A COMBINATORIAL ANALOGUE OF POINCARÉ'S DUALITY THEOREM

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Introduction. For a non-negative integer s and a finite simplicial complex K, let $\beta_s(K)$ denote the s-dimensional Betti number of K and let $f_s(K)$ denote the number of s-simplices of K. Our theorem, like Poincaré's, applies to combinatorial manifolds M, but it concerns the numbers $f_s(M)$ instead of the numbers $\beta_s(M)$. One of the formulae given below is used by the author in **(5)** to establish a sharp upper bound for the number of vertices of n-dimensional convex polytopes which have a given number i of $(n-1)$-faces. This amounts to estimating the size of the computation problem which may be involved in solving a system of i linear inequalities in n variables, and was the original motivation for our study.

A *combinatorial n-manifold*[1] is a finite simplicial n-complex M^n such that for each s-simplex $\sigma^s \in M^n$, the linked complex $L(\sigma^s, M^n)$ has the same homology groups as an $(n-s-1)$-sphere; analogously, an *Eulerian n-manifold* is defined here by the condition that $L(\sigma^s, M^n)$ always has the same *Euler characteristic* $1-(-1)^{n-s}$ as an $(n-s-1)$-sphere, where of course the Euler characteristic of a finite complex K is the alternating sum

$$\chi(K) = \sum_{s=0}^{\infty} (-1)^s f_s(K) \left(= \sum_{s=0}^{\infty} (-1)^s \beta_s(K)\right).$$

Let $\mathbf{E}^n$ $[\mathbf{C}^n]$ denote the class of all Eulerian [orientable combinatorial] n-manifolds, and for each $M \in \mathbf{E}^n$ let

$$\beta(M) = (\beta_0(M), \beta_1(M), \ldots, \beta_n(M)) \quad \textit{and} \quad f(M) = (f_0(M), f_1(M), \ldots, f_n(M)).$$

Then define

$$\beta(\mathbf{C}^n) = \{\beta(M) \colon M \in \mathbf{C}^n\} \subset \mathfrak{R}^{n+1} \textit{ and } f(\mathbf{E}^n) = \{f(M) \colon M \in \mathbf{E}^n\} \subset \mathfrak{R}^{n+1}.$$

Poincaré's theorem ($\beta_s(M) = \beta_{n-s}(M)$) implies that the linear span of the set $\beta(\mathbf{C}^n)$ is an $[(n+2)/2]$-dimensional subspace of $\mathfrak{R}^{n+1}$ (where $[k]$ denotes the greatest integer $\leqslant k$), and the theorem exhibits a convenient basis for that subspace. The same results are obtained here for the linear span of $f(\mathbf{E}^n)$, which has a convenient basis involving binomial coefficients in a simple way. For example, bases for the linear spans of $f(\mathbf{E}^6) \subset \mathbf{R}^7$ and $f(\mathbf{E}^7) \subset \mathbf{R}^8$ are as follows:

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[1]This is the definition of Lefschetz **(8)**; it is not the currently popular use of the term.

$\mathbf{E}^6$: $(2, 0, 0, 0, 0, 0, 0)$, $(1, 3, 2, 0, 0, 0, 0)$, $(0, 1, 4, 5, 2, 0, 0)$, $(0, 0, 1, 5, 9, 7, 2)$;

$\mathbf{E}^7$: $(1, 1, 0, 0, 0, 0, 0, 0)$, $(0, 1, 2, 1, 0, 0, 0, 0)$, $(0, 0, 1, 3, 3, 1, 0, 0)$,
$(0, 0, 0, 1, 4, 6, 4, 1)$.

(Note that $(1, 3, 2) = (1, 2, 1) + (0, 1, 1)$, $(1, 4, 5, 2) = (1, 3, 3, 1) + (0, 1, 2, 1)$, etc.)

Having a convenient basis for the linear span of $f(\mathbf{E}^n)$ leads to a useful characterization of the linear relations which must subsist among the numbers $f_s(M)$ for all $M \in \mathbf{E}^n$. It turns out that when $n = 2u - 1$ (whence $\chi(M) = 0$ for all $M \in \mathbf{E}^n$) the numbers $f_n(M), f_{n-1}(M), \ldots, f_u(M)$ can be expressed linearly in terms of $f_{u-1}(M), \ldots, f_1(M), f_0(M)$ (the expressions being valid for all $M \in \mathbf{E}^n$), while when $n = 2u - 2$, the numbers $f_n(M), f_{n-1}(M), \ldots, f_{u-1}(M)$ admit linear expressions in terms of $f_{u-2}(M), \ldots, f_0(M), \chi(M)$.

Our approach is of a purely combinatorial nature, involving neither subdivision nor homology. The arithmetical identities of §1 are used in §2 to prove the main result, a theorem concerning abstract incidence systems which exhibit some properties of those which are dual to Eulerian manifolds. Applications to Eulerian manifolds and convex polytopes appear in §3.

For the elementary properties of complexes and convex polytopes which are employed here, the reader may consult Alexandroff and Hopf **(1)** and Weyl **(6)**. For a treatment of the Euler characteristic which is well suited to the present elementary combinatorial approach, see Hadwiger **(3)** or Klee **(4)**.

Helpful comments were supplied by C. B. Allendoerfer, E. H. Spanier, and H. S. Zuckerman. D. Gale and E. C. Zeeman have observed that our approach has several points of similarity with those of Somerville **(9)** and Fieldhouse **(7)**.

1. Some arithmetical identities. This section contains some arithmetical identities involving binomial coefficients which are to be employed in §2. Although these may appear in the literature, we have not found them there, and thus include their proofs as an aid to the reader.[2] We agree that $\binom{n}{r}$ is defined in the usual way for *all* integers n and r—positive, zero, or negative (cf. **2**, p. 40), and shall use freely the basic recursion relation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}.$$

1.1. PROPOSITION. *For all non-negative integers j and k,*

$$\sum_{i=0}^{j} (-1)^i \binom{j}{i}\binom{i+j}{k} = (-1)^j \binom{j}{k-j}.$$

Proof. Let

$$V(i, j, k) = (-1)^i \binom{j}{i}\binom{i+j}{k},$$

whence

[2]Alternative (and simpler) proofs have been supplied to the author by J. Riordan.

$$\begin{aligned}V(i,j,k) &= (-1)^i\left[\binom{j-1}{i}+\binom{j-1}{i-1}\right]\left[\binom{i+j-1}{k}+\binom{i+j-1}{k-1}\right]\\ &= V(i,j-1,k)+V(i,j-1,k-1)\\ &\qquad - (-1)^{i-1}\binom{j-1}{i-1}\left[\binom{i+j-2}{k}+\binom{i+j-2}{k-1}\right.\\ &\qquad\qquad \left.+\binom{i+j-2}{k-1}+\binom{i+j-2}{k-2}\right]\\ &= V(i,j-1,k)+V(i,j-1,k-1)-V(i-1,j-1,k)\\ &\qquad -2V(i-1,j-1,k-1)-V(i-1,j-1,k-2).\end{aligned}$$

Now let

$$U(j,k) = \sum_{i=0}^{j} V(i,j,k).$$

Corresponding to the five terms $V(\cdot, j-1, \cdot)$ in the above expression for $V(i,j,k)$, we obtain the five bracketed terms in the equation

$$\begin{aligned}U(j,k) &= [U(j-1,k)+V(j,j-1,k)]\\ &\qquad + [U(j-1,k-1)+V(j,j-1,k-1)]\\ &\qquad - [V(-1,j-1,k)+U(j-1,k)]\\ &\qquad - 2[V(-1,j-1,k-1)+U(j-1,k-1)]\\ &\qquad - [(V(-1,j-1,k-2)+U(j-1,k-2)]\\ &= -U(j-1,k-1)-U(j-1,k-2).\end{aligned}$$

Now, clearly, when $m = 0$,

$$U(m,k) = (-1)^m\binom{m}{k-m}$$

for all integers k (positive, negative, or zero). Suppose the same is known for $m = j-1$ and consider the case of $U(j,k)$. We have

$$\begin{aligned}U(j,k) &= -U(j-1,k-1)-U(j-1,k-2)\\ &= -(-1)^{j-1}\left[\binom{j-1}{k-j}+\binom{j-1}{k-j-1}\right] = (-1)^j\binom{j}{k-j},\end{aligned}$$

so the proof of 1.1 is completed by mathematical induction.

1.2. Proposition. *For* $0 < j < k$,

$$\sum_{i=2k-2j}^{2k-j} (-1)^i\binom{j}{2k-j-i}(2k-i)\binom{i-1}{k-1} = 0.$$

Proof. Let

$$h(k,i) = \frac{2k-i}{k}\binom{i-1}{k-1}$$

and note that

$$h(k-1, i) = h(k, i+1) - h(k, i). \tag{1}$$

Indeed, (1) asserts that

$$\frac{2k-i-2}{k-1}\binom{i-1}{k-2} = \frac{2k-i-1}{k}\left[\binom{i-1}{k-1} + \binom{i-1}{k-2}\right] - \frac{2k-i}{k}\binom{i-1}{k-1},$$

which reduces at once to

$$\frac{i-k+1}{k-1}\binom{i-1}{k-2} = \binom{i-1}{k-1},$$

which is easily verified.

Now let

$$T(k,j) = \sum_{i=2k-2j}^{2k-j} (-1)^i \binom{j}{2k-j-i} h(k, i).$$

We want to show that $T(k,j) = 0$ whenever $0 < j < k$. Since effective summation in the expression for $T(k,j)$ is over the range $\max(2k-2j, k) \leqslant i \leqslant 2k-j$, it is easily verified that $T(k, k-1) = 0$ for $k \geqslant 2$, and in particular $T(2, m) = 0$ whenever $0 < m < 2$. Now suppose it is known that $T(k-1, m) = 0$ whenever $0 < m < k-1$ and that $T(k, m) = 0$ whenever $j < m < k$ (where $j < k-1$). We can show that $T(k,j) = 0$ by proving that

$$T(k,j) + T(k,j+1) + T(k-1,j) + T(k-1,j-1) = 0. \tag{2}$$

To verify (2) we note that

$$T(k,j) = \sum_{2k-2j}^{2k-j} (-1)^i \binom{j}{2k-j-i} h(k, i)$$

and

$$T(k,j+1) = \sum_{2k-2j-2}^{2k-j-1} (-1)^i \binom{j+1}{2k-j-i-1} h(k, i),$$

with summation always on i, and from (1) it follows that

$$T(k-1,j) = \sum_{2k-2j-3}^{2k-j-3} (-1)^{i-1} \binom{j}{2k-j-i-1} h(k, i) - \sum_{2k-2j-2}^{2k-j-2} (-1)^i \binom{j}{2k-j-i-2} h(k, i),$$

$$T(k-1,j-1) = \sum_{2k-2j+1}^{2k-j} (-1)^{i-1} \binom{j-1}{2k-j-i} h(k, i) - \sum_{2k-2j}^{2k-j-1} (-1)^i \binom{j-1}{2k-j-i-1} h(k, i).$$

Then (2) is proved by showing that for each i, the net coefficient of $h(k, i)$ on the left side of (2) is equal to zero. For example, when $2k - 2j + 1 \leqslant i \leqslant 2k - j - 3$, this coefficient is equal to $(-1)^i$ times the number

$$\binom{j}{2k-j-i} + \binom{j+1}{2k-j-i-1} - \binom{j}{2k-j-i-1} - \binom{j}{2k-j-i-2} - \binom{j+1}{2k-j-i} - \binom{j-1}{2k-j-i-1},$$

which is equal to zero by the basic recursion used in justifying (1). The other cases are even simpler.

2. A theorem on certain incidence systems. By the term *incidence system* we shall mean a finite set X with an associated *incidence function* ϕ and *dimension function* δ; ϕ is a symmetric real-valued function on $X \times X$ (that is, $\phi(x, y) = \phi(y, x)$ for all $x, y \in X$) and δ is a function on X to a set of integers. For each element y of X and each integer i, we define

$$\mu_i(y) = \sum_{x \in X, \delta(x) = i} \phi(y, x).$$

In the case of special interest, ϕ assumes only the values 0 and 1 and is thus the characteristic function of an incidence relation (a symmetric subset of $X \times X$); in this case, $\mu_i(y)$ is merely the number of i-dimensional elements of X which are incident to y.

The *characteristic* $\chi(y)$ of an element $y \in X$ is defined as the alternating sum

$$\chi(y) = \sum_{i=0}^{\delta(y)} (-1)^i \mu_i(y).$$

For $d \geqslant 1$, the system (X, ϕ, δ) will be called a *d-system* provided it satisfies the following conditions:

(i) $\max\{\delta(x): x \in X\} = d - 1$;

(ii) $\chi(y) = 1$ for all $y \in X$ with $\delta(y) \geqslant 0$;

(iii) whenever $y \in X$ and $0 \leqslant \delta(y) \leqslant i \leqslant d - 1$, then

$$\mu_i(y) = \binom{d - \delta(y)}{i - \delta(y)}.$$

Note that these conditions are all satisfied when X is the simplest triangulation of a $(d - 1)$-sphere (that is, the system of all proper faces of a d-simplex), $\phi(x, y) = 1$ when x and y are incident ($x \subset y$ or $x \supset y$) and $= 0$, otherwise, and δ is the usual dimension function.

2.1. THEOREM. *Suppose the incidence system (X, ϕ, δ) is a d-system, with $d = 2u - 1$ or $d = 2u$. For $0 \leqslant s \leqslant d - 1$, let f_s denote the number of s-dimensional elements of X, and let*

$$f_d = \frac{1}{2} \sum_{s=0}^{d-1} (-1)^s f_s.$$

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For $1 \leqslant j \leqslant u$, let γ_j^d denote the $2u$-vector $(\gamma^d{}_{j0}, \ldots, \gamma^d{}_{j(2u-1)})$, where

$$\gamma^d_{js} = \begin{cases} (2u - s)\binom{j}{2u - j - s} & \text{when } d = 2u - 1, \\ \binom{j}{2u - j - s} & \text{when } d = 2u. \end{cases}$$

Then the vector $f = (f_0, \ldots, f_{2u-1})$ is linearly dependent on the u vectors $\gamma^d{}_1, \ldots, \gamma^d{}_u$. Further, $f_d = 0$ when $d = 2u$.

Proof. For i and j between 0 and $d - 1$, let

$$g_{ij} = \sum_{x,y \in X;\ \delta(x)=i,\ \delta(y)=j} \phi(x, y).$$

Then, of course, $g_{ij} = g_{ji}$. It follows from Condition (iii) that $\mu_{\delta(y)}(y) = 1$ for all $y \in X$, and then from Condition (ii) that

$$\sum_{i=0}^{\delta(y)-1} (-1)^i \mu_i(y) = 1 - (-1)^{\delta(y)}.$$

Using this equation in conjunction with (iii), we see that for $1 \leqslant m \leqslant d - 1$,

$$\begin{aligned}(1 - (-1)^m)f_m &= g_{m0} - g_{m1} + \ldots + (-1)^{m-1} g_{m(m-1)} \\ &= \binom{d}{d - m} f_0 - \binom{d-1}{d-m} f_1 + \ldots + (-1)^{m-1}\binom{d - m + 1}{d - m} f_m.\end{aligned}$$

Hence, we obtain the following equations E_m for $1 \leqslant m \leqslant d - 1$:

$$\text{(odd } m\text{) } E_m: \quad 0 = \binom{d}{d-m} f_0 - \binom{d-1}{d-m} f_1 + \ldots + \binom{d-m+1}{d-m} f_{m-1} - 2f_m;$$

$$\text{(even } m\text{) } E_m: \quad 0 = \binom{d}{d-m} f_0 - \binom{d-1}{d-m} f_1 + \ldots + \binom{d-m+1}{d-m} f_{m-1}.$$

And we have also

$$E_d: \quad 0 = f_0 - f_1 + \ldots + (-1)^{d-1} f_{d-1} - 2f_d.$$

These equations are redundant, and we shall be concerned only with those having odd indices, that is, with

$$E_1: \quad 0 = \binom{d}{d-1} f_0 - 2f_1,$$

$$E_3: \quad 0 = \binom{d}{d-3} f_0 - \binom{d-1}{d-3} f_1 + \binom{d-2}{d-3} f_2 - 2f_3,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

terminating with E_{2u-1}, or in other words with (odd d) $\quad E_d$: $\quad 0 = f_0 - f_1 + f_2 - f_3 + \ldots + f_{d-1} - 2f_d$

or

(even d) $\quad E_{d-1}$: $\quad 0 = df_0 - (d-1)f_1 + (d-2)f_2 - (d-3)f_3 + \ldots + 2f_{d-2} - 2f_{d-1}.$

For $1 \leqslant r \leqslant u$ and $0 \leqslant s \leqslant 2u-1$, let $\beta^d{}_{rs}$ be the coefficient of f_s in the equation E_{2r-1}, where, of course, $\beta^d{}_{rs} = 0$ for $s > 2r-1$. The u vectors

$$\beta^d{}_r = (\beta^d{}_{r0}, \ldots, \beta^d{}_{r(2u-1)}) \in \mathfrak{R}^{2u}, \qquad 1 \leqslant r \leqslant u,$$

are linearly independent because the $u \times u$ submatrix

$$(\beta^d_{rs}) \qquad (1 \leqslant r \leqslant u, 1 \leqslant \text{odd } s \leqslant 2u-1)$$

is triangular and has exclusively -2's along its main diagonal. Let L denote the u-dimensional linear subspace of $\mathfrak{R}^{2u}$ which is spanned by $\{\beta^d{}_1, \ldots, \beta^d{}_u\}$ and let L^0 denote the orthogonal supplement of L, consisting of all vectors $\gamma = (\gamma_0, \ldots, \gamma_{2u-1}) \in \mathfrak{R}^{2u}$ such that

$$\sum_{s=0}^{2u-1} \beta^d_{rs}\, \gamma_s = 0$$

for $1 \leqslant r \leqslant u$. Then, of course, $f \in L^0$, and we shall show below that $\{\gamma^d{}_1, \ldots, \gamma^d{}_u\} \subset L^0$. Since L^0 is a u-dimensional linear space and since the u vectors $\gamma^d{}_j$ $(1 \leqslant j \leqslant u)$ are easily seen to be linearly independent, it will follow that f is a linear combination of the $\gamma^d{}_j$'s. This is the first assertion of 2.1. The second assertion of 2.1 is that if $d = 2u$, then $f_d = 0$, or in other words the $2u$-vector $(1, -1, \ldots, 1-1)$ is orthogonal to the $2u$-vector $f = (f_0, \ldots, f_{2u-1})$. For this it suffices (in view of the first assertion) to show that $(1, -1, \ldots, 1-1)$ is orthogonal to each of the vectors $\gamma^d{}_j$ $(1 \leqslant j \leqslant u)$. But recalling the definition of the vectors $\gamma^d{}_j$, we note that if $d = 2u$ and $1 \leqslant j \leqslant u$, then

$$\sum_{s=0}^{2u-1} (-1)^s \gamma^d_{js} = \sum_{s=0}^{2u-1} (-1)^s \binom{j}{2u-j-s} = (-1)^{2u-j} \sum_{i=1-j}^{2u-j} (-1)^i \binom{j}{i}$$
$$= (-1)^j \sum_{i=0}^{j} (-1)^i \binom{j}{i} = 0,$$

where the final equality follows from 1.1. with $k = 0$.

To complete the proof we must show that

$$[d, r, j]: \qquad \sum_{s=0}^{2u-1} \beta^d_{rs}\, \gamma^d_{js} = 0 \qquad \text{for } 1 \leqslant r \leqslant u, 1 \leqslant j \leqslant u.$$

Recalling the definition of $\beta^d{}_{rs}$, we see that $\beta^d{}_{r(2r-1)} = -2$, while

$$\beta^d_{rs} = (-1)^s \binom{d-s}{d-2r+1} \qquad \text{for } s \neq 2r-1.$$

When $d = 2u-1$ and $2r-1$ is not between $2u-2j$ and $2u-j$, the left side of $[d, r, j]$ is given by

$$\sum_{s=0}^{2u-1} \beta^d_{rs}(2u-s)\binom{j}{2u-j-s} = \sum_{s=2u-2j}^{2u-j} \beta^d_{rs}(2u-s)\binom{j}{2u-j-s}$$
$$= \sum_{s=2u-2j}^{2u-j} (-1)^s \binom{j}{2u-j-s}(2u-s)\binom{d-s}{d-2r+1}$$
$$= (-1)^{2u-j} \sum_{i=0}^{j} (-1)^i \binom{j}{i}(i+j)\binom{i+j-1}{d-2r+1},$$

where the last equality comes from the substitution $i = 2u - j - s$. But

$$(i+j)\binom{i+j-1}{d-2r+1} = (d-2r+2)\binom{i+j}{d-2r+2},$$

so the above sum is equal to

$$(-1)^{2u-j}(d-2r+2) \sum_{i=0}^{j} (-1)^i \binom{j}{i}\binom{i+j}{d-2r+2}$$
$$= (-1)^{2u-j}(d-2r+2)(-1)^j \binom{j}{d-2r+2-j} = 0,$$

where the next-to-last equality comes from 1.1 and the final equality results from the fact that

$$\binom{j}{d-2r+2-j} = 0$$

when $2r - 1$ is not between $2u - 2j$ and $2u - j$.

Now suppose that $d = 2u - 1$ but $2u - 2j \leqslant 2r - 1 \leqslant 2u - j$. Correcting the preceding computation to account for the special value of $\beta^d{}_{r(2r-1)}$, we see that the left side of $[d, r, j]$ is equal to

$$(d-2r+2)\binom{j}{d-2r+2-j} + \gamma^d_{j(2r-1)}\left(-2 - (-1)^{2r-1}\binom{d-(2r-1)}{d-2r+1}\right)$$
$$= (d-2r+2)\binom{j}{d-2r+2-j}$$
$$- (2u-(2r-1))\binom{j}{2u-j-(2r-1)} = 0.$$

Suppose, finally, that $d = 2u$. When $2r - 1$ is not between $2u - 2j$ and $2u - j$, the left side of $[d, r, j]$ is given by

$$\sum_{s=d-2j}^{d-j} (-1)^s \binom{j}{d-j-s}\binom{d-s}{d-2r+1}$$
$$= (-1)^{d-j} \sum_{i=0}^{j} (-1)^i \binom{j}{i}\binom{i+j}{d-2r+1}$$
$$= (-1)^{d-j}(-1)^j \binom{j}{d-2r+2-j} = 0,$$

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where we have used 2.1 and the fact that $d - 2r + 1 - j$ is <0 or $>j$. When $d - 2j \leqslant 2r - 1 \leqslant d - j$, correction for the special value of $\beta^d{}_{r(2r-1)}$ leads again to the value 0, as in the preceding paragraph. This completes the proof of 2.1.

2.2. Corollary. *Suppose that d is a positive integer with $d = 2u - 1$ or $d = 2u$, and I is a set of integers which includes at least one from each of the u pairs $\{0, 1\}, \{2, 3\}, \ldots, \{2u - 2, 2u - 1\}$. If two d-systems (χ, ϕ, δ) and (X', ϕ', δ') are such that $f_i = f_i'$ for all $i \in I$ (where the numbers f_i and f_i' are as in 2.1), then $f_s = f_s'$ for $0 \leqslant s \leqslant 2u - 1$.*

Proof. Upon examination of the basis system $\gamma^d{}_1, \ldots, \gamma^d{}_u$, this is seen to follow at once from 2.1.

The following is also an immediate consequence of 2.1.

2.3. Corollary. *With hypotheses and notation as in 2.1, let Ξ denote the set of all vectors $\xi = (\xi_0, \ldots, \xi_{2u-1}) \in \Re^{2u}$ such that*

$$\sum_{s=0}^{2u-1} \xi_s f_s = 0.$$

Then Ξ includes all vectors ξ such that

$$\sum_{s=0}^{2u-1} \xi_s \gamma^d_{js} = 0$$

for $1 \leqslant j \leqslant u$.

The next theorem is the one whose dual (given in 3.2 below) will be applied in **(5)**.

2.4. Theorem. *Suppose d is a positive integer with $d = 2u - 1$ or $d = 2u$, and t is an integer with $0 \leqslant t \leqslant u - 1$. Then there is a vector*

$$\xi^{d\,t} = (\xi^{d\,t}_u, \ldots, \xi^{d\,t}_{2u-1}) \qquad \textit{such that} \qquad f_t = \sum_{i-u}^{2u-1} \xi^{d\,t}_i f_i$$

whenever the numbers f_s are obtained from a d-system as in 2.1. In particular,

$$f_0 = \sum_{i=u}^{d} (-1)^{i-u} 2 \binom{i-1}{u-1} f_i \qquad \textit{when } d = 2u - 1,$$

and

$$f_0 = \sum_{i=u}^{d-1} (-1)^{i-u} \left(2 - \frac{i}{u}\right) \binom{i-1}{u-1} f_i \qquad \textit{when } d = 2u.$$

Proof. The first assertion of 2.4 is an easy consequence of 2.3. To justify the specific formulae for f_0 it suffices (in view of 2.3) to show that:

for $d = 2u - 1$ and $1 \leqslant j \leqslant u$,

$$\sum_{i=u}^{2u-1} (-1)^{i-u} 2 \binom{i-1}{u-1} \gamma^d_{ji} = -\gamma^d_{j0};$$

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for $d = 2u$ and $1 \leqslant j \leqslant u$,

$$\sum_{i=u}^{2u-1} (-1)^{i-u} \left(2 - \frac{i}{u}\right)\binom{i-1}{u-1}\gamma_{ji}^{d} = -\gamma_{j0}^{d}.$$

Recalling the formulae for $\gamma^{d}{}_{js}$ (which depend on the parity of d), we see that the statements are easily verified when $j = u$, while for $1 \leqslant j < u$ they both amount to the assertion that

$$\sum_{i=u}^{2u-1} (-1)^{i} \binom{j}{2u-j-i}(2u-i)\binom{i-1}{u-1} = 0.$$

But here the effective range of summation is only for $2u - 2j \leqslant i \leqslant 2u - j$, since otherwise

$$\binom{j}{2u-j-i} = 0,$$

and the desired conclusion follows from 1.2.

3. Application to Eulerian manifolds and convex polytopes. A *cell-complex* is a finite family K of convex polytopes (the *cells* of K) such that each face of a member of K is a member of K, and the intersection of any two members of K is a face of both. An n-dimensional cell-complex K^n will be called a *simple n-manifold* provided that for $0 \leqslant s \leqslant i \leqslant n$, each s-cell of K^n is a face of $\binom{n+1-s}{i-s}$ i-cells of K^n.

3.1. PROPOSITION. *Suppose K is a simple n-manifold and $d = n + 1$. For $\sigma, \tau \in K$, let $\phi(\sigma, \tau) = 1$ when $\sigma \subset \tau$ or $\sigma \supset \tau$, and $\phi(\sigma, \tau) = 0$ otherwise. Let δ be the usual dimension function. Then (K, ϕ, δ) is a d-system and hence the results* 2.1–2.4 *apply to the numbers $f_0, \ldots, f_d$, where f_s is the number of s-cells of K for $0 \leqslant s \leqslant n$, and*

$$f_d = \sum_{s=0}^{n} (-1)^{s} f_s.$$

Proof. Conditions (i) and (iii) (in the definition of a d-system) are obviously satisfied, and Condition (ii) follows from the fact that when a cell-complex is formed in the natural way from a convex polytope, its Euler characteristic must be equal to 1.

Now we recall (from the Introduction) the notion of an *Eulerian n-manifold.* This is a finite simplicial n-complex M^n such that for each s-simplex $\sigma^s \in M^n$, the Euler characteristic of the linked complex $L(\sigma^s, M^n)$ is equal to $1 - (-1)^{n-s}$. Here, as usual, $L(\sigma^s, M^n)$ is the set of all simplexes σ of M^n such that $\sigma \cap \sigma^s = \emptyset$ and the join of σ and σ^s is a simplexo f M^n.

3.2. THEOREM. *Let E^n denote the class of all Eulerian n-manifolds. For*

$M \in \mathbf{E}^n$ and $0 \leqslant s \leqslant n$, let $f_s(M)$ denote the number of s-simplices of M and let $\chi(M)$ denote the Euler characteristic

$$\sum_{s=0}^{n} (-1)^s f_s(M).$$

If $n = 2u - 1$ and $M \in \mathbf{E}^n$, then $\chi(M) = 0$ and the $2u$-vector $(f_0(M), \ldots, f_n(M))$ is a linear combination of the u row-vectors of the $u \times (2u)$ matrix J_n:

$$\begin{bmatrix} 1 & 1 & & & & & & & \\ & 1 & 2 & 1 & & & & & \\ & & 1 & 3 & 3 & 1 & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ & & & \binom{u}{0} & \binom{u}{1} & \binom{u}{2} & \cdots & \binom{u}{u-2} & \binom{u}{u-1} & \binom{u}{u} \end{bmatrix}$$

(where zeros have been omitted). Further,

$$f_n = \sum_{j=0}^{u-1} (-1)^{u-1-j} \frac{j+1}{u} \binom{n-j-1}{u-1} f_j.$$

If $n = 2u - 2$ and $M \in \mathbf{E}^n$, then the $2u$-vector $(\frac{1}{2}\chi(M), f_0(M), \ldots, f_n(M))$ is a linear combination of the u row-vectors of the $u \times (2u)$ matrix J_n:

$$\begin{bmatrix} 1 & 2 & & & \\ & 1 & 3 & 2 & \\ & & 1 & 4 & 5 \quad 2 \\ & & & \cdot \quad \cdot \quad \cdot & \\ & & & \binom{u}{0}, \binom{u}{1} + \binom{u-1}{0}, \binom{u}{2} + \binom{u-1}{1} \cdots & \\ & & & \binom{u}{u-2} + \binom{u-1}{u-3}, \binom{u}{u-1} + \binom{u-1}{u-2}, \binom{u}{u} + \binom{u-1}{u-1} & \end{bmatrix}$$

(where zeros have been omitted). Further,

$$f_n = (-1)^{u+1} \binom{n}{u-1} \chi + \sum_{j=0}^{u-2} (-1)^{u-j} 2 \binom{n-j-1}{u-1} f_j.$$

Proof. For $\sigma, \tau \in M$ let $\phi(\sigma, \tau) = 1$ when $\sigma \subset \tau$ or $\sigma \supset \tau$, and $\phi(\sigma, \tau) = 0$ otherwise. For each $\sigma \in M$ let $\delta(\sigma) = n - \dim \sigma$, where dim is the usual dimension function. With $d = n + 1$, we claim that $(M \sim \{\emptyset\}, \phi, \delta)$ is a d-system. Since $\min\{\dim \sigma : \sigma \in M \sim \{\emptyset\}\} = 0$, Condition (i) is evident. To verify Condition (ii) we note that if $\sigma \in M$, then relative to the system $(M \sim \{\emptyset\}, \phi, \delta)$ the characteristic $\chi(\sigma)$ of σ (in the sense of §2) is the alternating sum

$$\sum_{i=0}^{\delta(\sigma)} (-1)^i \mu_i(\sigma),$$

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where $\mu_i(\sigma)$ is the number of simplices $\tau \in M$ for which $\sigma \subset \tau$ and $\delta(\tau) = i$. Since $\delta(\tau) = n - \dim \tau$, each simplex $\tau \supset \sigma$ contributes $(-1)^{n-\dim \tau}$ to the formation of $\chi(\sigma)$. The choice $\tau = \sigma$ contributes nothing to the formation of $\chi L(\sigma, M)$, but each $\tau \in M$ which properly contains σ corresponds to a simplex of dimension $\dim \tau - \dim \sigma - 1$ whose join with σ is equal to τ, and thus with $s = \dim \sigma$ each such simplex τ contributes $(-1)^{\dim \tau - s - 1}$ to the formation of $\chi L(\sigma, M)$. Since

$$(-1)^{n-\dim \tau} = (-1)^{n-s-1}(-1)^{\dim \tau - s - 1},$$

we have

$$\chi(\sigma) = (-1)^{n-s-1}\chi L(\sigma, M) + (-1)^{n-s}.$$

But M is an Eulerian n-manifold, so $\chi L(\sigma, M) = 1 - (-1)^{n-s}$ and

$$\chi(\sigma) = (-1)^{n-s-1}[1 - (-1)^{n-s}] + (-1)^{n-s} = 1.$$

This establishes Condition (ii). Condition (iii) follows at once from the relevant definitions in conjunction with the fact that M is a *simplicial* complex. Thus (M, ϕ, δ) is a d-system with $d = n + 1$. It is then a routine matter to derive the assertions of 3.2 from 2.1 and 2.4.

Of course the results 2.2 and 2.3 can also be dualized so as to apply to Eulerian manifolds, but this is immediate and will be left to the reader. We shall describe explicitly the application to convex polytopes, for this will be required in **(5)**.

An n-dimensional convex polytope P will be called *simplicial* provided all of its $(n - 1)$-faces are simplices, and it will be called *simple* provided each of its vertices is on exactly n edges (or, equivalently, on exactly n $(n - 1)$-faces). From the standard polarity theory **(6)** it follows that if P is an n-dimensional convex polytope in $\mathfrak{R}^n$ and $0 \in \operatorname{int} P$, then P is simplicial if and only if the polar body P^0 is simple, where

$$P^0 = \left\{x \in \mathfrak{R}^n : \sup_{y \in P} \sum_{i=1}^{n} x_i y_i \leqslant 1\right\}.$$

3.3. Proposition. *Suppose P is a convex polytope of dimension $n + 1$ and M is the cell-complex consisting of all faces of P which are of dimension $\leqslant n$. If P is simple, M is a simple n-manifold and is subject to* 3.1. *If P is simplicial, M is an Eulerian n-manifold and is subject to* 3.2.

Proof. First verify that M is a cell-complex; then clearly M is simplicial if and only if P is simplicial. It follows by polarity that M is a simple n-manifold when P is simple and then by a second use of polarity that M is actually an Eulerian n-manifold when P is simplicial.

Now let $f(\mathbf{E}^n) = \{f(M) : M \in \mathbf{E}^n\} \subset \mathfrak{R}^{n+1}$, where

$$f(M) = (f_0(M), \ldots, f_n(M)) \subset \mathfrak{R}^{n+1}.$$

Theorem 3.2 implies that both when $n = 2u - 1$ and when $n = 2u - 2$, the set $f(\mathbf{E}^n)$ *lies in* a u-dimensional linear subspace of $\mathfrak{R}^{n+1}$. Our final result shows that, in fact, the linear span of $f(\mathbf{E}^n)$ *is* u-dimensional, even when attention is restricted to those Eulerian n-manifolds which arise from $(n+1)$-dimensional convex polytopes.

3.4. PROPOSITION. *For $0 \leqslant r \leqslant n+1$, let $C^n{}_r$ denote the Eulerian n-manifold which is the join of the boundary B_r of an r-simplex and the boundary B_{n+1-r} of an $(n+1-r)$-simplex.*

When $n = 2u - 1$ the matrix

$$\begin{bmatrix} f(C_1^n) \\ f(C_2^n) \\ \cdot \\ \cdot \\ f(C_u^n) \end{bmatrix}$$

is of rank u, and when $n = 2u - 2$ the matrix

$$\begin{bmatrix} f(C_0^n) \\ f(C_1^n) \\ \cdot \\ \cdot \\ f(C_{u-1}^n) \end{bmatrix}$$

is of rank u.

Proof. Each s-simplex of $C^n{}_r$ is the join of a $(\lambda - 1)$-simplex (determined by λ vertices) of B_r and a $(\mu - 1)$-simplex (determined by μ vertices) of B_{n+1-r}, where $\lambda \in [0, r]$, $\mu \in [0, n + r - 1]$, and $\lambda + \mu = s + 1$; conversely, each such join is an s-face of $C^n{}_r$. Hence with $f^n{}_{rs} = f_s(C^n{}_r)$ we have

$$f_{rs}^n = \sum_{\lambda\epsilon[0,r],\ \mu\epsilon[0,n+1-r],\ \lambda+\mu=s+1} \binom{r+1}{\lambda}\binom{n+2-r}{\mu}.$$

Considering the expansion of the polynomial $(1+x)^{r+1}(1+x)^{n+2-r} = (1+x)^{n+3}$, we see that

$$\sum_{\lambda\geqslant 0,\ \mu\geqslant 0,\ \lambda+\mu=s+1} \binom{r+1}{\lambda}\binom{n+2-r}{\mu} = \binom{n+3}{s+1}.$$

It follows that

$$f_{rs}^n = \binom{n+3}{s+1}$$

whenever $\min(r, n+1-r) > s$ (and, in particular, when $n+1 \geqslant 2r > 2s$), while

$$f_{ss}^n = \binom{n+3}{s+1} - 1$$

when $n + 1 > 2s$.

Now suppose that $n = 2u - 1$ and consider the $u \times u$ matrix

$$(f^n_{rs}) \qquad (1 \leqslant r \leqslant u, 0 \leqslant s \leqslant u - 1).$$

Each element of its 0-column is equal to $n + 3$; its 1-column starts with $\binom{n+3}{2} - 1$ and has $\binom{n+3}{2}$ thereafter; . . . ; its s-column has $f^n{}_{ss} = \binom{n+3}{s+1} - 1$ but $\binom{n+3}{s+1}$ thereafter. Subtracting the last row from each of the others, we obtain a matrix $(g^n{}_{rs})$ in which the 0-column ends with $n + 3$ but has all its other entries equal to 0, while the matrix

$$(g^n_{rs}) \qquad (1 \leqslant r \leqslant u - 1, 1 \leqslant s \leqslant u - 1)$$

is triangular, with all 0's below its main diagonal and all -1's along the main diagonal. Hence the determinant of $(g^n{}_{rs})$ is equal to $n + 3$ and we have the desired conclusion for the case $n = 2u - 1$.

Suppose, finally, that $n = 2u - 2$ and note that since, for each r,

$$\sum_{s=0}^{n} (-1)^s f_s(C^n_r) = \chi(C^n_r) = 2,$$

the rank of the matrix with which we are concerned is not changed by adding a column of 1's. The augmented matrix has the $u \times u$ submatrix

$$\begin{bmatrix} 1 & f^n_{00} & f^n_{01} & . & . & . & f^n_{0(u-2)} \\ 1 & f^n_{10} & f^n_{11} & . & . & . & f^n_{1(u-2)} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & f^n_{(u-1)0} & f^n_{(u-1)1} & . & . & . & f^n_{(u-1)(u-2)} \end{bmatrix}$$

whose determinant is equal to 1 (as is verified by the method employed above). This completes the proof of 3.4.

References

1. Paul Alexandroff and Heinz Hopf, *Topologie* I (Berlin, 1935).
2. William Feller, *An introduction to probability theory and its applications* (New York, 1951).
3. H. Hadwiger, *Eulers Charakteristik und kombinatorische Geometrie*, J. Reine Angew. Math., *194* (1955), 101–110.
4. Victor Klee, *The Euler characteristic in combinatorial geometry*, Am. Math. Monthly, *70* (1963), 119–127.
5. ——— *The number of vertices of a convex polytope*; to appear in this Journal.
6. H. Weyl, *Elementare Theorie der konvexen Polyeder*, Comment. Math. Helv., *7* (1935), 290–306.

https://doi.org/10.4153/CJM-1964-053-0 Published online by Cambridge University Press

7. M. Fieldhouse, *Linear programming*, Ph.D. Thesis, Cambridge Univ. (1961). [Reviewed in Operations Res. *10* (1962), 740.]

8. S. Lefschetz, *Introduction to topology* (Princeton, 1949).

9. D. M. Y. Sommerville, *The relations connecting the angle sums and volume of a polytope in space of n dimensions*, Proc. Roy. Soc. London, Ser. A, *115* (1927), 103–119.

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