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A COMMUTATIVITY THEOREM FOR SEMI-PRIMITIVE RINGS

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In this brief note, we prove the following: Let R be a semi-primitive ring. Suppose that for each pair $x, y \in R$ there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that either $[x^m, (xy)^n - (yx)^n] = 0$ or $[x^m, (xy)^n + (yx)^n] = 0$. Then R is commutative.

Throughout, R will represent a ring with centre Z. Recently, in the main theorem of [5], Quadri and Ashraf proved the following: Let R be a semi-primitive ring. (1) If for each pair $x, y \in R$ there is a positive integer $n = n(x,y)$ such that $(xy)^n - (yx)^n \in Z$, then R is commutative. (2) If for each pair $x, y \in R$ there is a positive integer $n = n(x,y)$ such that $(xy)^n + (yx)^n \in Z$, then R is commutative. In order to prove (1), they gave a much shorter and simpler proof for [2, Theorem 1]. But, unfortunately, their proof is incomplete, because n is not fixed but depends on x and y. However, (1) itself

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is true and is involved in [*4*, Corollary 1] and [*1*, Theorem 2] (see also [*3*, Corollary 1]): (1)' Let R be a semi-primitive ring. If for each pair $x, y \in R$ there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that $[x^m, (xy)^n - (yx)^n] = 0$, then R is commutative. (1)" Let R be a ring without non-zero nil ideals. If for each pair $x, y \in R$ there is a positive integer $n = n(x,y)$ such that $(xy)^n - (yx)^n \in Z$, then R is commutative.

We shall improve the main theorem of [5] as follows:

THEOREM. *Let R be a semi-primitive ring. Suppose that for each pair $x, y \in R$ there exist positive integers $m = m(x,y)$ and $n = n(x,y)$ such that either*

$$[x^m, (xy)^n - (yx)^n] = 0 \quad \text{or} \quad [x^m, (xy)^n + (yx)^n] = 0$$

Then R is commutative.

Proof. Note that the hypothesis is inherited by all subrings and all homomorphic images of R. Note also that no complete matrix ring $(D)_t$ over a division ring $D (t > 1)$ satisfies the hypothesis, as a consideration of $x = E_{11}$ and $y = E_{11} + E_{12}$ shows. Because of these facts and the structure theorem of primitive rings, we may assume that R is a division ring. Let x, y be non-zero elements of R. By hypothesis, either

$$[x^m, (x \cdot x^{-1}y)^n - (x^{-1}y \cdot x)^n] = 0 \text{ or } [x^m, (x \cdot x^{-1}y)^n + (x^{-1}y \cdot x)^n] = 0$$

with some positive integers m, n. Then we see that either

$$[x^m, [x^2, y^n]] = x^2[x^m, y^n - x^{-1}y^n x] + x[x^m, y^n - x^{-1}y^n x]x = 0$$

or

$$[x^m, [x^2, y^n]] = x[x^m, [x, y^n + x^{-1}y^n x]] = x[x, [x^m, y^n + x^{-1}y^n x]] = 0$$

Hence R is commutative, by [*4*, Theorem 1].

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