ON THE MAXIMUM ORDER OF THE GROUP OF A TOURNAMENT

Myron Goldberg and J. W. Moon

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1. Introduction. A (round-robin) tournament T_n consists of n nodes $p_1, p_2, \ldots, p_n$ such that each pair of distinct nodes p_i and p_j is joined by one of the oriented arcs $\vec{p_i p_j}$ or $\vec{p_j p_i}$. If the arc $\vec{p_i p_j}$ is in T_n, then we say that p_i dominates p_j. The set of all dominance-preserving permutations α of the nodes T_n form a group, the automorphism group $G(T_n)$ of T_n. It is known (see [1]) that there exist tournaments T_n whose group $G(T_n)$ is abstractly isomorphic to a given group H if and only if the order $g(H)$ of H is odd.

If $g(T_n)$ denotes the order of the group $G(T_n)$, let $g(n)$ denote the maximum of $g(T_n)$ taken over all tournaments T_n. Our main object here is to prove the following result.

THEOREM. The limit of $g(n)^{1/n}$ as n tends to infinity exists and lies between $\sqrt{3}$ and 2.5, inclusive.

2. An Upper Bound. In this section we shall prove by induction that

$$(1) \qquad g(n) \leq \frac{(2.5)^n}{2n}, \quad \text{for } n \geq 4 .$$

It is not difficult to verify that this inequality holds when $4 \leq n \leq 9$ by using the exact values of $g(n)$ given in Table 1.

n	1	2	3	4	5	6	7	8	9
g(n)	1	1	3	3	5	9	21	21	81

Table 1

Consider any node p of an arbitrary tournament T_n, where $n \geq 10$. Let d denote the number of different nodes in the set

$$D = \{\alpha(p) : \alpha \in G(T_n)\} .$$

If T_d and T_{n-d} denote the subtournaments determined by the nodes that are in D and by the nodes that are not in D, then it is clear that

$$(2) \qquad g(T_n) \leq g(T_d) \cdot g(T_{n-d}) \leq g(d) \cdot g(n-d) .$$

If $3 < d < n-3$, then it follows from the induction hypothesis that

$$g(T_n) \leq \frac{(2.5)^d}{2d} \cdot \frac{(2.5)^{n-d}}{2(n-d)} \leq \frac{n}{8(n-4)} \cdot \frac{(2.5)^n}{2n} < \frac{(2.5)^n}{2n} .$$

If $d = 3$ or $n-3$, then

$$g(T_n) \leq 3 \cdot \frac{(2.5)^{n-3}}{2(n-3)} < \frac{(2.5)^n}{2n} ,$$

and if $d = 1, 2, n-2$ or $n-1$, then

$$g(T_n) \leq 1 \cdot \frac{2n}{5(n-2)} \cdot \frac{(2.5)^n}{2n} < \frac{(2.5)^n}{2n} .$$

A different argument must be used when $d = n$.

There are $n(n-1)/2$ arcs in the tournament T_n. Hence, if $d = n$ and the nodes of T_n are all similar to each other with respect to the group $G(T_n)$, it must be that each node dominates exactly $(n-1)/2$ other nodes. This can happen only when n is odd.

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Consider the subgroup H of automorphisms α of $G(T_n)$ such that $\alpha(p) = p$. It follows from a result in group theory (see [2]) that if $d = n$, then

$$g(T_n) = ng(H).$$

No element of H can transform one of the $(n-1)/2$ nodes that dominate p into one of the $(n-1)/2$ nodes dominated by p, since p is fixed. Hence,

$$g(H) \leq (g((n-1)/2))^2.$$

Therefore, if $d = n$, then

$$g(T_n) \leq n\left(\frac{(2.5)^{(n-1)/2}}{n-1}\right)^2 = \frac{4}{5}\left(\frac{n}{n-1}\right)^2 \cdot \frac{(2.5)^n}{2n} < \frac{(2.5)^n}{2n}.$$

(Notice that if $n \geq 11$, then $(n-1)/2 \geq 5$, so we are certainly entitled to apply the induction hypothesis to $g((n-1)/2)$.) This suffices to complete the proof of inequality (1) by induction.

An immediate consequence of inequality (1) is that

$$\text{(3)} \qquad \limsup g(n)^{1/n} \leq 2.5 .$$

We remark that equality holds in inequality (2) when the arcs joining nodes in T_d to nodes in T_{n-d} all have the same orientation. It follows that if n is even, then

$$\text{(4)} \qquad g(n) = \max\{g(d) \cdot g(n-d)\}, \quad d = 1, 3, 5, \ldots, n-1,$$

since d is odd. Hence, in determining exact values of $g(n)$ the only tournaments that need to be examined individually are those with an odd number of nodes in which all the nodes are similar to each other.

When n is odd and $d = n$, the inequality

$$g(T_n) \leq n(g((n-1)/2))^2$$

is best possible in the sense that equality holds for certain tournaments when $n = 3, 9, 27$ and, perhaps, for all higher powers of three.

Stronger forms of inequality (1) can be obtained by the same type of argument if one is willing to treat more special cases separately. For example, it can be shown that

$$g(n) \leq \frac{.45(2.03)^n}{n} \qquad \text{if } n \geq 13 ,$$

but this result has to be verified directly for $13 \leq n \leq 26$.

We conjecture that

$$g(n) \leq \sqrt{3}^{\,n-1} ,$$

with equality holding if and only if $n = 3^k$, $k = 0, 1, \ldots$.

3. <u>Proof of the existence of the limit.</u> If T_a and T_b are two arbitrary tournaments, consider the tournament T_{ab} obtained by replacing each node of T_a by a copy of T_b; if the node p dominates the node q in T_a originally, then in T_{ab} each node of the tournament that replaces p dominates each node of the tournament that replaces q. (When T_a and T_b are both 3-cycles, the tournament T_{ab} is illustrated in Figure 1.) It is not difficult to see that the orders of the groups of T_a,

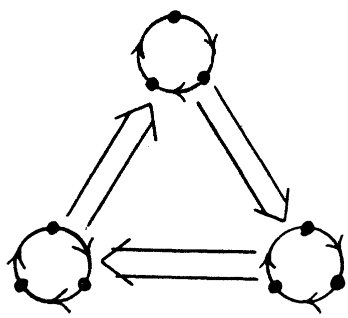

Figure 1.

T_b and T_{ab} satisfy the inequality

$$g(T_{ab}) \geq g(T_a)\,[g(T_b)]^a .$$

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Therefore,

(5) $$g(ab) \geq g(a)\,[g(b)]^{a}, \qquad \text{for all integers } a \text{ and } b.$$

In particular, since $g(3) = 3$, it follows by induction that

(6) $$g(n) \geq \sqrt{3}^{\,n-1} \quad \text{if} \quad n = 3^{k}, \quad k = 0, 1, \ldots .$$

Hence,

(7) $$\limsup g(n)^{1/n} \geq \sqrt{3}.$$

We now use inequality (5) to prove the following result.

LEMMA. If $g(m)^{1/m} > \gamma$, then $g(n)^{1/n} > \gamma - \epsilon$ for any positive ϵ and all sufficiently large n.

Proof. We assume that $\gamma > 1$ since the result is obvious otherwise. Let ℓ be the least integer such that $\gamma^{-1/\ell} > 1 - \epsilon/\gamma$. Every sufficiently large integer n can be written in the form $n = km + t$, where $k > \ell$ and $0 \leq t < m$. Then

$$\begin{aligned} g(n)^{1/n} &= g(km+t)^{1/km+t} \geq g(km)^{1/m(k+1)} \\ &\geq [g(m)^{1/m}]^{k/k+1} > \gamma^{k/k+1} \\ &> \gamma^{\ell/\ell+1} > \gamma^{1-1/\ell} \\ &= \gamma(1-\epsilon/\gamma) = \gamma - \epsilon, \end{aligned}$$

as required.

Let $\beta = \limsup g(n)^{1/n}$. (We know that $\sqrt{3} \leq \beta \leq 2.5$.) For every positive ϵ there exists an integer m such that

$$g(m)^{1/m} > \beta - \epsilon .$$

But then, according to the lemma,

$$g(n)^{1/n} > \beta - 2\epsilon$$

for all sufficiently large n. Hence,

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$$\liminf g(n)^{1/n} > \beta - 2\epsilon$$

for every positive ϵ. Therefore,

$$\liminf g(n)^{1/n} = \limsup g(n)^{1/n} . \tag{8}$$

The theorem stated in § 1 now follows from statements (3), (7) and (8).

ADDENDUM

Perhaps it should be pointed out that the problem of determining $g(n)$ is equivalent to the group-theoretic problem of determining the order of the largest subgroup of odd order of S_n, the symmetric group on n objects.

If H is any subgroup of odd order of S_n let H act on the nodes of the complete graph K_n with n nodes. Then H induces an equivalence relation on the edges of K_n. Assign an arbitrary orientation to one edge from each equivalence class and orient the images of these edges under H in the same way. The fact that H has odd order implies that the orientation of every edge of K_n is now uniquely determined. This procedure defines a tournament T_n and it is clear that H is a subgroup of $G(T_n)$. If H is chosen to be the largest subgroup of odd order in S_n then, since $g(T_n)$ is always odd, it must be that H is in fact isomorphic to $G(T_n)$. It follows, therefore, that $g(n)$ is the order of the largest subgroup H of odd order of S_n. It is not difficult to show that in determining the order of the largest subgroup H of odd order it is sufficient to consider the case that H is transitive.

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University of Alberta, Edmonton

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