Canad. J. Math. Vol. **55** (1), 2003 pp. 204–224

On the Nonsquare Constants of Orlicz Spaces with Orlicz Norm

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Abstract. Let l^Φ and $L^\Phi(\Omega)$ be the Orlicz sequence space and function space generated by N-function $\Phi(u)$ with Orlicz norm. We give equivalent expressions for the nonsquare constants $C_J(l^\Phi), C_J\big(L^\Phi(\Omega)\big)$ in sense of James and $C_S(l^\Phi), C_S\big(L^\Phi(\Omega)\big)$ in sense of Schäffer. We are devoted to get practical computational formulas giving estimates of these constants and to obtain their exact value in a class of spaces l^Φ and $L^\Phi(\Omega)$.

1 Introduction

Let $(X, \|\cdot\|)$ be a Banach space. $S(X) = \{x : \|x\| = 1, x \in X\}$ denotes the unit sphere of X. In 1990, Gao and Lau [4] defined the James nonsquare constant $C_J(X)$ and Schäffer nonsquare constant $C_S(X)$ as

$$C_J(X) = \sup\{\min(\|x+y\|, \|x-y\|) : x, y \in S(X)\}, \tag{1}$$

$$C_s(X) = \inf\{\max(\|x+y\|, \|x-y\|) : x, y \in S(X)\}. \tag{2}$$

Clearly, if $\dim X \geq 2$, then $1 \leq C_S(X) \leq \sqrt{2} \leq C_J(X) \leq 2$. Ji and Wang [6] (1994) asserted

$$C_J(X) \cdot C_S(X) = 2 \tag{3}$$

for $\dim X \geq 2$. Ji and Zhan [7] found the following formulas:

$$C_J(X) = \sup\{\|x+y\| : \|x-y\| = \|x+y\|, x, y \in S(X)\}, \tag{4}$$

$$C_S(X) = \inf\{\|x+y\| : \|x-y\| = \|x+y\|, x, y \in S(X)\}. \tag{5}$$

It was proved (see Chen [1], Hudzik [5], Wang and Chen [12]) that $C_J(X) = 2$ if and only if X is nonreflexive.

Let

$$\Phi(u) = \int_0^{|u|} \phi(t)\,dt \quad \text{and} \quad \Psi(v) = \int_0^{|v|} \psi(s)\,ds$$

be a pair of complementary N-functions, *i.e.*, $\phi(t)$ is right continuous, $\phi(0) = 0$, and $\phi(t) \nearrow \infty$ as $t \nearrow \infty$. We call $\Phi \in \Delta_2(0)$ (or $\Delta_2(\infty)$), if there exist $u_0 > 0$ and

Received by the editors September 7, 2001; revised May 28, 2002.
AMS subject classification: 46E30.
Keywords: James nonsquare constant, Schäffer nonsquare constant, Orlicz sequence space, Orlicz function space.

https://doi.org/10.4153/CJM-2003-009-1 Published online by Cambridge University Press

$k > 2$ such that $\Phi(2u) \le k\Phi(u)$ for $0 \le u \le u_0$ (or for $u \ge u_0$). The Orlicz sequence space is defined as the set

$$l^{\Phi} = \Big\{ x(i) : \rho_{\Phi}(\lambda x) = \sum_{n=1}^{\infty} \Phi\big(\lambda|x(i)|\big) < \infty \text{ for some } \lambda > 0 \Big\}.$$

The Orlicz function space $L^{\Phi}(\Omega)$ on a nonatomic measure space (Ω, Σ, μ) is defined as

$$L^{\Phi}(\Omega) = \Big\{ x(t) : \rho_{\Phi}(\lambda x) = \int_{\Omega} \Phi\big(\lambda|x(t)|\big)\, dt < \infty \text{ for some } \lambda > 0 \Big\}.$$

The Luxemburg norm and Orlicz norm (see [2]) are expressed as

$$\|x\|_{(\Phi)} = \inf\Big\{ c > 0 : \rho_{\Phi}\Big(\frac{x}{c}\Big) \le 1 \Big\}$$

and

$$\|x\|_{\Phi} = \inf_{k>0} \frac{1}{k}[1 + \rho_{\Phi}(kx)].$$

For the Orlicz spaces equipped with Luxemburg norm with Φ satisfying the Δ_2-condition, Ji and Wang [6] gave the expressions for function spaces. Latter on, Ji and Zhan [7] gave the corresponding results for sequences spaces. They showed:

(i) If $\phi(t)$ is a concave function, then

$$C_J(l^{(\Phi)}) = \sup\Big\{ k_x > 0 : \rho_{\Phi}\Big(\frac{x}{k_x}\Big) = \frac{1}{2}, \rho_{\Phi}(x) = 1 \Big\},$$

$$C_S\big(L^{(\Phi)}(\Omega)\big) = \inf\Big\{ k_x > 0 : \rho_{\Phi}\Big(\frac{2x}{k_x}\Big) = 2, \rho_{\Phi}(x) = 1 \Big\};$$

(iv) if $\phi(t)$ is convex, then

$$C_S(l^{(\Phi)}) = \inf\Big\{ k_x > 0 : \rho_{\Phi}\Big(\frac{x}{k_x}\Big) = \frac{1}{2}, \rho_{\Phi}(x) = 1 \Big\},$$

$$C_J\big(L^{(\Phi)}(\Omega)\big) = \sup\Big\{ k_x > 0 : \rho_{\Phi}\Big(\frac{2x}{k_x}\Big) = 2, \rho_{\Phi}(x) = 1 \Big\}.$$

The author [14] then found some formulas for computations of estimates. Some exact values of nonsquare constants was obtained. For example, for the pair of N-function

$$M(u) = e^{|u|} - |u| - 1 \quad \text{and} \quad N(v) = (1 + |v|)\ln(|v| + 1) - |v|, \tag{6}$$

the exact values of nonsquare constants for $l^{(M)}$ and $l^{(N)}$ with Luxemburg are:

$$C_J(l^{(M)}) = \frac{2M^{-1}(\frac{1}{2})}{M^{-1}(1)} \approx 1.49656; \quad C_J(l^{(N)}) = \frac{N^{-1}(1)}{N^{-1}(\frac{1}{2})} \approx 1.48699.$$

However, for the Orlicz spaces l^Φ and $L^\Phi(\Omega)$ equipped with Orlicz norm, the expressions for $C_J(l^\Phi)$ and $C_J\big(L^\Phi(\Omega)\big)$ have still remained unsolved and consequently little achievement about the estimates has been obtained since Ren [10](1997) produced the lower bounds. This paper is devoted to this problem, so as to get practical computation formulas for reasonable estimation.

In what follows, we will use Semenove and Simonenko indices of $\Phi(u)$:

$$\alpha_\Phi^0 = \liminf_{u\to 0^+} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_\Phi^0 = \limsup_{u\to 0^+} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{7}$$

$$\alpha_\Phi = \liminf_{u\to \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \beta_\Phi = \limsup_{u\to \infty} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \tag{8}$$

$$\bar{\alpha}_\Phi = \inf_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}, \quad \bar{\beta}_\Phi = \sup_{u>0} \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}; \tag{9}$$

and

$$A_\Phi^0 = \liminf_{t\to 0^+} \frac{t\phi(t)}{\Phi(t)}, \quad B_\Phi^0 = \limsup_{t\to 0^+} \frac{t\phi(t)}{\Phi(t)}, \tag{10}$$

$$A_\Phi = \liminf_{t\to \infty} \frac{t\phi(t)}{\Phi(t)}, \quad B_\Phi = \limsup_{t\to \infty} \frac{t\phi(t)}{\Phi(t)}, \tag{11}$$

$$\bar{A}_\Phi = \inf_{t>0} \frac{t\phi(t)}{\Phi(t)}, \quad \bar{B}_\Phi = \sup_{t>0} \frac{t\phi(t)}{\Phi(t)}. \tag{12}$$

The same indices can be applied to $\Psi(v)$. We will frequently extend the indices in the following context for the sequential usage. The author [13] obtained

$$2\alpha_\Phi^0\beta_\Psi^0 = 1 = 2\alpha_\Psi^0\beta_\Phi^0, \tag{13}$$

$$2\alpha_\Phi\beta_\Psi = 1 = 2\alpha_\Psi\beta_\Phi, \tag{14}$$

$$2\bar{\alpha}_\Phi\bar{\beta}_\Psi = 1 = 2\bar{\alpha}_\Psi\bar{\beta}_\Phi. \tag{15}$$

Rao and Ren [9] gave the following interrelations:

$$2^{-\frac{1}{A_\Phi^0}} \le \alpha_\Phi^0 \le \beta_\Phi^0 \le 2^{-\frac{1}{B_\Phi^0}}, \tag{16}$$

$$2^{-\frac{1}{A_\Phi}} \le \alpha_\Phi \le \beta_\Phi \le 2^{-\frac{1}{B_\Phi}}, \tag{17}$$

$$2^{-\frac{1}{\bar{A}_\Phi}} \le \bar{\alpha}_\Phi \le \bar{\beta}_\Phi \le 2^{-\frac{1}{\bar{B}_\Phi}}. \tag{18}$$

If the index function $F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)}$ is monotonic (increase or decrease) at a right neighborhood of 0 (or ∞), then the limit $C_\Phi^0 = \lim_{t\to 0^+} \frac{t\phi(t)}{\Phi(t)}$ (or $C_\Phi = \lim_{t\to\infty} \frac{t\phi(t)}{\Phi(t)}$, respectively) must exist, and hence

$$\alpha_\Phi^0 = \beta_\Phi^0 = 2^{-\frac{1}{C_\Phi^0}}, \quad \text{and} \quad \alpha_\Phi = \beta_\Phi = 2^{-\frac{1}{C_\Phi}}. \tag{19}$$

These relations will play important roles in our main results. We now introduce an auxiliary lemma.

Lemma 1 *Let $\phi(t)$ be the right derivative of N-function $\Phi(u)$. Then for any $x \geq y \geq 0$ and $k \geq h > 0$, we have*

(i) *if $\phi(t)$ is concave, then*

$$\frac{k+h}{2kh}\Big[\Phi\Big(\frac{2kh}{k+h}(x+y)\Big)+\Phi\Big(\frac{2kh}{k+h}(x-y)\Big)\Big] \leq \frac{2}{k}\Phi(kx)+\frac{2}{h}\Phi(hy), \tag{20}$$

$$\frac{k+h}{4kh}\Big[\Phi\Big(\frac{4kh}{k+h}x\Big)+\Phi\Big(\frac{4kh}{k+h}y\Big)\Big] \leq \frac{1}{k}\Phi\big(k(x+y)\big)+\frac{1}{h}\Phi\big(h(x-y)\big); \tag{21}$$

(ii) *if $\phi(t)$ is convex, then*

$$\frac{k+h}{kh}\Big[\Phi\Big(\frac{2kh}{k+h}x\Big)+\Phi\Big(\frac{2kh}{k+h}y\Big)\Big] \leq \frac{1}{k}\Phi\big(k(x+y)\big)+\frac{1}{h}\Phi\big(h(x-y)\big), \tag{22}$$

$$\frac{k+h}{2kh}\Big[\Phi\Big(\frac{2kh}{k+h}(x+y)\Big)+\Phi\Big(\frac{2kh}{k+h}(x-y)\Big)\Big] \leq \frac{1}{2k}\Phi(2kx)+\frac{1}{2h}\Phi(2hy). \tag{23}$$

Proof (i) We first observe that

$$\Phi(2t) \leq 4\Phi(t)$$

if $\phi(t)$ is concave. In fact, we have

$$\Phi(2t) = \int_0^{2t}\phi(s)\,ds = 2\int_0^t \phi(2r)\,dr \leq 4\int_0^t \phi(r)\,dr = 4\Phi(t).$$

For any real number k and h, fix y, denote

$$H(x) = \frac{k+h}{2kh}\Big[\Phi\Big(\frac{2kh}{k+h}(x+y)\Big)+\Phi\Big(\frac{2kh}{k+h}(x-y)\Big)\Big] - \frac{2}{k}\Phi(kx) - \frac{2}{h}\Phi(hy).$$

We first show that

$$H(y) = \frac{k+h}{2kh}\Big[\Phi\Big(\frac{4kh}{k+h}y\Big)\Big] - \frac{2}{k}\Phi(ky) - \frac{2}{h}\Phi(hy) \leq 0 \tag{24}$$

for $x \geq y$. Treat $H(y)$ as the function of $k (k \geq h > 0)$. When $k = h$, we have

$$H(y,k) \overset{\triangle}{=} H(y)|_{k=h} = \frac{1}{h}\Phi(2hy) - \frac{4}{h}\Phi(hy) = \frac{1}{h}[\Phi(2hy) - 4\Phi(hy)] \leq 0$$

since $\phi(t)$ is concave. Then we show $H'_k(y,k)$, the right derivative of $H(y)$ to k, is less than or equal to 0, so that $H(y,k)$ is decreasing and $H(y,k) \leq 0$. In fact, since

$$H'_k(y,k) = \frac{1}{2k^2}\Big[\frac{4kh}{k+h}y\phi\Big(\frac{4kh}{k+h}y\Big) + 4\Phi(ky) - \Phi\Big(\frac{4kh}{k+h}y\Big) - 4ky\phi(ky)\Big],$$

https://doi.org/10.4153/CJM-2003-009-1 Published online by Cambridge University Press

we need only to check

$$L(a,b)=\frac{4ab}{a+b}\phi\Big(\frac{4ab}{a+b}\Big)+4\Phi(a)-\Phi\Big(\frac{4ab}{a+b}\Big)-4a\phi(a)\le 0 \tag{25}$$

Notice $\frac{4ab}{a+b}\le 2a$, the function $f(t)=t\phi(t)-\Phi(t)$ is increasing, and $g(t)=2t\phi(2t)+4\Phi(t)-\Phi(2t)-4t\phi(t)\le 0$ ($g(0)=0$ and $g(t)$ is decreasing since $\phi(t)$ is concave), we have

$$L(a,b)\le 2a\phi(2a)+4\Phi(a)-\Phi(2a)-4a\phi(a)\le 0.$$

Therefore (25) holds, and hence (24) holds.

Finally, because $\phi(t)$ is concave, we have

$$\begin{aligned}H'(x)&=\phi\Big(\frac{2kh}{k+h}(x+y)\Big)+\phi\Big(\frac{2kh}{k+h}(x-y)\Big)-2\phi(kx)\\&\le 2\phi\Big(\frac{2kh}{k+h}x\Big)-2\phi(kx)\le 0\end{aligned}$$

for $k\ge h>0$. This implies $H(x)$ is decreasing on $[y,+\infty)$. It follows that (20) holds since we have already proved that $H(y)\le 0$.

Let $x=A+B$, $y=A-B$ in (20) we immediately get (21).

(ii) The proof is similar to that of (i). ■

It should be noted that when $k=h=1$, we can deduce Ji and Wang's [6] result, with which they produced the expressions for spaces equipped with Luxemburg norm:

$$\frac{1}{2}[\Phi(2a)+\Phi(2b)]\le\Phi(a+b)+\Phi(a-b)\le 2[\Phi(a)+\Phi(b)],\quad \text{if } \phi(t) \text{ is concave;}$$

$$2[\Phi(a)+\Phi(b)]\le\Phi(a+b)+\Phi(a-b)\le\frac{1}{2}[\Phi(2a)+\Phi(2b)],\quad \text{if } \phi(t) \text{ is convex.}$$

2 Expressions For Sequence and Function Spaces

Lemma 2 *Let $\Phi(u)$ be an N-function with $\Phi\in\Delta_2(0)$, then for every $x\in S(l^\Phi)$ and $k>1$ there is unique $d_{x,k}>1$ such that*

$$\rho_\Phi\Big(\frac{kx}{d_{x,k}}\Big)=\frac{k-1}{2}.$$

Proof Indeed, $\rho_\Phi(\frac{kx}{d})$, as a function of d, is continuous and strictly decreasing. Since $\rho_\Phi(kx)\ge\|kx\|_\Phi-1=k-1>\frac{k-1}{2}$, and $\lim_{d\to\infty}\rho_\Phi(\frac{kx}{d})=0<\frac{k-1}{2}$ when $\Phi\in\Delta_2(0)$, there is unique d such that $\rho_\Phi(\frac{kx}{d_{x,k}})=\frac{k-1}{2}$. ■

Now we give the formulas for $C_J(l^\Phi)$ and $C_S(l^\Phi)$. The idea of proof is refined from Wang [11], Cui [3], Ji and Wang [6], Ji and Zhan [7], when they studied packing constants, weakly convergent sequence coefficients and nonsquare constants.

Theorem 1 *Let $\Phi(u)$ be an N-function with $\Phi\in\Delta_2(0)$, and $\phi(t)$ be the right derivative of $\Phi(u)$. Then*

(i) $C_J(l^\Phi) = \sup_{\|x\|=1} \inf_{k>1}\{d_{x,k} : \rho_\Phi(\frac{kx}{d_{x,k}}) = \frac{k-1}{2}\}$ *if* $\phi(t)$ *is concave,*
(ii) $C_S(l^\Phi) = \inf_{\|x\|=1} \inf_{k>1}\{d_{x,k} : \rho_\Phi(\frac{kx}{d_{x,k}}) = \frac{k-1}{2}\}$ *if* $\phi(t)$ *is convex.*

Proof (i) Denote $d = \sup_{\|x\|=1} \inf_{k>1}\{d_{x,k} : \rho_\Phi(\frac{kx}{d_{x,k}}) = \frac{k-1}{2}\}$. Given $\varepsilon > 0$, there exists $x \in l^\Phi$ with $\|x\|_\Phi = 1$, and $d_{x,k}$ such that $d_{x,k} \geq d - \varepsilon$ for $k > 1$. Let $y = (x_1, 0, x_2, 0, x_3, \dots)$, and $z = (0, x_1, 0, x_2, 0, x_3, \dots)$. Obviously, $\|y\|_\Phi = \|z\|_\Phi = 1$, and we have

$$\inf_{k>1} \frac{1}{k}\Big[1 + \rho_\Phi\Big(\frac{k(y+z)}{d-\varepsilon}\Big)\Big] = \inf_{k>1} \frac{1}{k}\Big[1 + 2\rho_\Phi\Big(\frac{kx}{d-\varepsilon}\Big)\Big] \geq \inf_{k>1} \frac{1}{k}\Big[1 + 2\rho_\Phi\Big(\frac{kx}{d_{x,k}}\Big)\Big]$$
$$= \inf_{k>1} \frac{1}{k}\Big(1 + 2 \cdot \frac{k-1}{2}\Big) = 1.$$

Therefore,

$$\Big\|\frac{y+z}{d-\varepsilon}\Big\|_\Phi = \min\Big\{\inf_{0<k\leq 1} \frac{1}{k}\Big[1 + \rho_\Phi\Big(\frac{k(y+z)}{d-\varepsilon}\Big)\Big], \inf_{k>1} \frac{1}{k}\Big[1 + \rho_\Phi\Big(\frac{k(y+z)}{d-\varepsilon}\Big)\Big]\Big\}$$
$$\geq \min\Big\{1, \inf_{k>1} \frac{1}{k}\Big[1 + \rho_\Phi\Big(\frac{k(y+z)}{d-\varepsilon}\Big)\Big]\Big\} = 1.$$

It follows that $\|y+z\|_\Phi \geq d - \varepsilon$. Similarly we have $\|y-z\|_\Phi \geq d - \varepsilon$. By the arbitrariness of ε and the definition of $C_J(X)$ we have $C_J(l^\Phi) \geq d$.

On the other hand, for any pair of x and y with $\|x+y\|_\Phi = \|x-y\|_\Phi$, and for any $\varepsilon > 0$, there exist $k > 1$, $d_{x,k} < d+\varepsilon$ and $h > 1$, $d_{y,h} < d+\varepsilon$ such that $\rho_\Phi(\frac{kx}{d_{x,k}}) = \frac{k-1}{2}$, and $\rho_\Phi(\frac{hy}{d_{y,h}}) = \frac{h-1}{2}$. Since $\phi(t)$ is concave, we have from (20) that

$$\begin{aligned}
\Big\|\frac{x+y}{d+\varepsilon}\Big\|_\Phi + \Big\|\frac{x-y}{d+\varepsilon}\Big\|_\Phi &\leq \frac{k+h}{2kh}\Big[1 + \rho_\Phi\Big(\frac{2kh}{k+h}\Big(\frac{x+y}{d+\varepsilon}\Big)\Big)\Big] \\
&\quad + \frac{k+h}{2kh}\Big[1 + \rho_\Phi\Big(\frac{2kh}{k+h}\Big(\frac{x-y}{d+\varepsilon}\Big)\Big)\Big] \\
&= \frac{1}{k} + \frac{1}{h} + \frac{k+h}{2kh}\Big[\rho_\Phi\Big(\frac{2kh}{k+h}\Big(\frac{x+y}{d+\varepsilon}\Big)\Big) \\
&\quad + \rho_\Phi\Big(\frac{2kh}{k+h}\Big(\frac{x-y}{d+\varepsilon}\Big)\Big)\Big] \\
&\leq \frac{1}{k} + \frac{1}{h} + \frac{2}{k}\rho_\Phi\Big(\frac{kx}{d+\varepsilon}\Big) + \frac{2}{h}\rho_\Phi\Big(\frac{hy}{d+\varepsilon}\Big) \\
&< \frac{1}{k} + \frac{1}{h} + \frac{2}{k}\rho_\Phi\Big(\frac{kx}{d_{x,k}}\Big) + \frac{2}{h}\rho_\Phi\Big(\frac{hy}{d_{y,h}}\Big) \\
&= \frac{1}{k}\Big[1 + 2 \cdot \frac{k-1}{2}\Big] + \frac{1}{h}\Big[1 + 2 \cdot \frac{h-1}{2}\Big] = 2.
\end{aligned}$$

Therefore, we have $\|x+y\|_\Phi \le d+\varepsilon$ when $\|x+y\|_\Phi = \|x-y\|_\Phi$. By (4) and the arbitrariness of ε we have $C_J(l^\Phi) \le d$. Consequently, (i) is proved.

(ii) Let $c = \inf_{\|x\|=1} \inf_{k>1}\{d_{x,k} : \rho_\Phi(\frac{kx}{d_{x,k}}) = \frac{k-1}{2}\}$. Given $\varepsilon > 0$, there exists $x \in l^\Phi$ with $\|x\|_\Phi = 1$, such that $\inf_{k>1}\{d_{x,k} : \rho_\Phi(\frac{kx}{d_{x,k}}) = \frac{k-1}{2}\} < c + \frac{\varepsilon}{2}$. So, there are $k > 1$ and $d_{x,k} < c+\varepsilon$ such that $\rho_\Phi(\frac{kx}{d_{x,k}}) = \frac{k-1}{2}$. Put $y = (x_1, 0, x_2, 0, x_3, \dots)$, and $z = (0, x_1, 0, x_2, 0, x_3, \dots)$. We have

$$\begin{aligned}\Big\|\frac{y+z}{c+\varepsilon}\Big\|_\Phi = \Big\|\frac{y-z}{c+\varepsilon}\Big\|_\Phi &\le \frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{y+z}{c+\varepsilon}\Big)\Big]\\ &= \frac{1}{k}\Big[1+2\rho_\Phi\Big(\frac{kx}{c+\varepsilon}\Big)\Big] < \frac{1}{k}\Big[1+2\rho_\Phi\Big(\frac{kx}{d_{x,k}}\Big)\Big] = 1,\end{aligned}$$

which means that $\|y+z\|_\Phi = \|y-z\|_\Phi \le c+\varepsilon$, *i.e.*, by the definition of $C_S(X)$, that $C_S(l^\Phi) \ge c$, since ε is arbitrary.

Finally, we prove $C_S(l^\Phi) \le c$ if $\phi(t)$ is convex. By the definition of c, given x with $\|x\|_\Phi = 1$ and $k' > 1$ we have $d_{x,k'} \le c$ and $\rho_\Phi(\frac{k'x}{d_{x,k'}}) = \frac{k'-1}{2}$. Therefore, for any pair of $x, y \in S(l^\Phi)$ and $\varepsilon > 0$, there are k and h, such that

$$\begin{aligned}\Big\|\frac{x+y}{c}\Big\|_\Phi &> \frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(x+y)}{c}\Big)\Big] - \frac{\varepsilon}{2},\\ \Big\|\frac{x-y}{c}\Big\|_\Phi &> \frac{1}{h}\Big[1+\rho_\Phi\Big(\frac{h(x-y)}{c}\Big)\Big] - \frac{\varepsilon}{2}.\end{aligned}$$

By Lemma 1(ii)(22), we have for $k' = \frac{2kh}{k+h}$,

$$\begin{aligned}&\Big\|\frac{x+y}{c}\Big\|_\Phi + \Big\|\frac{x-y}{c}\Big\|_\Phi\\ &\quad> \frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(x+y)}{c}\Big)\Big] + \frac{1}{h}\Big[1+\rho_\Phi\Big(\frac{h(x-y)}{c}\Big)\Big] - \varepsilon\\ &\quad= \frac{k+h}{kh} + \frac{1}{k}\rho_\Phi\Big(\frac{k(x+y)}{c}\Big) + \frac{1}{h}\rho_\Phi\Big(\frac{h(x-y)}{c}\Big) - \varepsilon\\ &\quad\ge \frac{k+h}{kh} + \frac{k+h}{kh}\Big[\rho_\Phi\Big(\frac{2kh}{k+h}\frac{x}{c}\Big) + \rho_\Phi\Big(\frac{2kh}{k+h}\frac{y}{c}\Big)\Big] - \varepsilon\\ &\quad= \frac{k+h}{2kh}\Big[1+2\rho_\Phi\Big(\frac{2kh}{k+h}\frac{x}{c}\Big)\Big]\\ &\qquad+ \frac{k+h}{2kh}\Big[1+2\rho_\Phi\Big(\frac{2kh}{k+h}\frac{y}{c}\Big)\Big] - \varepsilon\\ &\quad\ge \frac{k+h}{2kh}\Big[1+2\rho_\Phi\Big(\frac{2kh}{k+h}\frac{x}{d_{x,\frac{2kh}{k+h}}}\Big)\Big]\\ &\qquad+ \frac{k+h}{2kh}\Big[1+2\rho_\Phi\Big(\frac{2kh}{k+h}\frac{y}{d_{x,\frac{2kh}{k+h}}}\Big)\Big] - \varepsilon\end{aligned}$$

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$$= \frac{k+h}{2kh}\Big[1 + 2\cdot\frac{\frac{2kh}{k+h}-1}{2}\Big] + \frac{k+h}{2kh}\Big[1 + 2\cdot\frac{\frac{2kh}{k+h}-1}{2}\Big] - \varepsilon$$
$$= 2 - \varepsilon.$$

Thus, when $\|x+y\|_\Phi = \|x-y\|_\Phi$ we have $\|x+y\|_\Phi \ge c(1-\frac{\varepsilon}{2})$. By the definition of $C_S(X)$ we have $C_S(l^\Phi) \ge c$.

The proof is completed. ■

To give the expressions for function spaces, we first show the following lemma:

Lemma 3 *Let $\Phi(u)$ be an N-function with $\Phi \in \Delta_2(\infty)$, then*

(i) *for every $x \in S\big(L^\Phi(\Omega)\big)$ and $k > 1$ there is a unique $c_{x,k} > 1$ such that*

$$\rho_\Phi\Big(\frac{2kx}{c_{x,k}}\Big) = 2k - 2;$$

(ii) *for every $x \in S\big(L^\Phi(\Omega)\big)$ and a number c, there are a pair of $y, z \in S\big(L^\Phi(\Omega)\big)$ such that*

$$\rho_\Phi\Big(\frac{y+z}{c}\Big) = \rho_\Phi\Big(\frac{y-z}{c}\Big) = \frac{1}{2}\rho_\Phi\Big(\frac{2x}{c}\Big).$$

Proof The proof for (i) is similar to that of Lemma 2. In fact, $\rho_\Phi(\frac{2kx}{c})$, as a continuous function, satisfies:

$$\rho_\Phi(2kx) \ge 2\rho_\Phi(kx) \ge 2(k\|x\|_\Phi - 1) = 2k - 2$$

and

$$\lim_{c\to\infty} \rho_\Phi\Big(\frac{2kx}{c}\Big) = 0 < 2k - 2$$

when $\Phi \in \Delta_2(\infty)$, there is unique $c_{x,k}$ such that $\rho_\Phi(\frac{2kx}{c_{x,k}}) = 2k-2$.

(ii) Suppose $\rho_\Phi(\frac{2x}{c}) = A$, *i.e.*, $\int_\Omega \Phi(\frac{2x}{c})\,dt = A$. This means there is a set $\Omega_1 \subset \Omega$ such that $\int_{\Omega_1} \Phi(\frac{2x}{c})\,dt = \frac{A}{2}$. Define

$$y(t) = x(t), \quad \text{and} \quad z(t) = \begin{cases} x(t), & t \in \Omega_1, \\ -x(t), & t \in \Omega \setminus \Omega_1. \end{cases}$$

Then we have

$$\int_\Omega \Phi\Big(\frac{y+z}{c}\Big)\,dt = \int_\Omega \Phi\Big(\frac{y-z}{c}\Big)\,dt$$
$$= \frac{1}{2}\int_{\Omega_1} \Phi\Big(\frac{2x}{c}\Big)\,dt = \frac{1}{2}\int_{\Omega\setminus\Omega_1} \Phi\Big(\frac{2x}{c}\Big)\,dt = \frac{A}{2}. \qquad ■$$

Theorem 2 *Let $\Phi(u)$ be an N-function with $\Phi \in \Delta_2(\infty)$, and $\phi(t)$ be the right derivative of $\Phi(u)$. Then*

(i) $C_S\big(L^\Phi(\Omega)\big) = \inf_{\|x\|=1}\inf_{k>1}\{c_{x,k} : \rho_\Phi(\frac{2kx}{c_{x,k}}) = 2k-2\}$ *if* $\phi(t)$ *is concave,*

(ii) $C_J\big(L^\Phi(\Omega)\big) = \sup_{\|x\|=1}\inf_{k>1}\{c_{x,k} : \rho_\Phi(\frac{2kx}{c_{x,k}}) = 2k-2\}$ *if* $\phi(t)$ *is convex.*

Proof (i) Let $a = \inf_{\|x\|=1}\inf_{k>1}\{c_{x,k} : \rho_\Phi(\frac{2kx}{c_{x,k}}) = 2k-2\}$. Given $\varepsilon > 0$, there exists $x \in S\big(L^\Phi(\Omega)\big)$, such that $\inf_{k>1}\{c_{x,k} : \rho_\Phi(\frac{2kx}{c_{x,k}}) = 2k-2\} < a + \frac{\varepsilon}{2}$. So, there are $k > 1$ and $c_{x,k} < a + \varepsilon$ such that $\rho_\Phi(\frac{2kx}{c_{x,k}}) = 2k-2$. By Lemma 3(ii) there exist $y, z \in S\big(L^\Phi(\Omega)\big)$ such that

$$\begin{aligned}\Big\|\frac{y+z}{a+\varepsilon}\Big\|_\Phi = \Big\|\frac{y-z}{a+\varepsilon}\Big\|_\Phi &\le \frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(y+z)}{a+\varepsilon}\Big)\Big]\\ &= \frac{1}{k}\Big[1+\frac{1}{2}\rho_\Phi\Big(\frac{2kx}{a+\varepsilon}\Big)\Big] < \frac{1}{k}\Big[1+\frac{1}{2}\rho_\Phi\Big(\frac{2kx}{c_{x,k}}\Big)\Big]\\ &= \frac{1}{k}\Big[1+\frac{1}{2}(2k-2)\Big] = 1,\end{aligned}$$

which means that $\|y+z\|_\Phi = \|y-z\|_\Phi \le a+\varepsilon$, *i.e.*, by the definition of $C_S(X)$, that $C_S\big(L^\Phi(\Omega)\big) \ge a$, since ε is arbitrary.

Next, we prove $C_S\big(L^\Phi(\Omega)\big) \le a$ if $\phi(t)$ is concave. By the definition of a, given x with $\|x\|_\Phi = 1$ and $k' > 1$ we have $c_{x,k'} \le a$ and $\rho_\Phi(\frac{2k'x}{c_{x,k'}}) = 2k'-2$. Therefore, for any pair of $x, y \in S\big(L^\Phi(\Omega)\big)$ and $\varepsilon > 0$, there are k and h, such that

$$\begin{aligned}\Big\|\frac{x+y}{a}\Big\|_\Phi &> \frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(x+y)}{a}\Big)\Big] - \frac{\varepsilon}{2},\\ \Big\|\frac{x-y}{a}\Big\|_\Phi &> \frac{1}{h}\Big[1+\rho_\Phi\Big(\frac{h(x-y)}{a}\Big)\Big] - \frac{\varepsilon}{2}.\end{aligned}$$

By Lemma 1(21), we have for $k' = \frac{2kh}{k+h}$,

$$\begin{aligned}&\Big\|\frac{x+y}{a}\Big\|_\Phi + \Big\|\frac{x-y}{a}\Big\|_\Phi\\ &\qquad > \frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(x+y)}{a}\Big)\Big] + \frac{1}{h}\Big[1+\rho_\Phi\Big(\frac{h(x-y)}{a}\Big)\Big] - \varepsilon\\ &\qquad = \frac{k+h}{kh} + \frac{1}{k}\rho_\Phi\Big(\frac{k(x+y)}{a}\Big) + \frac{1}{h}\rho_\Phi\Big(\frac{h(x-y)}{a}\Big) - \varepsilon\\ &\qquad \ge \frac{k+h}{kh} + \frac{k+h}{4kh}\Big[\rho_\Phi\Big(\frac{4kh}{k+h}\frac{x}{a}\Big) + \rho_\Phi\Big(\frac{4kh}{k+h}\frac{y}{a}\Big)\Big] - \varepsilon\\ &\qquad = \frac{k+h}{2kh}\Big[1+\frac{1}{2}\rho_\Phi\Big(\frac{2kh}{k+h}\frac{2x}{a}\Big)\Big]\\ &\qquad\quad + \frac{k+h}{2kh}\Big[1+\frac{1}{2}\rho_\Phi\Big(\frac{2kh}{k+h}\frac{2y}{a}\Big)\Big] - \varepsilon\end{aligned}$$

$$
\begin{aligned}
&\geq \frac{k+h}{2kh}\Big[1+\frac{1}{2}\rho_\Phi\Big(\frac{2kh}{k+h}\frac{2x}{c_{x,\frac{2kh}{k+h}}}\Big)\Big] \\
&\qquad + \frac{k+h}{2kh}\Big[1+\frac{1}{2}\rho_\Phi\Big(\frac{2kh}{k+h}\frac{2y}{c_{x,\frac{2kh}{k+h}}}\Big)\Big] - \varepsilon \\
&= \frac{k+h}{2kh}\Big[1+\frac{1}{2}\cdot\Big(2\frac{2kh}{k+h}-2\Big)\Big] \\
&\qquad + \frac{k+h}{2kh}\Big[1+\frac{1}{2}\cdot\Big(2\frac{2kh}{k+h}-2\Big)\Big] - \varepsilon \\
&= 2-\varepsilon.
\end{aligned}
$$

Thus, when $\|x+y\|_\Phi = \|x-y\|_\Phi$ we have $\|x+y\|_\Phi \geq a(1-\frac{\varepsilon}{2})$. By the definition of $C_S(X)$ we have $C_S\big(L^\Phi(\Omega)\big) \geq a$.

(ii) Denote $b = \sup_{\|x\|=1}\inf_{k>1}\{c_{x,k} : \rho_\Phi(\frac{2kx}{c_{x,k}}) = 2k-2\}$. Given $\varepsilon > 0$, there exists $x \in L^\Phi(\Omega)$ with $\|x\|_\Phi = 1$, and $c_{x,k}$ such that $c_{x,k} \geq b-\varepsilon$ for $k > 1$. By Lemma 3(ii) there are $y, z \in S\big(L^\Phi(\Omega)\big)$ such that

$$
\begin{aligned}
\inf_{k>1}\frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(y+z)}{b-\varepsilon}\Big)\Big] &= \inf_{k>1}\frac{1}{k}\Big[1+\frac{1}{2}\rho_\Phi\Big(\frac{2kx}{b-\varepsilon}\Big)\Big] \\
&\geq \inf_{k>1}\frac{1}{k}\Big[1+\frac{1}{2}\rho_\Phi\Big(\frac{kx}{c_{x,k}}\Big)\Big] \\
&= \inf_{k>1}\frac{1}{k}\Big(1+\frac{1}{2}\cdot(2k-2)\Big) = 1.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\Big\|\frac{y+z}{b-\varepsilon}\Big\|_\Phi &= \min\Big\{\inf_{0<k\leq 1}\frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(y+z)}{b-\varepsilon}\Big)\Big], \inf_{k>1}\frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(y+z)}{b-\varepsilon}\Big)\Big]\Big\} \\
&\geq \min\Big\{1, \inf_{k>1}\frac{1}{k}\Big[1+\rho_\Phi\Big(\frac{k(y+z)}{b-\varepsilon}\Big)\Big]\Big\} = 1.
\end{aligned}
$$

It follows that $\|y+z\|_\Phi \geq b-\varepsilon$. Similarly we have $\|y-z\|_\Phi \geq b-\varepsilon$. By the arbitrariness of ε and the definition of $C_J(X)$ we have $C_J\big(L^\Phi(\Omega)\big) \geq b$.

Finally, for any pair of x and y with $\|x+y\|_\Phi = \|x-y\|_\Phi$, and for any $\varepsilon > 0$, there exist $k > 1$, $c_{x,k} < b+\varepsilon$ and $h > 1$, $c_{y,h} < b+\varepsilon$ such that $\rho_\Phi(\frac{2kx}{c_{x,k}}) = 2k-2$, and $\rho_\Phi(\frac{2hy}{c_{y,h}}) = 2h-2$. Since $\phi(t)$ is convex, we have from (23) that

$$
\begin{aligned}
\Big\|\frac{x+y}{d+\varepsilon}\Big\|_\Phi + \Big\|\frac{x-y}{d+\varepsilon}\Big\|_\Phi \leq \frac{k+h}{2kh}\Big[1+\rho_\Phi\Big(\frac{2kh}{k+h}\Big(\frac{x+y}{d+\varepsilon}\Big)\Big)\Big] \\
+ \frac{k+h}{2kh}\Big[1+\rho_\Phi\Big(\frac{2kh}{k+h}\Big(\frac{x-y}{d+\varepsilon}\Big)\Big)\Big]
\end{aligned}
$$

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$$= \frac{1}{k} + \frac{1}{h} + \frac{k+h}{2kh}\left[\rho_\Phi\left(\frac{2kh}{k+h}\left(\frac{x+y}{b+\varepsilon}\right)\right) + \rho_\Phi\left(\frac{2kh}{k+h}\left(\frac{x-y}{b+\varepsilon}\right)\right)\right]$$
$$\leq \frac{1}{k} + \frac{1}{h} + \frac{1}{2k}\rho_\Phi\left(\frac{2kx}{b+\varepsilon}\right) + \frac{1}{2h}\rho_\Phi\left(\frac{2hy}{b+\varepsilon}\right)$$
$$< \frac{1}{k} + \frac{1}{h} + \frac{1}{2k}\rho_\Phi\left(\frac{2kx}{c_{x,k}}\right) + \frac{1}{2h}\rho_\Phi\left(\frac{2hy}{c_{y,h}}\right)$$
$$= \frac{1}{k}\left[1 + \frac{1}{2}(2k-2)\right] + \frac{1}{h}\left[1 + \frac{1}{2}(2h-2)\right] = 2.$$

Therefore, we have $\|x+y\|_\Phi \leq b+\varepsilon$ when $\|x+y\|_\Phi = \|x-y\|_\Phi$. By (4) and the arbitrariness of ε we have $C_J\big(L^\Phi(\Omega)\big) \leq b$. Consequently, (ii) is proved.

The proof is completed. ■

3 Bounds of $C_J(l^\Phi)$ For Sequence Spaces

In view of (3), we now mainly deal with $C_J(l^\Phi)$. We first estimate the lower bound of it. This work is an extension of Ren [10], who obtained that

$$\max\left(2\beta_\Psi^0, \frac{1}{\alpha_\Psi^0}\right) \leq C_J(l^\Phi), \tag{26}$$

where $\alpha_\Psi^0, \beta_\Psi^0$ are defined as in (7). We first extend these indices for $\Psi(v)$, the complementary function of $\Phi(u)$, and denote

$$\alpha'_\Psi = \inf\left\{\frac{\Psi^{-1}(\frac{1}{2k})}{\Psi^{-1}(\frac{1}{k})} : k = 1, 2, \dots\right\}, \quad \beta'_\Psi = \sup\left\{\frac{\Psi^{-1}(\frac{1}{2k})}{\Psi^{-1}(\frac{1}{k})} : k = 1, 2, \dots\right\}. \tag{27}$$

Theorem 3 *Let $\Phi(u)$ be an N-function. Then the nonsquare constant of sequence space l^Φ generated by Φ equipped with Orlicz norm satisfies:*

$$\max\left(2\beta'_\Psi, \frac{1}{\alpha'_\Psi}\right) \leq C_J(l^\Phi). \tag{28}$$

Proof First we show

$$2\beta'_\Psi \leq C_J(l^\Phi). \tag{29}$$

For any natural number k, put

$$x = \Big(\overbrace{\frac{1}{k\Psi^{-1}(\frac{1}{k})}, \dots, \frac{1}{k\Psi^{-1}(\frac{1}{k})}}^{k}, 0, 0, \dots\Big)$$

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and

$$y = \Big(\overbrace{0, \dots, 0}^{k}, \overbrace{\frac{1}{k\Psi^{-1}(\frac{1}{k})}, \dots, \frac{1}{k\Psi^{-1}(\frac{1}{k})}}^{k}, 0, 0, \dots\Big).$$

Then we have $\|x\|_\Phi = \|y\|_\Phi = 1$ and

$$\|x-y\|_\Phi = \|x+y\|_\Phi = \frac{1}{k\Psi^{-1}(\frac{1}{k})} \cdot 2k\Psi^{-1}\Big(\frac{1}{2k}\Big) = \frac{2\Psi^{-1}(\frac{1}{2k})}{\Psi^{-1}(\frac{1}{k})}.$$

Therefore,

$$\min(\|x-y\|_\Phi, \|x+y\|_\Phi) \geq \frac{2\Psi^{-1}(\frac{1}{2k})}{\Psi^{-1}(\frac{1}{k})} \quad (k = 1, 2, \dots).$$

We have proved (29) by the definition (1) and (27).

Secondly, we prove

$$\frac{1}{\alpha'_\Psi} \leq J(l^\Phi). \tag{30}$$

Given a natural number k, put $t = \frac{1}{2k\Psi^{-1}(\frac{1}{2k})}$. Denote

$$x = (\overbrace{t, \dots, t}^{k}, \overbrace{t, \dots, t}^{k}, 0, 0, \dots)$$

and

$$y = (\overbrace{t, \dots, t}^{k}, \overbrace{-t, \dots, -t}^{k}, 0, 0, \dots).$$

Then $\|x\|_\Phi = \|y\|_\Phi = 1$ and

$$\|x-y\|_\Phi = \|x+y\|_\Phi = 2t \cdot k\Psi^{-1}\Big(\frac{1}{k}\Big) = \frac{\Psi^{-1}(\frac{1}{k})}{\Psi^{-1}(\frac{1}{2k})}.$$

Therefore,

$$\min(\|x-y\|_\Phi, \|x+y\|_\Phi) \geq \frac{\Psi^{-1}(\frac{1}{k})}{\Psi^{-1}(\frac{1}{2k})} \quad (k = 1, 2, \dots)$$

and we obtain (30). Finally, (28) follows from (29) and (30). ∎

We now deal with the upper bound of $C_J(l_\Phi)$.

Theorem 4 *Let $\Phi(u)$ be an N-function, $\phi(t)$ being its right derivative. We have*

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(i) *if $\phi(t)$ is concave, then*

$$C_J(l^\Phi) \le \frac{1}{\alpha_\Phi^*}; \tag{31}$$

(ii) *if $\phi(t)$ is convex, then*

$$C_J(l^\Phi) \le 2\beta_\Phi^*, \tag{32}$$

where

$$\alpha_\Phi^* = \inf\left\{ \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u \le \frac{1}{2}(Q_\Phi - 1) \right\}, \tag{33}$$

$$\beta_\Phi^* = \sup\left\{ \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)} : 0 < u \le \frac{1}{2}(Q_\Phi - 1) \right\}, \tag{34}$$

with

$$Q_\Phi = \sup_{\|x\|_\Phi = 1} \left\{ k_x > 1 : \|x\|_\Phi = \frac{1}{k_x}[1 + \rho_\Phi(k_x x)] \right\}. \tag{35}$$

Proof If $\phi(t)$ is concave, then $\Phi \in \Delta_2(0)$ (see Krasnosel'skii and Rutickii [8]). If $\phi(t)$ is convex and $\Phi \notin \Delta_2(0)$, then $C_J(l^\Phi) = 2$ while $2\beta_\Phi^* \ge 2\beta_\Phi^0 = 2$, which means that (32) holds. Therefore, it suffices for us to check (31) and (32) when $\Phi \in \Delta_2(0)$.

(i) Define the index function $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ for $u > 0$. Then

$$\Phi[G_\Phi(u)\Phi^{-1}(2u)] = u.$$

For any given $x = (x_1, x_2, \dots) \in S(l^\Phi)$, there exists $k > 1$ such that

$$1 = \|x\|_\Phi = \frac{1}{k}[1 + \rho_\Phi(kx)].$$

Let $u_i = \frac{1}{2}\Phi(k|x_i|)$ for all $x_i \ne 0$. Then $\Phi(k|x_i|) \le \rho_\Phi(kx) = k - 1 \le Q_\Phi - 1$, and so $u_i \le \frac{1}{2}(Q_\Phi - 1)$. It follows from (33) that

$$\begin{aligned}\rho_\Phi(\alpha_\Phi^* kx) &= \sum_{i=1}^\infty \Phi(\alpha_\Phi^* k|x_i|) \le \sum_{i=1}^\infty \Phi\left[G_\Phi\left(\frac{1}{2}\Phi(k|x_i|)\right) k|x_i| \right] \\ &= \frac{1}{2}\sum_{i=1}^\infty \Phi(k|x_i|) = \frac{k-1}{2}.\end{aligned}$$

Because of Theorem 1(i), we obtain (31).

(ii) Analogously, one can prove

$$C_J(l^\Phi) \ge \frac{1}{\beta_\Phi^*}.$$

By (3), we obtain (32). ∎

To improve the above results, we give:

Lemma 4 *Let Φ and Ψ be a pair of complementary N-functions, $\phi(t)$ be the right derivative of Φ. We have*

(i) *if $\phi(t)$ is concave, then*

$$\frac{1}{\alpha'_\Psi} \le \sqrt{2} \le 2\beta'_\Psi; \tag{36}$$

(ii) *if $\phi(t)$ is convex, then*

$$2\beta'_\Psi \le \sqrt{2} \le \frac{1}{\alpha'_\Psi}. \tag{37}$$

Proof (i) It is easy to check that $\bar{B}_\Phi \le 2$, since the function $M(t) = t\phi(t) - 2\Phi(t)$ is decreasing on $[0,\infty)$ and ≤ 0, when $\phi(t)$ is concave. Therefore, by (15) and (18):

$$2\beta'_\Psi \ge 2\bar{\alpha}_\Psi = \frac{1}{\bar{\beta}_\Phi} \ge 2^{\frac{1}{\bar{B}_\Phi}} \ge 2^{\frac{1}{2}} = \sqrt{2},$$

$$\frac{1}{\alpha'_\Psi} \le \frac{1}{\bar{\alpha}_\Psi} = 2\bar{\beta}_\Phi \le 2 \cdot 2^{-\frac{1}{\bar{B}_\Phi}} \le 2 \cdot 2^{-\frac{1}{2}} = \sqrt{2},$$

which implies (36).

(ii) Observe that $\bar{A}_\Phi \ge 2$ by the same reason as in (i) when $\phi(t)$ is convex. Therefore, (37) holds since

$$\frac{1}{\alpha'_\Psi} \ge \frac{1}{\bar{\beta}_\Psi} = 2\bar{\alpha}_\Phi \ge 2 \cdot 2^{-\frac{1}{\bar{A}_\Phi}} \ge 2 \cdot 2^{-\frac{1}{2}} = \sqrt{2},$$

and

$$2\beta'_\Psi \le 2\bar{\beta}_\Psi = \frac{1}{\bar{\alpha}_\Phi} \le 2^{\frac{1}{\bar{A}_\Phi}} \le 2^{\frac{1}{2}} = \sqrt{2}.$$ ∎

Theorem 5 *Let $\Phi(u)$ be an N-function, $\phi(t)$ being its right derivative. We have*

(i) *if $\phi(t)$ is concave, then*

$$2\beta'_\Psi \le C_J(l^\Phi) \le \frac{1}{\alpha^*_\Phi}; \tag{38}$$

(ii) *if $\phi(t)$ is convex, then*

$$\frac{1}{\alpha'_\Psi} \le C_J(l^\Phi) \le 2\beta^*_\Phi. \tag{39}$$

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Proof (38) directly results from (28), (31) and (36), while (39) from (28), (32) and (37). ■

To make Theorem 4 easier to use, we shall further improve it. First of all, note that the author [13] gave a fine estimate of Q_Φ in (35):

$$Q_\Phi \le b^*_\Psi, \tag{40}$$

where

$$b^*_\Psi = \sup\left\{ \frac{s\psi(s)}{\Psi(s)} : 0 < s \le \Psi^{-1}(1) \right\}. \tag{41}$$

Denote the index functions by

$$F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)},\ G_\Phi(c,u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(cu)} \quad (c > 1). \tag{42}$$

Then the author [13] proved that

Lemma 5 *Suppose Φ, Ψ be a pair of complementary N-functions.*

(i) *$F_\Phi(t)$ is increasing (decreasing) on $(0, \Phi^{-1}(u_0)]$ if and only if $G_\Phi(c,u)$ is increasing (decreasing) on $(0, \frac{u_0}{c}]$ for every $c > 1$.*
(ii) *$F_\Phi(t)$ is increasing(decreasing) on $(0, \psi(C)]$ if and only if $F_\Psi(s) = \frac{s\psi(s)}{\Psi(s)}$ is decreasing (increasing) on $(0, C]$.*

Theorem 6 *Let Φ, Ψ be a pair of N-functions, $\phi(t)$ and $\psi(s)$ be their right derivatives, respectively.*

(i) *If $\phi(t)$ is concave, we have*
  (A) *if $F_\Phi(t)$ is increasing on $\big(0, \psi[\Psi^{-1}(1)]\big]$, then*

$$C_J(l^\Phi) = 2^{\frac{1}{C^0_\Phi}}, \quad C^0_\Phi = \lim_{t\to 0^+} F_\Phi(t); \tag{43}$$

  (B) *if $F_\Phi(t)$ is decreasing on $\big(0, \psi[\Psi^{-1}(1)]\big]$, then*

$$\frac{2\Psi^{-1}(\frac{1}{2})}{\Psi^{-1}(1)} \le C_J(l^\Phi) \le \frac{\psi[\Psi^{-1}(1)]}{\Phi^{-1}\big\{ \frac{1}{2}\Phi\big(\psi[\Psi^{-1}(1)]\big) \big\}}. \tag{44}$$

(ii) *If $\phi(t)$ is convex, we have*
  (A) *if $F_\Phi(t) =$ is increasing on $\big(0, \psi[\Psi^{-1}(1)]\big]$, then*

$$\frac{\Psi^{-1}(1)}{\Psi^{-1}(\frac{1}{2})} \le C_J(l^\Phi) \le \frac{2\Phi^{-1}(\frac{C^0_\Psi - 1}{2})}{\Phi^{-1}(C^0_\Psi - 1)}, \quad C^0_\Psi = \lim_{s\to 0^+} F_\Phi(s); \tag{45}$$

(B) *if* $F_\Phi(t)$ *is decreasing on* $\big(0, \psi[\Psi^{-1}(1)]\big]$, *then*

$$C_J(l^\Phi) = 2^{1-\frac{1}{C_\Phi^0}}. \tag{46}$$

Proof Under the condition of (A) in (i), $F_\Psi(s) = \frac{s\psi(s)}{\Psi(s)}$ is decreasing on $(0, \Psi^{-1}(1)]$, which implies that $G_\Psi(v) = \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}$ is decreasing on $(0, \frac{1}{2}]$, and $G_\Phi(u) = \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}$ is increasing on $(0, \frac{1}{2}\Phi(\psi[\Psi^{-1}(1)])]$, and hence

$$\begin{aligned} b_\Psi^* &= [F_\Psi(s)]_{s=\Psi^{-1}(1)} = \Psi^{-1}(1)\psi[\Psi^{-1}(1)] \\ &= \Phi\{\psi[\Psi^{-1}(1)]\} + \Psi[\Psi^{-1}(1)] \\ &= \Phi\{\psi[\Psi^{-1}(1)]\} + 1, \end{aligned}$$

i.e.,

$$b_\Psi^* - 1 = \Phi\{\psi[\Psi^{-1}(1)]\}.$$

It follows that

$$\Big(0, \frac{1}{2}(Q_\Phi - 1)\Big] \subset \Big(0, \frac{1}{2}(b_\Psi^* - 1)\Big] = \Big(0, \frac{1}{2}\Phi\big(\psi[\Psi^{-1}(1)]\big)\Big].$$

By (13) and (19), we have

$$2\beta_\Psi' = 2\beta_\Psi^0 = \frac{1}{\alpha_\Phi^0} = 2^{\frac{1}{C_\Phi^0}},$$

and

$$\frac{1}{\alpha_\Phi^*} = \frac{1}{\alpha_\Phi^0} = 2^{\frac{1}{C_\Phi^0}}.$$

It follows that $C_J(l^\Phi) = 2^{\frac{1}{C_\Phi^0}}$ from (38). So (43) is proved. Similarly, since $G_\Phi(u)$ is decreasing, one has

$$\alpha_\Phi^* = \frac{\Phi^{-1}\{\frac{1}{2}(Q_\Phi - 1)\}}{\Phi^{-1}(Q_\Phi - 1)} \geq \frac{\Phi^{-1}\big\{\frac{1}{2}\Phi\big(\psi[\Psi^{-1}(1)]\big)\big\}}{\psi[\Psi^{-1}(1)]}.$$

Then by (38), we have

$$C_J(l^\Phi) \leq \frac{1}{\alpha_\Phi^*} \leq \frac{\psi[\Psi^{-1}(1)]}{\Phi^{-1}\big\{\frac{1}{2}\Phi\big(\psi[\Psi^{-1}(1)]\big)\big\}},$$

and

$$C_J(l^\Phi) \geq 2\beta_\Psi' = \frac{2\Psi^{-1}(\frac{1}{2})}{\Psi^{-1}(1)}$$

since $G_\Psi(v)$ is increasing on $(0, \frac{1}{2}]$. Thus, (44) holds.

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(ii)(A) If $F_\Phi(t)$ is increasing on $\left(0, \psi[\Psi^{-1}(1)]\right]$, by the argument as in (i), we have

$$Q_\Phi \le b_\Psi^* = C_\Psi^0.$$

Hence, it follows that β_Φ^* satisfies

$$\beta_\Phi^* \le \frac{\Phi^{-1}(u)}{\Phi^{-1}(2u)}\Bigg|_{\frac{C_\Psi^0-1}{2}}.$$

Finally one has

$$C_J(l^\Phi) \le \frac{2\Phi^{-1}(\frac{C_\Psi^0-1}{2})}{\Phi^{-1}(C_\Psi^0-1)}$$

and

$$C_J(l^\Phi) \ge \frac{1}{\alpha'_\Psi} = \frac{\Psi^{-1}(1)}{\Psi^{-1}(\frac{1}{2})}$$

by (39). So (45) holds.

(B) It can be proved analogously as in (i)(A). In fact, observe that

$$2\beta_\Phi^* = 2\beta_\Phi^0 = 2\cdot 2^{-\frac{1}{C_\Phi^0}} = 2^{1-\frac{1}{C_\Phi^0}},$$

and

$$\frac{1}{\alpha'_\Psi} = \frac{1}{\alpha_\Psi^0} = 2\beta_\Phi^0 = 2^{1-\frac{1}{C_\Phi^0}}.$$

Therefore, (46) is proved from (39). The proof is completed. ∎

Example 1 Let the N-function be $\Phi(u) = \frac{1}{p}|u|^p$, $p > 1$, which generates l^p space. The complementary N-function is $\Psi(v) = \frac{1}{q}|v|^q$, with $\frac{1}{p} + \frac{1}{q} = 1$.

Since $\phi(t) = t^{p-1}$, we see that $\phi(t)$ is concave for $1 < p \le 2$ and is convex for $p \ge 2$. Because $F_\Phi(t) = \frac{t\phi(t)}{\Phi(t)} = p$ is a constant function and can be considered as either increasing or decreasing, we have immediately from (43) and (46) that

$$C_J(l^p) = 2^{\frac{1}{p}}, \quad 1 < p \le 2; \tag{47}$$

$$C_J(l^p) = 2^{1-\frac{1}{p}}, \quad p \ge 2. \tag{48}$$

Example 2 Let a pair of N-functions be defined as (8), *i.e.*,

$$M(u) = e^{|u|} - |u| - 1 \quad \text{and} \quad N(v) = (1+|v|)\ln(|v|+1) - |v|.$$

It is easy to check that $M'(t) = e^t - t$ is convex and $N'(t) = \ln(1+t)$ is concave and $F_M(t)$ is increasing on $[0,\infty)$ while $F_N(t)$ is decreasing. Now we estimate $C_J(l^M)$ and

$C_J(l^N)$ with (45) and (44), respectively. It is easy to check that

$$\frac{N^{-1}(1)}{N^{-1}(\frac{1}{2})} \le C_J(l^M) \le \frac{2M^{-1}(\frac{C_N^0-1}{2})}{M^{-1}(C_N^0-1)},$$
$$\frac{2M^{-1}(\frac{1}{2})}{M^{-1}(1)} \le C_J(l^N) \le \frac{M'[M^{-1}(1)]}{N^{-1}\{\frac{1}{2}N\big(M'[M^{-1}(1)]\big)\}}.$$

Note that $C_M^0 = 2$. By a simple computation we have:

$$1.487 \le C_J(l^M) \le 1.497; \quad 1.496 \le C_J(l^N) \le 1.498. \tag{49}$$

4 Bounds of $C_J(L^\Phi[0,1])$ For Function Spaces

Similar to last section, we mainly deal with $C_J\big(L^\Phi(\Omega)\big)$. For the sake of convenience, we assume $\Omega = [0,1]$. We now estimate the lower bound of it. Observe that Ren [10] obtained the following result:

$$\max\Big(2\beta_\Psi, \frac{1}{\alpha_\Psi}\Big) \le C_J(L^\Phi[0,1]), \tag{50}$$

where α_Ψ, β_Ψ are the limits of $G_\Psi = \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}$ at the neighborhood of ∞ (see (8)). We first extend these indices for $\Psi(v)$, and denote

$$\alpha_{\Psi[1,\infty)} = \inf_{v\in[1,\infty)} \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}, \quad \beta_{\Psi[1,\infty)} = \sup_{v\in[1,\infty)} \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}. \tag{51}$$

Theorem 7 *Let $\Phi(u)$ be an N-function. Then the nonsquare constant of function space $L^\Phi[0,1]$ generated by Φ equipped with Orlicz norm satisfies:*

$$\max\Big(2\beta_{\Psi[1,\infty)}, \frac{1}{\alpha_{\Psi[1,\infty)}}\Big) \le C_J(L^\Phi[0,1]). \tag{52}$$

Proof For any $v \in [1,\infty)$, choose on $[0,1]$ a pair of subsets G_1, and G_2 such that $G_1 \cap G_2 = \varnothing$, $\mu(G_1) = \mu(G_2) = \frac{1}{2v}$. Denote

$$x(t) = \frac{2v}{\Psi^{-1}(2v)}\chi_{G_1}(t) \quad \text{and} \quad y(t) = \frac{2v}{\Psi^{-1}(2v)}\chi_{G_2}(t).$$

Since

$$\|\chi_{G_1}\|_\Phi = \|\chi_{G_2}\|_\Phi = \mu(G_1)\Psi^{-1}\Big(\frac{1}{\mu(G_1)}\Big) = \frac{1}{2v}\Psi^{-1}(2v),$$

we have $\|x\|_\Phi = \|y\|_\Phi = 1$ and $\|x-y\|_\Phi = \|x+y\|_\Phi = \frac{2\Psi^{-1}(v)}{\Psi^{-1}(2v)}$. Taking the supreme over $v \in [1,\infty)$, we have

$$J(L^\Phi[0,1]) \ge 2\beta_{\Psi[1,\infty)}.$$

https://doi.org/10.4153/CJM-2003-009-1 Published online by Cambridge University Press

Next we show

$$\frac{1}{\alpha_{\Psi[1,\infty)}} \leq J(L^{\Phi}[0,1]).$$

Given $v \in [1,\infty)$, define $E_1, E_2 \subset [0,1]$, satisfying $E_1 \cap E_2 = \varnothing$ and $\mu(E_1) = \mu(E_2) = \frac{1}{2v}$. Put

$$x(t) = \frac{v}{\Psi^{-1}(v)}[\chi_{E_1}(t) + \chi_{E_2}(t)], \quad y(t) = \frac{v}{\Psi^{-1}(v)}[\chi_{E_1}(t) - \chi_{E_2}(t)],$$

Then $\|x\|_\Phi = \|y\|_\Phi = 1$, and $\|x-y\|_\Phi = \|x+y\|_\Phi = \frac{\Psi^{-1}(2v)}{\Psi^{-1}(v)}$. Take supreme over $v \in [1,\infty)$, we immediately have

$$J(L^{\Phi}[0,1]) \geq \sup_{v\in[1,\infty)} \frac{\Psi^{-1}(2v)}{\Psi^{-1}(v)} = \frac{1}{\inf_{v\in[1,\infty)} \frac{\Psi^{-1}(v)}{\Psi^{-1}(2v)}} = \frac{1}{\alpha_{\Psi[1,\infty)}}.$$

The proof is completed. ■

We now deal with the upper bound of $C_J(L^{\Phi}[0,1])$.

Theorem 8 *Let $\Phi(u)$ be an N-function, $\phi(t)$ being its right derivative. We have*

(i) *if $\phi(t)$ is concave, then*

$$C_J(L^{\Phi}[0,1]) \leq \frac{1}{\bar{\alpha}_\Phi}, \tag{53}$$

(ii) *if $\phi(t)$ is convex, then*

$$C_J(L^{\Phi}[0,1]) \leq 2\bar{\beta}_\Phi \tag{54}$$

Proof If $\phi(t)$ is concave, then $\Phi \in \Delta_2(\infty)$ (see Krasnosel'skii and Rutickii [8]). If $\phi(t)$ is convex and $\Phi \notin \Delta_2(\infty)$, then $C_J(L^{\Phi}[0,1]) = 2$ while $2\bar{\beta}_\Phi = 2$, which means that (54) holds. Therefore, it suffices for us to check (53) and (54) when $\Phi \in \Delta_2(\infty)$.

(i) Define the index function $H_\Phi(u) = \frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)}$ for $u > 0$. Then

$$\Phi[H_\Phi(u)\Phi^{-1}(u)] = 2u.$$

For any given $x = x(t) \in S(L^{\Phi}[0,1])$, there exists $k > 1$ such that

$$1 = \|x\|_\Phi = \frac{1}{k}[1 + \rho_\Phi(kx)].$$

Put $u(t) = \Phi(k|x(t)|)$ for all $x(t) \neq 0$ then

$$2\Phi\big(k|x(t)|\big) = \Phi\Big[H\Big(\Phi\big(k|x(t)|\big)\Big) \cdot k|x(t)|\Big].$$

Therefore, when $u = \Phi\big(x(t)\big) \geq 0$ we have

$$\rho_\Phi\left(\frac{2kx(t)}{2\bar{\alpha}_\Phi}\right) = \rho_\Phi\left(\frac{kx(t)}{\bar{\alpha}_\Phi}\right) \geq \rho_\Phi\left(\frac{\Phi^{-1}(2u)}{\Phi^{-1}(u)} \cdot k|x(t)|\right)$$
$$= \rho_\Phi[H(u) \cdot k|x(t)|] = 2\rho_\Phi\big(kx(t)\big) = 2k - 2.$$

Because of Theorem 2(i), we obtain

$$C_S(L^\Phi[0,1]) \geq 2\bar{\alpha}_\Phi,$$

which implies (53), by (3).

(ii) Analogously, one can prove (54). ■

Observe the right sides of (53) and (54) can be changed to be $2\bar{\beta}_\Psi$ and $1/\bar{\alpha}_\Psi$ by (15), respectively. From the above two theorems and the same reason as in Lemma 4 and 5, we deduce the results parallel to Theorem 5 and 6:

Theorem 9 *Let Φ, Ψ be a pair of N-functions, $\phi(t)$ being the right derivative of Φ. We have*

(i) *if $\phi(t)$ is concave, then*

$$2\beta_{\Psi[1,\infty)} \leq C_J(L^\Phi[0,1]) \leq 2\bar{\beta}_\Psi; \tag{55}$$

(ii) *if $\phi(t)$ is convex, then*

$$\frac{1}{\alpha_{\Psi[1,\infty)}} \leq C_J(L^\Phi[0,1]) \leq \frac{1}{\bar{\alpha}_\Psi}. \tag{56}$$ ■

Theorem 10 *Let Φ, Ψ be a pair of N-functions, $\phi(t)$ be the right derivative of Φ. Then*

(i) *If $\phi(t)$ is concave, we have*

(A) *if $F_\Phi(t)$ is increasing on $(0,\infty)$, then*

$$\frac{2\Psi^{-1}(1)}{\Psi^{-1}(2)} \leq C_J(L^\Phi[0,1]) \leq 2^{\frac{1}{C_\Phi^0}}, \tag{57}$$

(B) *if $F_\Phi(t)$ is decreasing on $(0,\infty)$, then*

$$C_J(L^\Phi[0,1]) = 2^{\frac{1}{C_\Phi}}; \tag{58}$$

(ii) *If $\phi(t)$ is convex, we have*

(A) *if $F_\Phi(t)$ is increasing on $(0,\infty)$, then*

$$C_J(L^\Phi[0,1]) = 2^{1-\frac{1}{C_\Phi}}, \tag{59}$$

(B) *if* $F_\Phi(t)$ *is decreasing on* $(0,\infty)$, *then*

$$\frac{\Psi^{-1}(2)}{2\Psi^{-1}(1)} \le C_J(L^\Phi[0,1]) \le 2^{1-\frac{1}{C_\Phi^0}}, \tag{60}$$

where C_Φ^0 *and* C_Φ *are defined as in (19).* ■

Remark For the N-function $\Phi(u) = \frac{1}{p}|u|^p$, $p > 1$, which generates L^p space, by the argument as in Example 1, we have from (58) and (59) (or (57) and (60)) that

$$C_J(L^p) = 2^{\frac{1}{p}} \quad (1 < p \le 2) \text{ and } C_J(L^p) = 2^{1-\frac{1}{p}} \quad (p \ge 2). \tag{61}$$

For the pair of N-functions described in Example 2, *i.e.*,

$$M(u) = e^{|u|} - |u| - 1 \quad \text{and} \quad N(v) = (1 + |v|)\ln(|v| + 1) - |v|.$$

Since $C_M = \lim_{t\to\infty} F_M(t) = \infty, C_N = \lim_{t\to\infty} F_N(t) = 1$, which implies $M \notin \Delta_2(\infty)$, $N \notin \nabla_2(\infty)$, we have from (59) and (58) that

$$J(L^M[0,1]) = 2^{1-\frac{1}{C_M}} = 2; \quad J(L^N[0,1]) = 2^{\frac{1}{C_N}} = 2. \tag{62}$$

This result agrees with the fact that the spaces $L^M[0,1]$ and $L^N[0,1]$ are both nonreflexive.

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