RELATIVISTIC STELLAR DYNAMICS

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Abstract: Three main areas of relativistic stellar dynamics are reviewed: (a) Relativistic clusters, (b) Systems containing a massive black hole, and (c) Perturbed expanding Universes. The emphasis is on the use of orbit perturbations.

There are three main areas of research that belong to the field of Relativistic Stellar Dynamics, namely:

(a) The dynamics of clusters, or nuclei of galaxies, of very high density,

(b) The dynamics of systems containing a massive black hole, and

(c) The dynamics of particles (and photons) in an expanding Universe.

(a) Up to now the emphasis has been put on the first subject. It is known that clusters evolve by condensing their central parts while expanding their outer parts. If the density contrast between the center and the envelope is sufficiently high the evolution produces a collapse of the center to theoretically infinite density. This is what Lynden-Bell calls a "gravothermal catastrophe" (Lynden-Bell and Wood, 1968).

Although infinite densities cannot be realized in practice, one expects that the core of the cluster will become relativistic. Relativistic effects become significant for densities of the order of $10^{12} M_\odot/pc$, in which case the stars are at distances similar to those of the planets in the solar system.

A review of the dynamics of relativistic clusters was provided by Ipser (1975). The main topic in this review was the stability of clusters. It seems that highly relativistic clusters are unstable in general.

A recent development in this direction was provided by Vandervoort

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and Ipser (1982). These authors proved that a large class of clusters are unstable because of gravitational radiation. This result led them to the conjecture that gravitational radiation makes all clusters unstable.

A similar theorem in the case of fluid systems (stars) was formulated a few years ago by Friedman (1978; see also Friedman and Shutz, 1978). Friedman proved that all stars are unstable, because of gravitational radiation. The unstable modes have angular dependence of the form $\exp(im\theta)$ where m (≥ 2) must be sufficiently large. The growth rate of the instability is proportional to $(c/v)^{2m+1}$, where v is a characteristic velocity.

On the other hand the unstable modes of the stellar axisymmetric systems considered by Vandervoort and Ipser are of the Dedekind type (ellipsoids with fixed boundaries in an inertial system, and with a strong internal circulation), i.e. they have $m = 2$. The instability appears for all angular velocities $\Omega > 0$. Therefore we may state that relativistic stellar systems are more unstable than the corresponding fluid systems.

(b) The second subject of interest is the dynamics of a system under the influence of a large central black hole. In very dense nuclei of clusters, or galaxies, one may have frequent collisions that may lead to the formation of a massive central black hole. Thus it is quite possible that the dynamics of the nuclei of some clusters, or galaxies, are dominated by a large black hole. This black hole should be in general of the Kerr type (rotating black hole).

The study of the orbits of particles and photons (geodesics) around Kerr black holes has received a great impetus after the discovery by Carter (1968) that such systems are completely integrable. Carter proved that the equations of motion of particles (or photons) around a Kerr black hole have one more integral of motion besides the classical ones (rest mass, energy and angular momentum). By using Carter's integral one can solve explicitly the equations of motion, using elliptic integrals (for a review see Sharp, 1979; see also Chandrasekhar, 1982, and Contopoulos, 1982). One can distinguish the following main types of orbits:

1) Orbits staying outside the ergosphere,

2) Orbits entering the ergosphere and coming out again,

3) Orbits staying always in the ergosphere,

4) Orbits entering the outer horizon of the black hole; such orbits do not come out again in finite coordinate time.

5) Orbits inside the inner horizon, which, however, are of no importance for the outside observer.

The relative position of the orbits with respect to the ergosphere is of interest in connection with the Penrose effect (Penrose, 1969). A particle in the ergosphere may split into one particle of negative energy and one of positive energy. The first particle enters the horizon, while the second one acquires a larger energy than the energy of the incoming particle. The energy excess is taken from the black hole itself, therefore it can be larger than the rest mass ($E = mc^2$) of the particle. The Penrose process has been invoked as a source of energy in the nuclei of galaxies. Although it is known now that the Penrose process is much less effective than previously thought (Bardeen et al., 1972; Chandrasekhar, 1982) the problem is of great theoretical and practical importance.

One problem of theoretical interest is whether negative energy particles can ever go out of the ergosphere. If this would happen, we might have closed time-like geodesics, which would lead to causality violation. However it can be proved that if we impose the condition that the coordinate time and the proper time increase together ($dt/d\tau > 0$) no negative energy orbit can go out of the ergosphere. However, this condition cannot be derived from the local dynamics of the splitting of a particle in the ergosphere.

Among the orbits around black holes of special interest are the nearly circular orbits. In fact if a quasi-stationary stellar system is formed around the black hole such orbits should contain most of the mass. Such orbits are the counterparts of the familiar epicyclic orbits of classical galactic dynamics. They have two basic frequencies, the rotation frequency Ω, and the epicyclic frequency κ of radial oscillations.

We come now to the problem of the perturbations of a Kerr black hole.

The linearized problem of perturbed Kerr black holes has been solved recently by Chandrasekhar (1982). This problem involves the solution of 76 coupled differential equations containing 50 unknown functions. It seems almost miraculous that this problem has an explicit solution.

The perturbations depend on the time through factors of the form $\exp(i\sigma t)$, where σ is complex and can be considered as an eigenvalue of the system.

From now on the problem takes the familiar aspect of galactic dynamics. The perturbations can be considered as due to a density distribution of matter (and radiation) around the black hole, which we call the imposed density. On the other hand the same distribution produces deviations in the orbits of particles (or photons) around the black hole. Thus the study of the orbits around perturbed black holes gives the response density, which must be equal to the imposed density. This condition is the well known self-consistency equation of galactic

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dynamics

$$\rho^{imposed} = \rho^{response} . \qquad (1)$$

Its solution will give the eigenvalues of the problem, i.e. the allowed values of σ.

The perturbed epicyclic orbits can be written in the form

$$r = r_o + \Sigma F_{mn} \exp[i(m\Omega + n\kappa + \sigma)t] . \qquad (2)$$

This form is similar to that of the orbits of stars in a spiral galaxy, if we set

$$\sigma = -m(\Omega_s + i\Omega_i) , \qquad (3)$$

where Ω_s is the angular velocity of the spiral pattern and $m\Omega_i$ its growth rate.

If we solve now the collisionless Boltzmann equation we find the response density in the form

$$\frac{\exp[i(m\Omega + n\kappa + \sigma)t]}{m\Omega + n\kappa + \sigma} \qquad (4)$$

which contains the expression $(m\Omega + n\kappa + \sigma)$ in the denominator. If σ has a zero, or small, imaginary part, then $(m\Omega + n\kappa + \sigma)$ may be zero, or close to zero, i.e. we have a resonance. The values of Ω and κ depend on the radius r, therefore at particular radii we have particular resonances. The most important resonances are the Lindblad resonances, where $m/n = \mp 2$, or

$$\frac{\kappa}{\Omega - \Omega_s} = \pm 2 , \qquad (5)$$

and the particle resonance (or corotation) where $n = 0$, or

$$\Omega = \Omega_s . \qquad (6)$$

Such resonances play an important role in spiral and barred galaxies and also in the rings of Saturn. Namely, when such resonances appear, the perturbations are so large that gaps are formed in the distribution of matter, both in galaxies and in Saturn's rings. We expect therefore a similar behavior in the distribution of matter around black holes; e.g., if an accretion disk is formed, it should have the form of the rings of Saturn with several gaps here and there. The same should be the case for a cluster, or a galaxy, that is dominated by the central black hole.

A similar problem refers to the dynamics of a system dominated by a massive central body with axial symmetry. Such is the case of a dense elliptical galaxy. In such a system we can consider perturbations in the same way as in the case of the third integral of motion of galactic dynamics (Contopoulos, 1960). This problem has been treated recently in the post-Newtonian approximation by Spyrou and Varvoglis (1982), and can be considered as a first approach to the problems of relativistic galactic dynamics.

(c) The third type of problems of relativistic stellar dynamics refers to perturbations in an expanding Universe.

It is well known that the equations of motion in a Friedmann Universe are integrable. Thus, we can write explicitly the solutions for the motions of particles (and photons) in such a Universe.

A more general integrable case is given by the metric

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-\frac{2\mu}{r}-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{7}$$

which may represent a spherical condensation in an expanding Universe. This formula includes as limiting cases the Friedmann Universes if $\mu = 0$ and $k = -1$, 0 or +1.

The geodesics in such a metric are plane and we can take $\theta = \pi/2$. Thus the Lagrangian can be written

$$\mathcal{L} = \frac{1}{2}\left\{\dot{t}^2 - R^2\left(\frac{\dot{r}^2}{1-\frac{2\mu}{r}-kr^2} + r^2\dot{\phi}^2\right)\right\}, \tag{8}$$

where dots mean derivatives with respect to the affine parameter τ. The conserved angular momentum is

$$R^2r^2\dot{\phi} = L \tag{9}$$

and we have also the Eulerian equation

$$\frac{d}{d\tau}\left(\frac{\partial\mathcal{L}}{\partial\dot{r}}\right) = \frac{\partial\mathcal{L}}{\partial r}, \tag{10}$$

that gives

$$\frac{d}{d\tau}\left(\frac{R^2\dot{r}}{1-\frac{2\mu}{r}-kr^2}\right) = \frac{R^2\dot{r}^2(kr-\mu/r^2)}{(1-\frac{2\mu}{r}-kr^2)^2} + \frac{L^2}{R^2r^3}. \tag{11}$$

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Thus we find

$$\frac{R^4\dot{r}^2}{1-\frac{2\mu}{r}-kr^2} = \Gamma^2 - \frac{L^2}{r^2}\,, \tag{12}$$

where Γ is a constant, and we derive

$$R^2\dot{r} = \Gamma\left\{\left(1-\frac{B^2}{r^2}\right)\left(1-\frac{2\mu}{r}-kr^2\right)\right\}^{\frac{1}{2}}\,, \tag{13}$$

where $B = L/\Gamma$.

On the other hand R is assumed to be given by the Friedmann equation with zero cosmological constant

$$\left(\frac{dR}{dt}\right)^2 = \frac{q}{R} - k\,. \tag{14}$$

Thus, R is a given function of t; hence, the above equations can be solved to give r and ϕ implicitly as functions of t.

Then the problem of perturbations is dealt with as in the case of a black hole. If we consider, in particular, nearly circular motions, we can define the rotational end epicyclic frequencies Ω and κ. Then we consider perturbations proportional to $\exp(i\sigma t)$, and find a response of the form (4). Thus we can search for eigenvalues σ, that define self-consistent solutions of the linearized Einstein equations and collisionless Boltzmann equations in the same way as in the case of a central black hole.

This type of research is still in its first stages. However, it is a useful approach if we want to understand the results of the N-body calculations in an expanding Universe of the kind presented to us by the movies of R. Miller and others during this Symposium.

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Discussion

Vignato: Is it possible by Lindblad resonance theory to take into account nonmonotonic density distribution in a spherically symmetric system?

Contopoulos: Yes, the theory can take care of all cases.

McCrea: Why should causality not be violated?

Contopoulos: This is a philosophical question and I will not answer it here. But we may discuss it privately and I have a few things to tell you.

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