\bar{t}, \bar{z}) &= \mathrm{E}_{t_1}\left(\prod_{j=1}^{N} P_k(\bar{t} - \tau_j, \bar{z})\right) + \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}(t_i < L \le t_{i+1}) \\ &\quad - \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{E}\left(1 - \prod_{j=1}^{N} P_k(\bar{t} - \tau_j, \bar{z}); t_i < L \le t_{i+1}\right). \end{aligned} \tag{39}$$

Using this relation we establish the following counterpart of Lemma 1.

Lemma 5. *Consider the function* (36) *and put* $P_k(t) := 1 - Q_k(t) = P_k\big(t + \bar{t}, \bar{z}\big)$. *For* $0 < u < t$, *the relation*

$$\begin{aligned} \Phi\big(h_k t^{-1}\big) = \mathrm{P}(L > t) - \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}\big(t + t_i < L \le t + t_{i+1}\big) \\ + \mathrm{E}_u\left(\sum\nolimits_{j=1}^{N} Q_k\left(t - \tau_j\right)\right) - Q_k(t) + \mathrm{E}_u(W_k(t)) + D_k(u, t) \end{aligned} \tag{40}$$

holds with $t_{k+1} = \infty$,

$$W_k(t) := \left(1 - h_k t^{-1}\right)^N + N h_k t^{-1} - \sum\nolimits_{j=1}^{N} Q_k\left(t - \tau_j\right) - \prod\nolimits_{j=1}^{N} P_k\left(t - \tau_j\right), \tag{41}$$

and

$$\begin{aligned} D_k(u, t) := \mathrm{E}\Big(1 - \prod\nolimits_{j=1}^{N} P_k\left(t - \tau_j\right); u < L \le t\Big) + \mathrm{E}\Big(\big(1 - h_k t^{-1}\big)^N - 1 + N h_k t^{-1}; L > u\Big) \\ + \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{E}\left(1 - \prod_{j=1}^{N} P_k\left(t - \tau_j\right); t + t_i < L \le t + t_{i+1}\right). \end{aligned} \tag{42}$$

Proof. According to (39),

$$\begin{aligned} P_k(t) = \mathrm{E}_u\left(\prod_{j=1}^{N} P_k\left(t - \tau_j\right)\right) + \mathrm{E}\Big(\prod\nolimits_{j=1}^{N} P_k\left(t - \tau_j\right); u < L \le t\Big) \\ + \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{P}\big(t + t_i < L \le t + t_{i+1}\big) \\ - \sum_{i=1}^{k} z_1 \cdots z_i \mathrm{E}\left(1 - \prod_{j=1}^{N} P_k\left(t - \tau_j\right); t + t_i < L \le t + t_{i+1}\right). \end{aligned}$$

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By the definition of $\Phi(\cdot)$,

$$\Phi\left(h_k t^{-1}\right) + 1 = \mathrm{E}_u\left(\left(1 - h_k t^{-1}\right)^N + N h_k t^{-1}\right) + \mathrm{P}(L > t)$$
$$+ \mathrm{E}\left(\left(1 - h_k t^{-1}\right)^N - 1 + N h_k t^{-1}; L > u\right) + \mathrm{P}(u < L \le t),$$

and after subtracting the two last equations, we get

$$\Phi\left(h_k t^{-1}\right) + Q_k(t) = \mathrm{E}_u\left(\left(1 - h_k t^{-1}\right)^N + N h_k t^{-1} - \prod_{j=1}^N P_k\left(t - \tau_j\right)\right) + \mathrm{P}(L > t)$$
$$- \sum_{i=1}^k z_1 \cdots z_i \mathrm{P}(t + t_i < L \le t + t_{i+1}) + D_k(u, t),$$

with $D_k(u, t)$ satisfying (42). After a rearrangement, the relation (40) follows together with (41). □

With Lemma 5 in hand, the convergence (38) is proven by applying almost exactly the same argument as used in the proof of $tQ(t) \to h$. An important new feature emerges because of the additional term in the asymptotic relation defining the limit h_k. Let $1 = y_1 < y_2 < \ldots < y_k < y_{k+1} = \infty$. Since

$$\sum_{i=1}^k z_1 \cdots z_i \mathrm{P}\left(t y_i < L \le t y_{i+1}\right) \sim d t^{-2} \sum_{i=1}^k z_1 \cdots z_i \left(y_i^{-2} - y_{i+1}^{-2}\right),$$

we see that

$$\mathrm{P}(L > t) - \sum_{i=1}^k z_1 \cdots z_i \mathrm{P}\left(t y_i < L \le t y_{i+1}\right) \sim d g_k t^{-2},$$

where g_k is defined by (37). Assuming $0 \le z_1, \ldots, z_k < 1$, we ensure that $g_k > 0$, and as a result, we arrive at a counterpart of the quadratic equation (29),

$$b h_k^2 = a h_k + d g_k,$$

which gives

$$h_k = \frac{a + \sqrt{a^2 + 4 b d g_k}}{2b} = h \frac{1 + \sqrt{1 + c g_k}}{1 + \sqrt{1 + c}},$$

justifying our definition (37). We conclude that for $k \ge 1$,

$$\frac{Q_k(t\bar{y}, \bar{z})}{Q(t)} \to \frac{1 + \sqrt{1 + c \sum_{i=1}^k z_1 \cdots z_{i-1}(1 - z_i) y_i^{-2}}}{1 + \sqrt{1 + c}},$$
$$1 = y_1 < \ldots < y_k, \quad 0 \le z_1, \ldots, z_k < 1. \tag{43}$$

4.2. Conditioned generating functions

To finish the proof of Theorem 1, consider the generating functions conditioned on the survival of the GWO process. Given (5) with $j \ge 1$, we have

$$Q(t) \mathrm{E}\left(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)} | Z(t) > 0\right) = \mathrm{E}(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)}; Z(t) > 0)$$
$$= P_k(t\bar{y}, \bar{z}) - \mathrm{E}\left(z_1^{Z(ty_1)} \cdots z_k^{Z(ty_k)}; Z(t) = 0\right) \overset{(35)}{=} Q_j^*\left(t, \bar{y}, \bar{z}\right) - Q_k(t\bar{y}, \bar{z}),$$

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and therefore,

$$\mathrm{E}\Big(z_1^{Z(ty_1)}\cdots z_k^{Z(ty_k)}|Z(t)>0\Big)=\frac{Q_j^*\big(t,\bar{y},\bar{z}\big)}{Q(t)}-\frac{Q_k(t\bar{y},\bar{z})}{Q(t)}.$$

Similarly, if (5) holds with $j=0$, then

$$\mathrm{E}\Big(z_1^{Z(ty_1)}\cdots z_k^{Z(ty_k)}|Z(t)>0\Big)=1-\frac{Q_k(t\bar{y},\bar{z})}{Q(t)}.$$

Letting $t'=ty_1$, we get

$$\frac{Q_k(t\bar{y},\bar{z})}{Q(t)}=\frac{Q_k(t',t'y_2/y_1,\ldots,t'y_k/y_1)}{Q(t')}\frac{Q(ty_1)}{Q(t)},$$

and applying the relation (43), we have

$$\frac{Q_k(t\bar{y},\bar{z})}{Q(t)}\to\frac{1+\sqrt{1+\sum_{i=1}^k z_1\cdots z_{i-1}(1-z_i)\Gamma_i}}{\big(1+\sqrt{1+c}\big)y_1},$$

where $\Gamma_i=c(y_1/y_i)^2$. On the other hand, since

$$Q_j^*\big(t,\bar{y},\bar{z}\big)=Q_{j+1}(ty_1,\ldots,ty_j,t;z_1,\ldots,z_j,0),\quad j\geq 1,$$

we also get

$$\frac{Q_j^*\big(t,\bar{y},\bar{z}\big)}{Q(t)}\to\frac{1+\sqrt{1+\sum_{i=1}^j z_1\cdots z_{i-1}(1-z_i)\Gamma_i+cz_1\cdots z_jy_1^2}}{\big(1+\sqrt{1+c}\big)y_1}.$$

We conclude that as stated in Section 2,

$$\mathrm{E}\Big(z_1^{Z(ty_1)}\cdots z_k^{Z(ty_k)}|Z(t)>0\Big)\to\mathrm{E}\Big(z_1^{\eta(y_1)}\cdots z_k^{\eta(y_k)}\Big).$$

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