Forum of Mathematics, Sigma (2022), **Vol. 10**:e90 1–39
doi:10.1017/fms.2022.78

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RESEARCH ARTICLE

Bogoliubov theory in the Gross-Pitaevskii limit: a simplified approach

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Received: 11 April 2022; **Revised:** 19 August 2022; **Accepted:** 6 September 2022

Abstract
We show that Bogoliubov theory correctly predicts the low-energy spectral properties of Bose gases in the Gross-Pitaevskii regime. We recover recent results from [6, 7]. While our main strategy is similar to the one developed in [6, 7], we combine it with new ideas, taken in part from [15, 25]; this makes our proof substantially simpler and shorter. As an important step towards the proof of Bogoliubov theory, we show that low-energy states exhibit complete Bose-Einstein condensation with optimal control over the number of orthogonal excitations.

1. Introduction

We consider a Bose gas consisting of N particles moving in the box $\Lambda = [-1/2; 1/2]^3$ with periodic boundary conditions. In the Gross-Pitaevskii regime, particles interact through a potential with scattering length of the order $1/N$. The Hamilton operator acts on the Hilbert space $L^2_s(\Lambda^N)$ of permutation symmetric complex-valued square integrable functions on Λ^N, and it has the form

$$H_N = \sum_{i=1}^N -\Delta_i + \sum_{i<j}^N V_N(x_i - x_j) \tag{1}$$

where

$$V_N(x) := N^2 V(Nx),$$

for a $V \in L^2(\mathbb{R}^3)$ nonnegative, radial and compactly supported. We denote the scattering length of V by $\mathfrak{a} > 0$. Following [15, 16], we define it through the formula

$$4\pi\mathfrak{a} = \frac{1}{2}\int_{\mathbb{R}^3} V(x)\mathrm{d}x - \left\langle \frac{1}{2}V, \frac{1}{-\Delta + \frac{1}{2}V}\frac{1}{2}V\right\rangle. \tag{2}$$

As first proven in [22, 20], the ground state energy E_N of (1) satisfies

$$E_N/N \to 4\pi\mathfrak{a} \tag{3}$$

in the limit $N \to \infty$. In particular, to leading order, the ground state energy only depends on the interaction potential through its scattering length $\mathfrak{a}$. In [18, 21, 24], it was also shown that the ground state of the Hamiltonian H_N in equation (1) and, in fact, every normalised sequence $\psi_N \in L^2_s(\Lambda^N)$ of approximate ground states with

$$\frac{1}{N}\langle \psi_N, H_N \psi_N\rangle \to 4\pi\mathfrak{a},$$

exhibits complete Bose-Einstein condensation in the zero-momentum state $\phi_0(x) = 1$, for all $x \in \Lambda$, in the sense that the corresponding one-particle reduced density matrix γ_N (normalised so that $\mathrm{Tr}\,\gamma_N = 1$) satisfies

$$\lim_{N\to\infty}\langle \phi_0, \gamma_N \phi_0\rangle = 1\,.$$

Recently, a rigorous version of Bogoliubov theory [8] has been developed in [4, 5, 6, 7] to provide more precise information about the low-energy spectrum of H_N in equation (1), resolving the ground state energy and low-lying excitations up to errors that vanish in the limit $N \to \infty$; and about the corresponding eigenvectors, showing Bose-Einstein condensation with optimal control over the number of orthogonal excitations. Analogous results have also been established for Bose gases trapped by external potentials in the Gross-Pitaevskii regime [10, 11, 23, 25] and for Bose gases in scaling limits interpolating between the Gross-Pitaevskii regime and the thermodynamic limit [1, 9]. Very recently, the upper bound for the ground state energy has also been extended to the case of hard-sphere interaction, as announced in [2].

In this paper, we propose a new and substantially simpler proof of the results established in [6, 7]. Our approach follows some of the ideas in the proof of Bose-Einstein condensation with optimal bounds on the number of excitations obtained in [15]. Moreover, it makes use of some ideas introduced in [25] for the case of particles trapped by an external potential. The next theorem is our main result; it describes the low-energy spectrum of H_N in equation (1).

Theorem 1. *Let $V \in L^2(\mathbb{R}^3)$ be nonnegative, radial and compactly supported, and let E_N denote the ground state energy of H_N in equation (1). Then the spectrum of $H_N - E_N$ below a threshold $\Theta \le N^{1/17}$ consists of eigenvalues having the form*

$$\sum_{p\in 2\pi\mathbb{Z}^3\setminus\{0\}} n_p\sqrt{|p|^4 + 16\pi\mathfrak{a}p^2} + \mathcal{O}(N^{-1/17}\Theta) \tag{4}$$

with $n_p \in \mathbb{N}$, for all $p \in 2\pi\mathbb{Z}^3\setminus\{0\}$.

Remark. Our analysis also provides a precise estimate for the ground state energy E_N of the Hamiltonian H_N in equation (1), showing that

$$E_N = 4\pi\mathfrak{a}_N(N-1) + \frac{1}{2}\sum_{p\in 2\pi\mathbb{Z}^3\setminus\{0\}}\left[\sqrt{|p|^4 + 16\pi\mathfrak{a}p^2} - p^2 - 8\pi\mathfrak{a} + \frac{(8\pi\mathfrak{a})^2}{2p^2}\right] + \mathcal{O}(N^{-1/17}) \tag{5}$$

with a 'box scattering length' $\mathfrak{a}_N$ (defined in the next section) satisfying $|\mathfrak{a}_N - \mathfrak{a}| \lesssim N^{-1}$. This immediately implies that E_N is given by equation (5), with $\mathfrak{a}_N$ replaced by the true scattering length $\mathfrak{a}$, up to an error that remains bounded as $N \to \infty$. In [6], the order-one correction arising from $N(\mathfrak{a}_N - \mathfrak{a})$ was also computed. Here we skip this step to keep our presentation as simple as possible. Note that an estimate similar to equation (5) has recently been shown to hold in the thermodynamics limit; see [13, 14] for the lower bound and [27, 3] for the upper bound.

The main strategy we use to prove Theorem 1 is similar to the one developed in [6]. First we switch to the formalism of second quantisation, expressing the Hamilton operator in momentum space in terms of creation and annihilation operators. Then we renormalise the Hamilton operator, conjugating

https://doi.org/10.1017/fms.2022.78 Published online by Cambridge University Press

it first with a generalised Bogoliubov transformation (the exponential of a quadratic expression in the modified creation and annihilation operators $b_p^\dagger = a_p^\dagger a_0/\sqrt{N}$, $b_p = a_0^\dagger a_p/\sqrt{N}$) and afterwards with the exponential of a cubic expression in (modified) creation and annihilation operators. Effectively, these conjugations regularise the interaction potential. As a last step, we diagonalise the resulting quadratic Hamiltonian; this allows us to establish Bose-Einstein condensation with optimal bounds on the number of excitations and to compute the low-energy spectrum, proving the estimate (4).

Compared with [6], our approach has the following advantages. First, we make a different choice for the coefficients φ_p of the quadratic and cubic transformations used to renormalise the Hamiltonian, which should model correlations among particles. Instead of the ground state of a Neumann problem on a ball of radius $\ell > 0$, we consider here the solution of an appropriate zero-energy scattering equation, describing scattering processes inside the box Λ. This simplifies the proof of important properties of φ and improves cancellations between different terms arising in the many-body analysis. Second, we restrict the quadratic conjugation to momenta $|p| > N^\alpha$ for some $0 < \alpha < 1$. Consequently, it is enough to expand its action to first or, in a few cases, second order; higher-order contributions are negligible. This is a substantial advantage compared with [6], where no cutoff was imposed and all contributions had to be computed precisely (in contrast to standard Bogoliubov transformations, the action of generalised Bogoliubov transformations is not explicit). The presence of the cutoff means the interaction is regularised only up to length scales $\ell \leq N^{-\alpha}$; this needs to be compensated at the end when we diagonalise the quadratic Hamiltonian resulting from the renormalisation procedure. Another important simplification of the analysis concerns the final diagonalisation. As in [6], we implement it through a generalised Bogoliubov transformation defined (like the first quadratic transformation) in terms of the modified creation and annihilation operators $b_p^\dagger, b_p$. Here, however, instead of expanding the action of the generalised Bogoliubov transformation to all orders, we compare it directly with the explicit action of the corresponding standard Bogoliubov transformation, using an appropriate interpolation. Finally, we use the tool of localisation in the number of particles not only to show Bose-Einstein condensation (similarly to [7]) but also to compute the spectrum and prove Theorem 1. This makes the analysis substantially simpler (but provides a worse estimate of the error).

2. Fock space formalism

We introduce the bosonic Fock space

$$\mathcal{F} = \bigoplus_{n\geq 0} L_s^2(\Lambda^n).$$

For a momentum $p \in \Lambda^* = 2\pi\mathbb{Z}^3$ and denoting $u_p(x) = e^{ip\cdot x}$, we define $a_p^\dagger = a^\dagger(u_p)$ and $a_p = a(u_p)$, where $a^\dagger$ and a are the usual creation and annihilation operators. They satisfy the canonical commutation relations

$$\left[a_p, a_q^\dagger\right] = \delta_{p,q}, \qquad \left[a_p, a_q\right] = \left[a_p^\dagger, a_q^\dagger\right] = 0. \tag{6}$$

We denote, in configuration space, the creation and annihilation operator-valued distributions by $\check{a}_x^\dagger, \check{a}_x$; they satisfy $a_p^\dagger = \int e^{ip\cdot x}\check{a}_x^\dagger \mathrm{d}x$, $a_p = \int e^{-ip\cdot x}\check{a}_x \mathrm{d}x$. The number-of-particles operator $\mathcal{N}$ on $\mathcal{F}$ is given by

$$\mathcal{N} = \sum_{p\in\Lambda^*} a_p^\dagger a_p.$$

In the formalism of second quantisation, the Hamilton operator in equation (1) takes the form

$$H_N = \sum_{p\in\Lambda^*} p^2 a_p^\dagger a_p + \frac{1}{2}\sum_{r,p,q\in\Lambda^*} \hat{V}_N(r) a_{p+r}^\dagger a_q^\dagger a_p a_{q+r}, \tag{7}$$

with

$$\hat{V}_N(r) = \frac{1}{N}\hat{V}(r/N). \tag{8}$$

To recover the expression in equation (1), we have to restrict equation (7) to the sector with $\mathcal{N} = N$.

Because of the presence of Bose-Einstein condensation, the mode with $p = 0$ plays a special role when considering states with low energy. We introduce the notation $\mathcal{N}_0 = a_0^\dagger a_0$ and $\mathcal{N}_+ = \mathcal{N} - \mathcal{N}_0$ for the operators measuring the number of particles in the condensate and the number of excitations, respectively. Following Bogoliubov [8], we decompose equation (7) according to the number of $a_0, a_0^\dagger$ operators. Since (on $\{\mathcal{N} = N\}$)

$$a_0^\dagger a_0^\dagger a_0 a_0 = \mathcal{N}_0(\mathcal{N}_0 - 1) = (N - \mathcal{N}_+)(N - \mathcal{N}_+ - 1) = N(N-1) - \mathcal{N}_+(2N-1) + \mathcal{N}_+^2,$$

we can rewrite equation (7) as

$$H_N = H_0 + H_1 + H_2 + Q_2 + Q_3 + Q_4, \tag{9}$$

where

$$H_0 = \frac{\hat{V}_N(0)}{2} N(N-1), \qquad H_1 = \sum_{p \neq 0} p^2 a_p^\dagger a_p,$$
$$H_2 = \sum_{p\neq 0} \hat{V}_N(p) a_p^\dagger a_p (N - \mathcal{N}_+) - \frac{\hat{V}_N(0)}{2}\mathcal{N}_+(\mathcal{N}_+ - 1).$$

and

$$\begin{aligned} Q_2 &= \frac{1}{2}\sum_{p\neq 0} \hat{V}_N(p)[a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.}], \\ Q_3 &= \sum_{q,r,q+r\neq 0} \hat{V}_N(r)\Big[a_{q+r}^\dagger a_{-r}^\dagger a_q a_0 + \text{h.c.}\Big], \\ Q_4 &= \frac{1}{2}\sum_{p,q\neq 0, r\neq -p, r\neq -q} \hat{V}_N(r) a_{p+r}^\dagger a_q^\dagger a_p a_{q+r}. \end{aligned} \tag{10}$$

Since we isolated the contributions of the zero modes, from now on we follow the convention that the indices appearing in creation and annihilation operators are always nonzero except when stated otherwise.

Naive power counting, based on the fact that $a_0, a_0^\dagger \simeq \sqrt{N}$ due to the presence of Bose-Einstein condensation and on the scaling (8) of the interaction, suggests that the terms Q_3 and Q_4 are small. For this reason, Bogoliubov neglected these contributions and diagonalised the remaining quadratic terms. This led to expressions similar to equations (4) and (5) for the low-energy spectrum of H_N, but with the scattering length replaced by its first and second Born approximations. In fact, because of the slow decay of the potential in Fourier space, the operators Q_3 and Q_4 are not small. They instead contain important terms that effectively renormalise the interaction and produce the scattering length appearing in the formulas in equations (4) and (5). To obtain a rigorous proof of Theorem 1, it is therefore crucial that we first extract the large contributions to the energy hidden in the cubic and quartic operators Q_3, Q_4; only afterwards can we diagonalise the remaining quadratic terms.

Let us give a little more detail about the main ideas of the proof. Following the strategy of [6], we will first conjugate equation (9) with a unitary operator of the form $e^{\mathcal{B}_2}$, where $\mathcal{B}_2$ is a quadratic expression in creation and annihilation operators $a_p, a_p^\dagger$, associated with momenta $p \neq 0$. The goal of this conjugation is to extract contributions that regularise the off-diagonal term Q_2 and, at the same time,

https://doi.org/10.1017/fms.2022.78 Published online by Cambridge University Press

reconstruct the leading-order ground state energy $4\pi\mathfrak{a}_N N$ when combined with H_0. Roughly speaking, neglecting several error terms, we will find

$$\begin{aligned} e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} \simeq 4\pi\mathfrak{a}_N (N-1) + \sum_{|p|\le N^\alpha} \frac{(4\pi\mathfrak{a}_N)^2}{p^2} \\ + \sum_{p\in\Lambda^*} (p^2 + 2\hat{V}(0) - 8\pi\mathfrak{a}_N) a_p^\dagger a_p + \sum_{|p|\le N^\alpha} 4\pi\mathfrak{a}_N [a_p^\dagger a_{-p}^\dagger + \text{h.c.}] + Q_3 + Q_4. \end{aligned} \tag{11}$$

As explained in the introduction, an important difference, compared with [6], is that here we impose an infrared cutoff in $\mathcal{B}_2$, defined in equation (13), letting it act only on momenta $|p| > N^\alpha$. On one hand, this choice simplifies the computation of the action of $\mathcal{B}_2$ (it allows us to expand it; important contributions arise only from the first and second commutators). On the other hand, it produces terms, like the sum on the first line and the regularised off-diagonal quadratic term on the second line of equation (11), which contribute to the energy to order N^α; these terms are larger than the precision we are looking for and will need to be compensated for with the second quadratic transformation. Notice that the idea of using an infrared cutoff in the quadratic conjugation already appeared in the proof of complete Bose-Einstein condensation given in [4] and, more recently, in the proof of the validity of Bogoliubov theory for Bose gases trapped by an external potential obtained in [25].

Observing equation (11), it is clear that we still have to renormalise the diagonal quadratic term (proportional to $\hat{V}(0)$) and the cubic term Q_3. To this end, we will introduce a unitary transformation $e^{\mathcal{B}_3}$, with $\mathcal{B}_3$, defined in equation (59), cubic in the operator $a_p, a_p^\dagger$, with $p \neq 0$. Up to several negligible errors, conjugation with $e^{\mathcal{B}_3}$ will lead us to

$$\begin{aligned} e^{-\mathcal{B}_3} e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} e^{\mathcal{B}_3} \simeq 4\pi\mathfrak{a}_N (N-1) + \sum_{|p|\le N^\alpha} \frac{(4\pi\mathfrak{a}_N)^2}{p^2}. \\ + \sum_{p\in\Lambda^*} (p^2 + 8\pi\mathfrak{a}_N) a_p^\dagger a_p + \sum_{|p|\le N^\alpha} 4\pi\mathfrak{a}_N [a_p^\dagger a_{-p}^\dagger + \text{h.c.}] + Q_4 . \end{aligned} \tag{12}$$

The only term on the right-hand side of the last equation where we still have the original, singular, potential $\hat{V}_N$ is Q_4; all other terms have been renormalised and are now expressed in terms of the scattering length $\mathfrak{a}_N$. Fortunately, Q_4 is positive; for this reason, we do not need to renormalise it (for lower bounds, it can be neglected; for upper bounds, it only needs to be controlled on special trial states). Finally, in section 5, we will apply a second quadratic transformation $e^{\mathcal{B}_4}$ to diagonalise the remaining quadratic Hamiltonian on the right-hand side of equation (12). This will lead us to

$$\begin{aligned} e^{-\mathcal{B}_4} e^{-\mathcal{B}_3} e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} e^{\mathcal{B}_3} e^{\mathcal{B}_4} \simeq 4\pi\mathfrak{a}_N (N-1) + \frac{1}{2}\sum_p \left[\sqrt{p^4 + 16\pi\mathfrak{a}_N p^2} - p^2 - 8\pi\mathfrak{a}_N + \frac{(8\pi\mathfrak{a}_N)^2}{2p^2} \right] \\ + \sum_p \sqrt{p^4 + p^2 16\pi\mathfrak{a}_N}\, a_p^\dagger a_p + Q_4, \end{aligned}$$

which will allow us to show Theorem 1. To control error terms, we use the tool of localisation in the number of particles to show Bose-Einstein condensation (similarly to [7, 25]).

3. Quadratic renormalisation

Starting with the quadratic transformation, we conjugate the Hamiltonian H_N in equation (7) with the unitary $e^{\mathcal{B}_2}$, where

$$\mathcal{B}_2 = \frac{1}{2}\sum_p \tilde{\varphi}_p [a_p^\dagger a_{-p}^\dagger a_0 a_0 - \text{h.c.}]. \tag{13}$$

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We are going to fix the coefficients $\tilde{\varphi}_p$ so that the commutator $[H_1 + Q_4, \mathcal{B}_2]$ arising from the action of the unitary $e^{\mathcal{B}_2}$ (13) renormalises the off-diagonal quadratic term Q_2 (effectively replacing the singular potential V_N with a regularised interaction having the same scattering length). To this end, we choose φ_p satisfying the relations

$$p^2\varphi_p + \frac{1}{2}\sum_{q\neq 0}\hat{V}_N((p-q))\varphi_q = -\frac{1}{2}\hat{V}_N(p)\,, \tag{14}$$

for all $p \in \Lambda_+^* = \Lambda^*\backslash\{0\}$. Equation (14) is a truncated version of the zero-energy scattering equation for the potential V_N on the whole space $\mathbb{R}^3$.

To prove the existence of a solution for equation (14), we consider the operator

$$\mathfrak{h} = -\Delta + \frac{1}{2}V_N$$

acting on the one-particle space $L^2(\Lambda)$ (for N large enough, V_N is supported in $[-1/2; 1/2]^3$ and can be periodically extended to define a function on the torus). Denoting by $P_0^\perp$ the orthogonal projection onto the orthogonal complement of the zero-momentum mode φ_0 in $L^2(\Lambda)$, we find (since $V_N \geq 0$) that $P_0^\perp \mathfrak{h} P_0^\perp \geq C > 0$, and therefore that $P_0^\perp \mathfrak{h} P_0^\perp$ is invertible. Thus, we can define $\check{\varphi} \in L^2(\Lambda)$ through

$$\check{\varphi} = -\frac{1}{2}P_0^\perp\Big[P_0^\perp\Big(-\Delta + \frac{1}{2}V_N\Big)P_0^\perp\Big]^{-1}P_0^\perp V_N\,. \tag{15}$$

It is then easy to check that the Fourier coefficients of $\check{\varphi}$ satisfy the relations in equation (14).

Using the sequence $\{\varphi_p\}_{p\in 2\pi\mathbb{Z}^3\backslash\{0\}}$, we can define the 'box scattering length' of V_N by

$$8\pi\mathfrak{a}_N := N\Big[\hat{V}_N(0) + \sum_p \hat{V}_N(p)\varphi_p\Big] = \hat{V}(0) + N\sum_p \hat{V}_N(p)\varphi_p. \tag{16}$$

As proven in [15], we have that $|\mathfrak{a}_N - \mathfrak{a}| \lesssim N^{-1}$.

As explained earlier, we first renormalise the high-momenta part of Q_2; for this reason, we use a cutoff version of φ_p to momenta $|p| > N^\alpha$ for some $0 < \alpha < 1$. We therefore define

$$\tilde{\varphi}_p = \varphi_p\chi_{|p|>N^\alpha}. \tag{17}$$

The next lemma lists some important properties of the sequences $\varphi, \tilde{\varphi}$ and the scattering length $\mathfrak{a}_N$ that will be useful for our analysis.

Lemma 2. *Let $V \in L^2(\mathbb{R}^3)$ be nonnegative and compactly supported. Define $\check{\varphi}$ as in equation (15), and denote by φ_p the corresponding Fourier coefficients. Then $\varphi_p \in \mathbb{R}$, $\varphi_{-p} = \varphi_p$ and*

$$|\varphi_p| \lesssim \frac{1}{Np^2} \tag{18}$$

for all $p \in 2\pi\mathbb{Z}\backslash\{0\}$. Moreover, with equation (17), we have

$$\|\tilde{\varphi}\|_2 \lesssim N^{-1-\alpha/2}, \qquad \|\tilde{\varphi}\|_\infty \lesssim N^{-1-2\alpha}, \qquad \|\tilde{\varphi}\|_1 \lesssim 1,$$

and

$$N\sum_p \hat{V}_N(p)\tilde{\varphi}_p = 8\pi\mathfrak{a}_N - \hat{V}(0) + \mathcal{O}(N^{\alpha-1}). \tag{19}$$

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Proof. Multiplying equation (14) by φ_p, summing over p and using that $V \geq 0$, we obtain

$$2\|p\varphi\|_2^2 = -\sum_p \hat{V}_N(p)\varphi_p - \sum_{p,q} \hat{V}_N(p-q)\varphi_p\varphi_q \leq -\sum_p \hat{V}_N(p)\varphi_p. \tag{20}$$

On one hand, this implies that $\|p\varphi\|_2 \lesssim \|V_N\|_2\|\varphi\|_2 < \infty$ (the last bound is not uniform in N; it follows from equation (15)). On the other hand, the estimate (20) leads to

$$2\|p\varphi\|_2^2 \leq \|V_N/|p|\|_2\|p\varphi\|_2.$$

Dividing by $\|p\varphi\|_2$ and squaring, we obtain

$$\|p\varphi\|_2^2 \lesssim \sum_p \frac{|\hat{V}_N(p)|^2}{p^2} \lesssim \|\hat{V}_N\|_\infty^2 \||p|^{-2}\chi_{|p|<N}\|_1 + \|\hat{V}_N\|_\infty\|\hat{V}_N\|_2\||p|^{-2}\chi_{|p|>N}\|_2 \lesssim N^{-1}. \tag{21}$$

Using equation (14) again, we obtain the pointwise bound

$$|p^2\varphi_p| \leq |\hat{V}_N(p)| + \Big[\sum_q \frac{|\hat{V}_N(p-q)|^2}{q^2}\Big]^{1/2}\|q\varphi\|_2 \lesssim N^{-1}, \tag{22}$$

where we proceeded as in (21) to bound $\||\hat{V}_N|^2 * |q|^{-2}\|_\infty$. This proves the bound (18) and immediately implies the bounds for $\|\tilde{\varphi}\|_2, \|\tilde{\varphi}\|_\infty$. To obtain the bound on $\|\tilde{\varphi}\|_1$, we divide equation (14) by $|p|^2$. Proceeding as in (21), we obtain

$$\sum_{p\neq 0} \frac{|\hat{V}_N(p)|}{|p|^2} \lesssim 1\,,$$

and hence we only have to bound $\||p|^{-2}(\hat{V}_N * \varphi)\|_1$. Iterating equation (14) and using the regularising estimate $\||p|^{-2}\hat{V}_N * g\|_{6p/(6+p)+\varepsilon} \leq C_\varepsilon\|\hat{V}_N\|_2\|g\|_p$ for all $\varepsilon > 0$, $p \geq 6/5$, $g \in \ell^p(\Lambda^*)$ and some $C_\varepsilon > 0$, we obtain that $\|\varphi\|_1 < \infty$. Separating high and low momenta, we obtain for $A \geq 1$ and $\varepsilon > 0$

$$\begin{aligned}\|\varphi\|_1 &\lesssim 1 + \|\chi_{|p|>AN}|p|^{-2}\|_2\|V_N\check{\varphi}\|_2 + \|\chi_{|p|\leq AN}|p|^{-2}\|_1\|\hat{V}_N * \varphi\|_\infty\\ &\lesssim 1 + A^{-\frac{1}{2}}\|\varphi\|_1 + A\,,\end{aligned}$$

where we used that $\|\check{\varphi}\|_\infty \leq \|\varphi\|_1$ and the Hölder inequality as in (22) to estimate $\|\hat{V}_N * \varphi\|_\infty$. Taking A sufficiently large but fixed, we obtain $\|\varphi\|_1 \lesssim 1$.

The estimate (19) follows by noticing that, from the definition given in equation (16),

$$\Big|8\pi\mathfrak{a}_N - \hat{V}(0) - N\sum_p V_N(p)\tilde{\varphi}_p\Big| \leq N\sum_{|p|\leq N^\alpha} |\hat{V}_N(0)||\varphi_p| \lesssim \frac{1}{N}\sum_{|p|\leq N^\alpha} \frac{1}{|p|^2} \lesssim N^{-1+\alpha},$$

where we use equation (18) and $\|\hat{V}_N\|_\infty \lesssim N^{-1}$. □

Using the bounds in Lemma 2, we can control the growth of the number of excitations w.r.t. the action of $\mathcal{B}_2$; the proof of the next lemma can be found, for example, in [12, Lemma 3.1].

Lemma 3. *For every $n \in \mathbb{N}$ and $|s| \leq 1$, we have*

$$\begin{aligned}\pm(e^{-s\mathcal{B}_2}\mathcal{N}_+e^{s\mathcal{B}_2} - \mathcal{N}_+) &\lesssim CN^{-\alpha/2}(\mathcal{N}_+ + 1),\\ e^{-s\mathcal{B}_2}(\mathcal{N}_+ + 1)^n e^{s\mathcal{B}_2} &\lesssim (\mathcal{N}_+ + 1)^n.\end{aligned}$$

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In the next proposition, we describe the action of the operator $\mathcal{B}_2$, defined as in equation (13), on the Hamilton operator in equation (9).

Proposition 4. *We have*

$$e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} = 4\pi \mathfrak{a}_N (N-1) + \sum_{|p|\le N^\alpha} \frac{(4\pi\mathfrak{a}_N)^2}{p^2} + H_1 + \tilde{H}_2 + \tilde{Q}_2 + Q_3 + Q_4 + \mathcal{E}_{\mathcal{B}_2}, \tag{23}$$

with

$$\tilde{H}_2 = (2\hat{V}(0) - 8\pi\mathfrak{a}_N)\mathcal{N}_+, \tag{24}$$

$$\tilde{Q}_2 = \sum_{|p|\le N^\alpha} 4\pi\mathfrak{a}_N \left[a_p^\dagger a_{-p}^\dagger \frac{a_0 a_0}{N} + \text{h.c.} \right], \tag{25}$$

and

$$\pm \mathcal{E}_{\mathcal{B}_2} \lesssim N^{-\alpha/2} Q_4 + \left[N^{-\alpha/2} + N^{-1+5\alpha/2} \right] (\mathcal{N}_+ + 1) + N^{-1+\alpha} \mathcal{N}_+^2 + N^{-2} H_1.$$

To show Proposition 4, we define

$$\Gamma_2 := [H_1 + Q_4, \mathcal{B}_2] + Q_2 - \tilde{Q}_2' \tag{26}$$

with

$$\tilde{Q}_2' = \sum_p \hat{W}(p) a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.}, \tag{27}$$

and

$$\hat{W}(p) = \frac{1}{2} \chi_{|p|\le N^\alpha} \left[\sum_q \hat{V}_N(p-q)\varphi_q + \hat{V}_N(p) \right] - \frac{1}{2} \sum_{|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q.$$

We observe that

$$\begin{aligned}
& e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} \\
&= H_0 + H_1 + Q_4 + \int_0^1 e^{-t\mathcal{B}_2} [H_1 + Q_4, \mathcal{B}_2] e^{t\mathcal{B}_2} \mathrm{d}t + e^{-\mathcal{B}_2} Q_2 e^{\mathcal{B}_2} + e^{-\mathcal{B}_2} (H_2 + Q_3) e^{\mathcal{B}_2} \\
&= H_0 + H_1 + Q_4 + \int_0^1 e^{-t\mathcal{B}_2} (-Q_2 + \tilde{Q}_2' + \Gamma_2) e^{t\mathcal{B}_2} \mathrm{d}t + e^{-\mathcal{B}_2} Q_2 e^{\mathcal{B}_2} + e^{-\mathcal{B}_2} (H_2 + Q_3) e^{\mathcal{B}_2} \\
&= H_0 + H_1 + \tilde{Q}_2' + Q_4 + \int_0^1 \int_s^1 e^{-t\mathcal{B}_2} [Q_2, \mathcal{B}_2] e^{t\mathcal{B}_2} \mathrm{d}t\mathrm{d}s + \int_0^1 \int_0^s e^{-t\mathcal{B}_2} \tilde{Q}_2' e^{t\mathcal{B}_2} \mathrm{d}t\mathrm{d}s \\
&\quad + \int_0^1 e^{-t\mathcal{B}_2} \Gamma_2 e^{t\mathcal{B}_2} \mathrm{d}t + e^{-\mathcal{B}_2} (H_2 + Q_3) e^{\mathcal{B}_2},
\end{aligned} \tag{28}$$

where, in the last step, we use

$$\int_0^1 e^{-t\mathcal{B}_2} \tilde{Q}_2' e^{t\mathcal{B}_2} \mathrm{d}t = \tilde{Q}_2' + \int_0^1 \int_0^s e^{-t\mathcal{B}_2} [\tilde{Q}_2', \mathcal{B}_2] e^{t\mathcal{B}_2} \mathrm{d}t\mathrm{d}s$$

$$e^{-\mathcal{B}_2} Q_2 e^{\mathcal{B}_2} - \int_0^1 e^{-t\mathcal{B}_2} Q_2 e^{t\mathcal{B}_2} \mathrm{d}t = \int_0^1 \int_s^1 e^{-t\mathcal{B}_2} [Q_2, \mathcal{B}_2] e^{t\mathcal{B}_2} \mathrm{d}t\mathrm{d}s.$$

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The proof of Proposition 4 now follows by controlling the terms on the right-hand side of equation (28). This is accomplished through a series of lemmas. We start by controlling the contribution arising from H_2.

Lemma 5. *On* $\{\mathcal{N} = N\}$, *we have*

$$e^{-\mathcal{B}_2} H_2 e^{\mathcal{B}_2} = \hat{V}(0)\mathcal{N}_+ + \mathcal{E}_{H_2}, \tag{29}$$

with

$$\pm\mathcal{E}_{H_2} \lesssim N^{-\alpha/2}(\mathcal{N}_+ + 1) + \mathcal{N}_+^2/N + N^{-2}H_1.$$

Proof. We have

$$\begin{aligned} e^{-\mathcal{B}_2} H_2 e^{\mathcal{B}_2} = N \sum_p \hat{V}_N(p)\Big(a_p^\dagger a_p + \int_0^1 e^{-s\mathcal{B}_2}[a_p^\dagger a_p, \mathcal{B}_2] e^{s\mathcal{B}_2} \mathrm{d}s\Big) \\ - e^{-\mathcal{B}_2}\Big(\sum_p \hat{V}_N(p) a_p^\dagger a_p \mathcal{N}_+ + \frac{\hat{V}_N(0)}{2}\mathcal{N}_+(\mathcal{N}_+ - 1)\Big)e^{\mathcal{B}_2}. \end{aligned} \tag{30}$$

The term on the second line is controlled using Lemma 3 by $N^{-1}\mathcal{N}_+^2$. For the second term in parenthesis in the first line, we use Lemma 3 to estimate

$$\begin{aligned} \pm \sum_p N\hat{V}_N(p)[a_p^\dagger a_p, \mathcal{B}_2] &= \pm \sum_p N\hat{V}_N(p)\tilde{\varphi}_p a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.} \\ &\le C\|\tilde{\varphi}\|_2(\mathcal{N}_0 + 1)(\mathcal{N}_+ + 1) \le N^{-\alpha/2}(\mathcal{N}_+ + 1), \end{aligned}$$

where we used that $\mathcal{N}_0 \lesssim N$ and Lemma 2. Finally, since V_N is even, we obtain

$$N|\hat{V}_N(p) - \hat{V}_N(0)| \le \|x^2 V\|_1 N^{-2} p^2 \le C N^{-2} p^2,$$

which gives

$$\pm\Big(N\sum_p \hat{V}_N(p) a_p^\dagger a_p - \hat{V}(0)\mathcal{N}_+\Big) \le N^{-2}H_1. \qquad \square$$

The estimate of the term involving Q_3 is obtained analogously to [7, 15]. We repeat the proof for the sake of completeness.

Lemma 6. *We have*

$$e^{-\mathcal{B}_2} Q_3 e^{\mathcal{B}_2} = Q_3 + \mathcal{E}_{Q_3},$$

with

$$\pm\mathcal{E}_{Q_3} \lesssim N^{-\alpha/2}(Q_4 + \mathcal{N}_+ + 1).$$

Proof. We can rewrite

$$e^{-\mathcal{B}_2} Q_3 e^{\mathcal{B}_2} = \sum_{q,r} \hat{V}_N(r)\Big[e^{-\mathcal{B}_2} a_{q+r}^\dagger a_{-r}^\dagger e^{\mathcal{B}_2} e^{-\mathcal{B}_2} a_q a_0 e^{\mathcal{B}_2} + \text{h.c.}\Big]. \tag{31}$$

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Via Duhamel's formula, we have

$$e^{-\mathcal{B}_2} a^\dagger_{q+r} a^\dagger_{-r} e^{\mathcal{B}_2} = a^\dagger_{q+r} a^\dagger_{-r} + \int_0^1 e^{-s\mathcal{B}_2} [a^\dagger_{q+r} a^\dagger_{-r}, \mathcal{B}_2] e^{s\mathcal{B}_2} \mathrm{d}s \tag{32}$$

and

$$e^{-\mathcal{B}_2} a_q a_0 e^{\mathcal{B}_2} = a_q a_0 + \int_0^1 e^{-s\mathcal{B}_2} (\tilde{\varphi}_q a^\dagger_{-q} a_0 a_0 a_0 - a_q \sum_l \tilde{\varphi}_l a_{-l} a_l a^\dagger_0) e^{s\mathcal{B}_2} \mathrm{d}s.$$

The product of the first terms, combined with its hermitian conjugate, corresponds to Q_3 in the statement of the lemma. All other terms will be estimated in three steps.

Step 1. Passing to x-space, we find

$$\begin{aligned}
&\Big| \sum_{q,r} \tilde{\varphi}_q \hat{V}_N(r) \int_0^1 \langle \xi, a^\dagger_{q+r} a^\dagger_{-r} e^{-s\mathcal{B}_2} a^\dagger_{-q} a_0 a_0 a_0 e^{s\mathcal{B}_2} \xi \rangle \,\mathrm{d}s \Big| \\
&\quad = \int_0^1 \int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y) \big\langle \xi, \check{a}^\dagger_x \check{a}^\dagger_y e^{-s\mathcal{B}_2} \check{a}^\dagger(\check{\tilde{\varphi}}_x) a_0 a_0 a_0 e^{s\mathcal{B}_2} \xi \big\rangle \,\mathrm{d}s \\
&\quad \le \Big(\int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y) \|\check{a}_x \check{a}_y \xi\|^2 \Big)^{1/2} \\
&\qquad\qquad \times \Big(\int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y) \int_0^1 \|\check{a}^\dagger(\check{\tilde{\varphi}}_x) a_0 a_0 a_0 e^{s\mathcal{B}_2} \xi\|^2 \mathrm{d}s \Big)^{1/2} \\
&\quad \le C N^{3/2} \|V_N\|_1^{1/2} \|\tilde{\varphi}\|_2 \|Q_4^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^{1/2} \xi\| \\
&\quad \le C N^{-\alpha/2} \langle \xi, (Q_4 + \mathcal{N}_+ + 1) \xi \rangle,
\end{aligned} \tag{33}$$

where we used Lemma 2, Lemma 3 and the bound $\mathcal{N}_0 \le N$.

Step 2. Similarly, we have

$$\begin{aligned}
&\Big| \sum_{q,r} \hat{V}_N(r) \sum_l \tilde{\varphi}_l \int_0^1 \mathrm{d}s \,\langle \xi, a^\dagger_{q+r} a^\dagger_{-r} e^{-s\mathcal{B}_2} a_q a_{-l} a_l a^\dagger_0 e^{s\mathcal{B}_2} \xi \rangle \Big| \\
&\quad = \Big| \sum_l \tilde{\varphi}_l \int_0^1 \mathrm{d}s \int_{\Lambda^2} \mathrm{d}x\mathrm{d}y \, V_N(x-y) \langle \xi, a^\dagger_x a^\dagger_y e^{-s\mathcal{B}_2} a_x a_{-l} a_l a^\dagger_0 e^{s\mathcal{B}_2} \xi \rangle \Big| \\
&\quad \le C \|\tilde{\varphi}\|_2 \|V_N\|_1^{1/2} \|Q_4^{1/2} \xi\| \|(\mathcal{N}_+ + 1)^2 \xi\| \le C N^{-\alpha/2} \langle \xi, Q_4 + \mathcal{N}_+ + 1) \xi \rangle.
\end{aligned} \tag{34}$$

Step 3. The remaining term has the form

$$\sum_{q,r} \hat{V}_N(r) \int_0^1 e^{-s\mathcal{B}_2} [a^\dagger_{q+r} a^\dagger_{-r}, \mathcal{B}_2] e^{(s-1)\mathcal{B}_2} a_q a_0 e^{\mathcal{B}_2} \mathrm{d}s.$$

A straightforward computation gives

$$[a^\dagger_{q+r} a^\dagger_{-r}, \mathcal{B}_2] = -\Big(\tilde{\varphi}_r a^\dagger_{q+r} a_r + \tilde{\varphi}_{-r-q} a^\dagger_{-r} a_{-q-r} \Big) a^\dagger_0 a^\dagger_0.$$

The contributions of the two terms in parenthesis can be handled similarly. Let us consider, for example, the expectation

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$$
\begin{aligned}
\Big|\sum_{q,r}\hat{V}_N(r)\tilde{\varphi}_r\int_0^1\langle\xi,e^{-s\mathcal{B}_2}a^\dagger_{q+r}a_ra^\dagger_0a^\dagger_0e^{(s-1)\mathcal{B}_2}a_qa_0e^{\mathcal{B}_2}\xi\rangle\,\mathrm{d}s\Big|&\\
\lesssim\frac{1}{N}\sum_{r,q}|\tilde{\varphi}_r|\int_0^1\|a_{q+r}a_0a_0e^{(s-1)\mathcal{B}_2}\xi\|\|a_re^{s\mathcal{B}_2}a_qa_0e^{\mathcal{B}_2}\xi\|\mathrm{d}s&\\
\lesssim\|\tilde{\varphi}\|_2\|\mathcal{N}_+^{1/2}\xi\|\int_0^1\Big(\sum_{r,q}\|a_re^{(s-1)\mathcal{B}_2}a_qa_0e^{\mathcal{B}_2}\xi\|^2\Big)^{1/2}\mathrm{d}s&\\
\lesssim\|\tilde{\varphi}\|_2\|\mathcal{N}_+^{1/2}\xi\|\|(\mathcal{N}_++1)^{3/2}\xi\|\leq N^{-\alpha/2}\langle\xi,(\mathcal{N}_++1)\xi\rangle,&
\end{aligned}
\tag{35}
$$

where we used Lemma 2 and Lemma 3. □

Next, we recall the definition of Γ_2 given in equation (26) and consider the term containing Γ_2 appearing on the right-hand side of equation (28).

Lemma 7. *We have*

$$
\Gamma_2=\sum_{r,p,q}\hat{V}_N(r)\tilde{\varphi}_p[a^\dagger_{p+r}a^\dagger_qa^\dagger_{-p}a_{q+r}a_0a_0+h.c.]
\tag{36}
$$

and

$$
\int_0^1e^{-t\mathcal{B}_2}\Gamma_2e^{t\mathcal{B}_2}\mathrm{d}t\lesssim N^{-\alpha/2}(\mathcal{Q}_4+\mathcal{N}_++1).
\tag{37}
$$

Proof. Straightforward calculations yield

$$
[H_1,\mathcal{B}_2]=\sum_p p^2\tilde{\varphi}_p[a^\dagger_pa^\dagger_{-p}a_0a_0+\text{h.c.}]
$$

and

$$
[\mathcal{Q}_4,\mathcal{B}_2]=\frac{1}{2}\sum_{p,q}\hat{V}_N(p-q)\tilde{\varphi}_qa^\dagger_pa^\dagger_{-p}a_0a_0+\sum_{r,p,q}\hat{V}_N(r)\tilde{\varphi}_pa^\dagger_{p+r}a^\dagger_qa^\dagger_{-p}a_{q+r}a_0a_0+\text{h.c.}
\tag{38}
$$

Hence,

$$
\begin{aligned}
[H_1+\mathcal{Q}_4,\mathcal{B}_2]+\mathcal{Q}_2=&\sum_{|p|>N^\alpha}\Big(p^2\varphi_p+\frac{1}{2}\sum_q\hat{V}_N(p-q)\varphi_q+\frac{1}{2}\hat{V}_N(p)\Big)(a^\dagger_pa^\dagger_{-p}a_0a_0+\text{h.c.})\\
&-\frac{1}{2}\sum_{|p|>N^\alpha,|q|\leq N^\alpha}\hat{V}_N(p-q)\varphi_q(a^\dagger_pa^\dagger_{-p}a_0a_0+\text{h.c.})\\
&+\frac{1}{2}\sum_{|p|\leq N^\alpha}\Big(\sum_{|q|>N^\alpha}\hat{V}_N(p-q)\varphi_q+\hat{V}_N(p)\Big)(a^\dagger_pa^\dagger_{-p}a_0a_0+\text{h.c.})\\
&+\sum_{r,p,q}\hat{V}_N(r)\tilde{\varphi}_pa^\dagger_{p+r}a^\dagger_qa^\dagger_{-p}a_{q+r}a_0a_0\\
=&\,\tilde{\mathcal{Q}}_2'+\sum_{r,p,q}\hat{V}_N(r)\tilde{\varphi}_pa^\dagger_{p+r}a^\dagger_qa^\dagger_{-p}a_{q+r}a_0a_0,
\end{aligned}
$$

where we used the scattering equation (14) and the definition of $\tilde{\mathcal{Q}}_2'$ given in equation (27). Comparing with equation (26), we find equation (36).

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To prove the estimate (37), we write

$$\int_0^1 e^{-t\mathcal{B}_2}\Gamma_2 e^{t\mathcal{B}_2}\mathrm{d}t = \sum_{p,q,r} \hat{V}_N(r)\tilde{\varphi}_p \int_0^1 e^{-t\mathcal{B}_2} a^\dagger_{p+r} a^\dagger_q e^{t\mathcal{B}_2} e^{-t\mathcal{B}_2} a^\dagger_{-p} a_{q+r} a_0 a_0 e^{t\mathcal{B}_2}\mathrm{d}t,$$

and we proceed similarly as in Lemma 6. We omit further details. □

Next, we focus on the contribution with the commutator $[Q_2, \mathcal{B}_2]$ on the right-hand side of equation (28).

Lemma 8. *On* $\{\mathcal{N} = N\}$, *we have*

$$\begin{aligned}&\int_0^1\int_s^1 e^{-t\mathcal{B}_2}[Q_2,\mathcal{B}_2]e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s\\ &\quad = -\frac{(N-1)}{2}(\hat{V}(0)-8\pi\mathfrak{a}_N) - \frac{N(N-1)}{2}\sum_{|p|\le N^\alpha}\hat{V}_N(p)\varphi_p + \mathcal{N}_+(\hat{V}(0)-8\pi\mathfrak{a}_N) + \mathcal{E}_{[Q_2,\mathcal{B}_2]},\end{aligned} \tag{39}$$

and

$$\pm\mathcal{E}_{[Q_2,\mathcal{B}_2]} \lesssim (N^{-\alpha/2}+N^{\alpha-1})\mathcal{N}_+ + N^{-1}(\mathcal{N}_++1)^2 + N^{-\alpha/2}Q_4. \tag{40}$$

Proof. First, we claim that

$$\int_0^1\int_s^1 e^{-t\mathcal{B}_2}[Q_2,\mathcal{B}_2]e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s = \frac{N(N-1)}{2}\sum_p \hat{V}_N(p)\tilde{\varphi}_p - N\mathcal{N}_+\sum_p\hat{V}_N(p)\tilde{\varphi}_p + \mathcal{E}'_{[Q_2,\mathcal{B}_2]} \tag{41}$$

with

$$\begin{aligned}\mathcal{E}'_{[Q_2,\mathcal{B}_2]} = -2N\sum_p \hat{V}_N(p)\tilde{\varphi}_p &\int_0^1\int_s^1 \left[e^{-t\mathcal{B}_2}\mathcal{N}_+ e^{t\mathcal{B}_2} - \mathcal{N}_+\right]\mathrm{d}t\mathrm{d}s\\ &+\sum_p \hat{V}_N(p)\tilde{\varphi}_p\int_0^1\int_s^1 e^{-t\mathcal{B}_2}\mathcal{N}_+(\mathcal{N}_++1)e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s\\ &+2\sum_p\hat{V}_N(p)\tilde{\varphi}_p\int_0^1\int_s^1 e^{-t\mathcal{B}_2}\mathcal{N}_0(\mathcal{N}_0-1)a^\dagger_p a_p e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s\\ &-\sum_{p,q}\hat{V}_N(p)\tilde{\varphi}_q\int_0^1\int_s^1 e^{-t\mathcal{B}_2}a^\dagger_p a^\dagger_{-p}a_{-q}a_q(2\mathcal{N}_0+1)e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s.\end{aligned} \tag{42}$$

To prove equation (42), we calculate

$$\begin{aligned}[][Q_2,\mathcal{B}_2] &= \frac{1}{4}\sum_{p,q}\hat{V}_N(p)\tilde{\varphi}_q[a^\dagger_p a^\dagger_{-p}a_0a_0 + a_{-p}a_p a^\dagger_0 a^\dagger_0, a^\dagger_q a^\dagger_{-q}a_0a_0 - a_{-q}a_q a^\dagger_0 a^\dagger_0]\\ &= \frac{1}{4}\sum_{p,q}\hat{V}_N(p)\tilde{\varphi}_q\Big([a_{-p}a_p a^\dagger_0 a^\dagger_0, a^\dagger_q a^\dagger_{-q}a_0a_0] - [a^\dagger_p a^\dagger_{-p}a_0a_0, a_{-q}a_q a^\dagger_0 a^\dagger_0]\Big).\end{aligned} \tag{43}$$

The two terms in brackets are hermitian conjugates. Hence, it suffices to compute the second one

$$-[a^\dagger_p a^\dagger_{-p}a_0a_0, a_{-q}a_q a^\dagger_0a^\dagger_0] = [a_{-q}a_q, a^\dagger_p a^\dagger_{-p}]a^\dagger_0a^\dagger_0a_0a_0 - a^\dagger_p a^\dagger_{-p}a_{-q}a_q[a_0a_0, a^\dagger_0a^\dagger_0],$$

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where

$$[a_{-p}a_p, a_q^\dagger a_{-q}^\dagger] = (\delta_{p,q} + \delta_{p,-q})(1 + a_p^\dagger a_p + a_{-p}^\dagger a_{-p}), \tag{44}$$

and

$$\begin{aligned} a_0^\dagger a_0^\dagger a_0 a_0 &= \mathcal{N}_0(\mathcal{N}_0 - 1) = N(N-1) - 2N\mathcal{N}_+ + \mathcal{N}_+(\mathcal{N}_+ + 1), \\ [a_0 a_0, a_0^\dagger a_0^\dagger] &= 2(2\mathcal{N}_0 + 1). \end{aligned} \tag{45}$$

Inserting these identities on the right-hand side of equation (43), conjugating with $e^{t\mathcal{B}_2}$ and integrating over t and s, we obtain equation (42).

With the definition given in equation (16), we write

$$\frac{N(N-1)}{2}\sum_p \hat{V}_N(p)\tilde{\varphi}_p = \frac{(N-1)}{2}(\hat{V}(0) - 8\pi\mathfrak{a}_N) - \frac{N(N-1)}{2}\sum_{|p|\le N^\alpha} \hat{V}_N(p)\varphi_p,$$

and we use the bound (19) to estimate

$$\pm\Big[-N\mathcal{N}_+\sum_p \hat{V}_N(p)\tilde{\varphi}_p + (8\pi\mathfrak{a}_N - \hat{V}(0))\mathcal{N}_+\Big] \lesssim N^{\alpha-1}\mathcal{N}_+.$$

Thus, Lemma 8 follows from equation (41), if we can prove that $\mathcal{E}'_{[\mathcal{Q}_2,\mathcal{B}_2]}$ satisfies the estimate given in (40).

Using the bound

$$\Big|\sum_p \hat{V}_N(p)\tilde{\varphi}_p\Big| \lesssim N^{-1}$$

and Lemma 3, we can bound the first term on the right-hand side of equation (42) by

$$\pm 2N\sum_p \hat{V}_N(p)\tilde{\varphi}_p \int_0^1\int_s^1 \big[e^{-t\mathcal{B}_2}\mathcal{N}_+ e^{t\mathcal{B}_2} - \mathcal{N}_+\big]\mathrm{d}t\mathrm{d}s \lesssim N^{-\alpha/2}(\mathcal{N}_+ + 1).$$

Also, the second term on the right-hand side of equation (42) can be bounded with Lemma 3; we find

$$\pm\sum_p \hat{V}_N(p)\tilde{\varphi}_p \int_0^1\int_s^1 e^{-t\mathcal{B}_2}\mathcal{N}_+(\mathcal{N}_+ + 1)e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s \lesssim N^{-1}(\mathcal{N}_+ + 1)^2.$$

For the third term, we use $\|\hat{V}_N\|_\infty \lesssim N^{-1}$, $\|\tilde{\varphi}\|_\infty \lesssim N^{-1-2\alpha}$ together with $\mathcal{N}_0(\mathcal{N}_0 - 1) \le N^2$ and Lemma 3 again to conclude that

$$\pm 2\sum_p \hat{V}_N(p)\tilde{\varphi}_p \int_0^1\int_s^1 e^{-t\mathcal{B}_2}\mathcal{N}_0(\mathcal{N}_0 + 1)e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s \lesssim N^{-2\alpha}\mathcal{N}_+.$$

To control the last term on the right-hand side of equation (42), we write

$$\begin{aligned} &\sum_{p,q} \hat{V}_N(p)\tilde{\varphi}_q \int_0^1\int_s^1 e^{-t\mathcal{B}_2}a_p^\dagger a_{-p}^\dagger a_{-q}a_q(2\mathcal{N}_0 + 1)e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s \\ &\qquad = \sum_p \hat{V}_N(p)\int_0^1\int_s^1 e^{-t\mathcal{B}_2}a_p^\dagger a_{-p}^\dagger e^{t\mathcal{B}_2}e^{-t\mathcal{B}_2}\Phi(2\mathcal{N}_0 + 1)e^{t\mathcal{B}_2}\mathrm{d}t\mathrm{d}s, \end{aligned} \tag{46}$$

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where we define $\Phi = \sum_q \tilde{\varphi}_q a_{-q} a_q$ so that, by Lemma 2,

$$\|\Phi\xi\| \lesssim \|\tilde{\varphi}\|_2 \|\mathcal{N}_+\xi\| \lesssim N^{-1-\alpha/2} \|\mathcal{N}_+\xi\|. \tag{47}$$

Next, we expand

$$e^{-t\mathcal{B}_2} a_p^\dagger a_{-p}^\dagger e^{t\mathcal{B}_2} = a_p^\dagger a_{-p}^\dagger + \int_0^t e^{-\tau\mathcal{B}_2} [a_p^\dagger a_{-p}^\dagger, \mathcal{B}_2] e^{\tau\mathcal{B}_2} d\tau. \tag{48}$$

Inserting this identity into equation (46), we obtain two contributions. The first contribution can be controlled by passing to position space. We find

$$\begin{aligned}
&\pm \sum_p \hat{V}_N(p) \int_0^1 \int_s^1 a_p^\dagger a_{-p}^\dagger e^{-t\mathcal{B}_2} \Phi(2\mathcal{N}_0 + 1) e^{t\mathcal{B}_2} \mathrm{d}t\mathrm{d}s \\
&\qquad = \pm \int_0^1 \int_s^1 \int_{\Lambda^2} \mathrm{d}x\mathrm{d}y \kappa V_N(x-y) \check{a}_x^\dagger \check{a}_y^\dagger e^{-t\mathcal{B}} \Phi(2\mathcal{N}_0 + 1) e^{t\mathcal{B}} \mathrm{d}t\mathrm{d}s + \text{h.c.} \\
&\qquad \lesssim \delta Q_4 + \delta^{-1} N^2 \|\varphi\|_{L^2}^2 \|V_N\|_1 (\mathcal{N}_+ + 1)^2 \lesssim N^{-\alpha/2} Q_4 + N^{-1-\alpha/2} (\mathcal{N}_+ + 1)^2.
\end{aligned}$$

On the other hand, the contribution arising from the second term on the right-hand side of equation (48) can be controlled by

$$\begin{aligned}
&\pm \sum_p \hat{V}_N(p) \tilde{\varphi}_p \int_0^1 \int_s^1 \int_0^t e^{-\tau\mathcal{B}_2} a_0^\dagger a_0^\dagger (2 a_p^\dagger a_p + 1) e^{\tau\mathcal{B}_2} e^{-t\mathcal{B}} \Phi(2\mathcal{N}_0 + 1) e^{t\mathcal{B}} \mathrm{d}t\mathrm{d}s + \text{h.c.} \\
&\qquad\qquad \lesssim N^{-1-\alpha/2} (\mathcal{N}_+ + 1)^2.
\end{aligned} \tag{49}$$

This concludes the proof of (39) and (40). □

Finally, we control the contribution with the commutator $[\tilde{Q}_2, \mathcal{B}_2]$ in equation (28).

Lemma 9. *We have*

$$\int_0^1 \int_0^s e^{-t\mathcal{B}_2} [\tilde{Q}_2, \mathcal{B}_2] e^{t\mathcal{B}_2} \mathrm{d}t\mathrm{d}s = -\frac{N(N-1)}{2} \sum_{|p|>N^\alpha, |q|\le N^\alpha} \hat{V}_N(p-q) \varphi_p \varphi_q + \mathcal{E}_{[\tilde{Q}_2, \mathcal{B}_2]}, \tag{50}$$

with

$$\pm \mathcal{E}_{[\tilde{Q}_2, \mathcal{B}_2]} \lesssim N^{-\alpha/2} Q_4 + N^{-1+\alpha} (\mathcal{N}_+ + 1)^2.$$

Proof. Using that $\chi_{|p|\le N^\alpha} \tilde{\varphi}_p = 0$, similar computations as in the proof of Lemma 41 yield

$$\begin{aligned}
&[\tilde{Q}_2, \mathcal{B}_2] \\
&= \frac{1}{4} \sum_{|p|>N^\alpha, |q|\le N^\alpha, r} \hat{V}_N(p-q) \varphi_q \tilde{\varphi}_r \\
&\quad \times (a_p^\dagger a_{-p}^\dagger a_r a_{-r} [a_0 a_0, a_0^\dagger a_0^\dagger] + [a_p^\dagger a_{-p}^\dagger, a_r a_{-r}] a_0^\dagger a_0^\dagger a_0 a_0) + \text{h.c.} \\
&= \sum_{|p|>N^\alpha, |q|\le N^\alpha} \frac{1}{4} \hat{V}_N(p-q) \varphi_q \\
&\quad \times \big((2 a_p^\dagger a_{-p}^\dagger \Phi(2\mathcal{N}_0 + 1) + \text{h.c.}) - 4\tilde{\varphi}_p (a_p^\dagger a_p + a_{-p}^\dagger a_{-p}) \mathcal{N}_0(\mathcal{N}_0 - 1) - 4\tilde{\varphi}_p \mathcal{N}_0(\mathcal{N}_0 - 1) \big),
\end{aligned} \tag{51}$$

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with the notation $\Phi = \sum_r \tilde{\varphi}_r a_r a_{-r}$. To bound the contribution arising from the first term in parentheses, we decompose

$$\begin{aligned} &\frac{1}{2}\sum_{|p|>N^\alpha,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q a_p^\dagger a_{-p}^\dagger \Phi(2\mathcal{N}_0+1) \\ &= \frac{1}{2}\sum_{p,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q a_p^\dagger a_{-p}^\dagger \Phi(2\mathcal{N}_0+1) - \frac{1}{2}\sum_{|p|,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q a_p^\dagger a_{-p}^\dagger \Phi(2\mathcal{N}_0+1). \end{aligned} \tag{52}$$

The first term can be controlled by switching to position space. With the notation $\check{\varphi}^<$ for the Fourier series of $\chi_{|q|\le N^\alpha}\varphi_q$, we find

$$\begin{aligned} &\pm\frac{1}{2}\sum_{p,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q a_p^\dagger a_{-p}^\dagger \Phi(2\mathcal{N}_0+1) \\ &\quad= \pm\int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y)\check{\varphi}^<(x-y)a_x^\dagger a_y^\dagger \Phi(2\mathcal{N}_0+1) + \text{h.c.} \\ &\quad\lesssim \delta Q_4 + \delta^{-1}N^{-\alpha}\|V_N^{1/2}\check{\varphi}^<\|_2^2(\mathcal{N}_+ +1)^2, \end{aligned}$$

where we used the bound (47) for Φ and $\mathcal{N}_0 \le N$. With

$$\|V_N^{1/2}\check{\varphi}^<\|_2^2 = \sum_{|p|,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_p\varphi_q \lesssim \frac{1}{N^3}\Big[\sum_{|p|\le N^\alpha}\frac{1}{|p|^2}\Big]^2 \lesssim N^{2\alpha-3} \tag{53}$$

and choosing $\delta = N^{-\alpha/2}$, we conclude (since $\alpha < 1$) that

$$\pm\frac{1}{2}\sum_{p,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q a_p^\dagger a_{-p}^\dagger \Phi(2\mathcal{N}_0+1) \lesssim N^{-\alpha/2}Q_4 + N^{-1-\alpha/2}(\mathcal{N}_+ +1)^2.$$

For the second term on the right-hand side of equation (52), we estimate

$$\pm\frac{1}{2}\sum_{|p|,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q a_p^\dagger a_{-p}^\dagger \Phi(2\mathcal{N}_0+1) + \text{h.c.} \lesssim N^{-\alpha/2}\|\chi_{|p|\le N^\alpha}\hat{V}_N * \varphi^<\|_2\|(\mathcal{N}_+ +1)^2,$$

where, again, we used the bound (47) and $\mathcal{N}_0 \le N$. With

$$\|\chi_{|p|\le N^\alpha}\hat{V}_N * \varphi^<\|_2 \le \|\chi_{|p|\le N^\alpha}\|_{L^2}\|V_N\check{\varphi}^<\|_1 \lesssim N^{3\alpha/2}\|V_N^{1/2}\|_2\|V_N^{1/2}\check{\varphi}^<\|_2 \lesssim N^{-2+5\alpha/2}$$

we conclude that

$$\pm\frac{1}{2}\sum_{|p|,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q a_p^\dagger a_{-p}^\dagger \Phi(2\mathcal{N}_0+1) + \text{h.c.} \lesssim N^{-2-2\alpha}(\mathcal{N}_+ +1)^2.$$

The contribution arising from the second term in parentheses on the right-hand side of equation (51) can be bounded by

$$\pm\sum_{|p|>N^\alpha,|q|\le N^\alpha} \hat{V}_N(p-q)\varphi_q\tilde{\varphi}_p a_p^\dagger a_p \mathcal{N}_0(\mathcal{N}_0-1) \lesssim N^{-1-\alpha}\mathcal{N}_+,$$

using that $\|\tilde{\varphi}(\hat{V}_N * \varphi^<)\|_{L^\infty} \lesssim \|\tilde{\varphi}\|_\infty\|V_N\check{\varphi}^<\|_1 \lesssim N^{-3-\alpha}$. For the contribution arising from the last term on the right-hand side of equation (51), we write $\mathcal{N}_0(\mathcal{N}_0-1) = N(N-1) - 2N\mathcal{N}_+ + \mathcal{N}_+(\mathcal{N}_+ +1)$.

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The contribution proportional to $N(N-1)$ produces the main term on the right-hand side of equation (50). The other contributions can be bounded, noticing that

$$\Big|\sum_{|p|>N^\alpha,|q|\le N^\alpha}\hat V_N(p-q)\varphi_p\varphi_q\Big|\le\|\check{\tilde\varphi}V_N\check\varphi^<\|_1\le\|\check{\tilde\varphi}\|_\infty\|V_N\check\varphi^<\|_1\lesssim N^{-2+\alpha},$$

where we used $\|\check{\tilde\varphi}\|_\infty\lesssim\|\tilde\varphi\|_1\lesssim 1$, by Lemma 2. □

We can now finish the proof of Proposition 4.

Proof of Proposition 4. Combining equation (28) with the bounds proven in Lemma 5, Lemma 6, Lemma 7, Lemma 8 and Lemma 9, we conclude that

$$e^{-\mathcal{B}_2}H_Ne^{\mathcal{B}_2}=4\pi\mathfrak{a}_N(N-1)+A_\alpha+H_1+\tilde H_2+\tilde Q_2'+Q_3+Q_4+\mathcal{E},\tag{54}$$

where

$$\pm\mathcal{E}\lesssim N^{-\alpha/2}Q_4+\left[N^{-\alpha/2}+N^{\alpha-1}\right](\mathcal{N}_++1)+N^{-1+\alpha}\mathcal{N}_+^2+N^{-2}H_1\tag{55}$$

and where we defined

$$\begin{aligned}A_\alpha&=-\frac{N(N-1)}{2}\Big[\sum_{|p|\le N^\alpha}\hat V_N(p)\varphi_p+\sum_{|p|>N^\alpha,|q|\le N^\alpha}\hat V_N(p-q)\varphi_p\varphi_q\Big]\\&=-\frac{N(N-1)}{2}\Big[\sum_{|p|\le N^\alpha}(\hat V_N(p)+\hat V_N*\varphi)\varphi_p-\sum_{|p|,|q|\le N^\alpha}\hat V_N(p-q)\varphi_p\varphi_q\Big].\end{aligned}$$

The second term in parentheses can be estimated as in (53). Setting, in position space, $f=1+\check\varphi$, we find

$$A_\alpha=-\frac{N(N-1)}{2}\sum_{|p|\le N^\alpha}(\hat V_N*\hat f)(p)\varphi_p+\mathcal{O}(N^{2\alpha-1}).\tag{56}$$

From equation (16), we have $\widehat{V_Nf}(0)=8\pi\mathfrak{a}_N$. Hence

$$|(\hat V_N*\hat f)(p)-8\pi\mathfrak{a}_N/N|\le\int_\Lambda V_N(x)f(x)|e^{-ip\cdot x}-1|\mathrm{d}x\le C|p|/N^2.\tag{57}$$

Moreover, from the scattering equation (14), we find

$$\varphi_p=-\frac{1}{2p^2}(\hat V_N*\hat f)(p),$$

which implies, by the bound (57),

$$\Big|\varphi_p+\frac{4\pi\mathfrak{a}_N}{Np^2}\Big|\le\frac{1}{2p^2}\big|(\hat V_N*\hat f)(p)-8\pi\mathfrak{a}_N\big|\le\frac{C}{|p|N^2}.$$

Inserting in equation (56), we obtain

$$A_\alpha=\sum_{|p|\le N^\alpha}\frac{(4\pi\mathfrak{a}_N)^2}{p^2}+\mathcal{O}(N^{2\alpha-1}).$$

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To conclude the proof of Proposition 4, we still have to compare the operator $\tilde{Q}_2'$ appearing on the right-hand side of equation (54) with the operator $\tilde{Q}_2$ defined in equation (25). From equation (27), we can write, using again the notation $f = 1 + \check{\varphi}$,

$$\tilde{Q}_2' - \tilde{Q}_2 = \frac{1}{2} \sum_{|p| \le N^\alpha} \left[(\hat{V}_N * \hat{f})(p) - \frac{8\pi \mathfrak{a}_N}{N} \right] a_p^\dagger a_{-p}^\dagger a_0 a_0 - \frac{1}{2} \sum_{p, |q| \le N^\alpha} \hat{V}_N(p-q) \varphi_q a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.} \tag{58}$$

The first term on the right-hand side of equation (58) can be bounded using the bound (57) by

$$\begin{aligned} \pm \sum_{|p| \le N^\alpha} &\left[(\hat{V}_N * \hat{f})(p) - \frac{8\pi \mathfrak{a}_N}{N} \right] (a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.}) \\ &\lesssim \frac{1}{N} \| |p| \chi_{|p| \le N^\alpha} \|_2 (\mathcal{N}_+ + 1) \lesssim N^{-1+5\alpha/2} (\mathcal{N}_+ + 1). \end{aligned}$$

For the second term on the right-hand side of equation (58), we set $\varphi_p^< = \varphi_p \chi_{|p| \le N^\alpha}$ and estimate, switching to position space,

$$\begin{aligned} \pm \sum_p &(\hat{V}_N * \varphi^<)(p)\, a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.} \\ &= \pm \int_{\Lambda^2} \mathrm{d}x \mathrm{d}y\, V_N(x-y) \check{\varphi}^<(x-y) a_x^\dagger a_y^\dagger a_0 a_0 + \text{h.c.} \\ &\lesssim \delta Q_4 + \delta^{-1} N^2 \|V_N\|_1 \|\check{\varphi}^<\|_\infty^2 \lesssim \delta Q_4 + \delta^{-1} N^{-1+2\alpha} \le N^{-\alpha/2} Q_4 + N^{-1+5\alpha/2}, \end{aligned}$$

since $\|\check{\varphi}^<\|_\infty \lesssim \|\varphi^<\|_1 \lesssim N^\alpha$, from Lemma 2 (in the last step, we chose $\delta = N^{-\alpha/2}$). The last two estimates show that the difference $\tilde{Q}_2' - \tilde{Q}_2$ can be added to the error (55) and therefore conclude the proof of Proposition 4. □

4. Cubic renormalisation

While conjugation with $e^{\mathcal{B}_2}$ allowed us to renormalise the quadratic part of the Hamiltonian H_N, regularising the off-diagonal term Q_2, it did not significantly change the cubic operator Q_3. To renormalise Q_3, we proceed with a second conjugation, with a unitary operator $e^{\mathcal{B}_3}$, where

$$\mathcal{B}_3 = \sum_{p,q} \tilde{\varphi}_p \chi_{|q| \le N^\alpha}\, a_{p+q}^\dagger a_{-p}^\dagger a_q a_0 - \text{h.c.}, \tag{59}$$

with the same $0 < \alpha < 1$ used in the definition given in equation (17) of $\tilde{\varphi}$. Similarly to what we did in Lemma 3 for the action of $\mathcal{B}_2$, it is important to notice that conjugation with $e^{\mathcal{B}_3}$ does not substantially change the number of excitations.

Lemma 10. *For all $s \in [-1; 1]$ and all $k \in \mathbb{N}$, we have*

$$\pm \left[e^{-s\mathcal{B}_3} \mathcal{N}_+ e^{s\mathcal{B}_3} - \mathcal{N}_+ \right] \lesssim N^{-\alpha/2} (\mathcal{N}_+ + 1), \tag{60}$$

$$e^{-s\mathcal{B}_3} (\mathcal{N}_+ + 1)^k e^{s\mathcal{B}_3} \lesssim (\mathcal{N}_+ + 1)^k. \tag{61}$$

Proof. We proceed similarly as in the proof of [7, Prop. 5.1]. For $\xi \in \mathcal{F}$, we set $f(s) = \langle \xi, e^{-s\mathcal{B}_3} (\mathcal{N}_+ + 1) e^{s\mathcal{B}_3} \xi \rangle$. For $s \in (0; 1)$, we find

$$\begin{aligned} f'(s) &= \langle \xi, e^{-s\mathcal{B}_3}[(\mathcal{N}_+ + 1), \mathcal{B}_3] e^{s\mathcal{B}_3}\xi\rangle \\ &= \sum_{p,q} \tilde{\varphi}_p \chi_{|q|\le N^\alpha} \langle \xi, e^{-s\mathcal{B}_3} a^\dagger_{p+q} a^\dagger_{-p} a_q a_0 e^{s\mathcal{B}_3}\xi\rangle + \text{h.c.} \\ &\lesssim \delta \sum_{p,q} \langle \xi, e^{-s\mathcal{B}_3} a^\dagger_{p+q} a^\dagger_{-p} a_{-p} a_{p+q} e^{s\mathcal{B}_3}\xi\rangle + \delta^{-1} \sum_{p,q} |\tilde{\varphi}_p|^2 \langle \xi, e^{-s\mathcal{B}_3} a^\dagger_q a_q a^\dagger_0 a_0 e^{s\mathcal{B}_3}\xi\rangle \\ &\lesssim \delta \langle e^{-s\mathcal{B}_3} \mathcal{N}_+^2 e^{s\mathcal{B}_3}\xi\rangle + C\delta^{-1} N \|\tilde{\varphi}\|_2^2 \langle \xi, e^{-s\mathcal{B}_3} \mathcal{N}_+ e^{s\mathcal{B}_3}\xi\rangle \lesssim N^{-\alpha/2} f(s), \end{aligned} \tag{62}$$

where we put $\delta = N^{-1-\alpha/2}$ and used that $\mathcal{N}_+, \mathcal{N}_0 \le N$, $\|\varphi\|_2 \lesssim N^{-1-\alpha/2}$, by Lemma 2. With Gronwall's lemma [26, Theorem 1.2.2], we obtain $f(s) \lesssim \langle \xi, (\mathcal{N}_+ + 1)\xi\rangle$ for all $s \in [-1; 1]$, proving the bound (61). Inserting this estimate on the right-hand side of (62) and integrating over s, we obtain the desired bound (60). For $k > 1$, the bound given in (61) can be shown similarly. □

The operator $\mathcal{B}_3$ is chosen (similarly as we did with $\mathcal{B}_2$ in Section 3) so that the commutator $[H_1 + Q_4, \mathcal{B}_3]$ arising from conjugation with $e^{\mathcal{B}_3}$ cancels the main part of Q_3. The goal of this section is to use this cancellation to prove the following proposition.

Proposition 11. *We have*

$$\begin{aligned} &e^{-\mathcal{B}_3} e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} e^{\mathcal{B}_3} \\ &\quad = 4\pi\mathfrak{a}_N(N-1) + \frac{1}{4}\sum_{|p|\le N^\alpha} \frac{(8\pi\mathfrak{a}_N)^2}{p^2} \\ &\qquad + \sum_p (p^2 + 8\pi\mathfrak{a}_N \chi_{|p|\le N^\alpha}) a^\dagger_p a_p + \frac{1}{2}\sum_{|p|\le N^\alpha} 8\pi\mathfrak{a}_N \Big[a^\dagger_p a^\dagger_{-p} \frac{a_0 a_0}{N} + h.c.\Big] + Q_4 + \mathcal{E}_{\mathcal{B}_3}, \end{aligned} \tag{63}$$

with

$$\begin{aligned} \pm\mathcal{E}_{\mathcal{B}_3} \lesssim\; & N^{-3\alpha/2} H_1 + N^{-\alpha/2} Q_4 + N^{-\alpha/2}(\mathcal{N}_+ + 1) \\ & + N^{(3\alpha-1)/2}(\mathcal{N}_+ + 1)^{3/2} + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2. \end{aligned} \tag{64}$$

To prove Proposition 11, we define

$$\Gamma_3 := \big[H_1 + Q_4, \mathcal{B}_3\big] + Q_3. \tag{65}$$

Starting from equation (23), we compute

$$\begin{aligned} e^{-\mathcal{B}_3} e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} e^{\mathcal{B}_3} - 4\pi\mathfrak{a}_N(N-1) - \sum_{|p|\le N^\alpha} \frac{(4\pi\mathfrak{a}_N)^2}{p^2} - e^{-\mathcal{B}_3}(\tilde{H}_2 + \tilde{Q}_2 + \mathcal{E}_{\mathcal{B}_2}) e^{\mathcal{B}_3} \\ = H_1 + Q_4 + \int_0^1 e^{-t\mathcal{B}_3}[H_1 + Q_4, \mathcal{B}_3] e^{t\mathcal{B}_2} \mathrm{d}t + e^{-\mathcal{B}_3} Q_3 e^{\mathcal{B}_3} \\ = H_1 + Q_4 + \int_0^1 e^{-t\mathcal{B}_3}(-Q_3 + \Gamma_3) e^{t\mathcal{B}_3} \mathrm{d}t + e^{-\mathcal{B}_3} Q_3 e^{\mathcal{B}_3}, \end{aligned}$$

which leads to

$$\begin{aligned} e^{-\mathcal{B}_3} e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} e^{\mathcal{B}_3} = 4\pi\mathfrak{a}_N(N-1) + \sum_{|p|\le N^\alpha} \frac{(4\pi\mathfrak{a}_N)^2}{p^2} + H_1 + Q_4 + e^{-\mathcal{B}_3}(\tilde{H}_2 + \tilde{Q}_2 + \mathcal{E}_{\mathcal{B}_2}) e^{\mathcal{B}_3} \\ + \int_0^1 e^{-t\mathcal{B}_3} \Gamma_3 e^{t\mathcal{B}_3} \mathrm{d}t + \int_0^1 \int_s^1 e^{-t\mathcal{B}_3}[Q_3, \mathcal{B}_3] e^{t\mathcal{B}_3} \mathrm{d}t \mathrm{d}s. \end{aligned} \tag{66}$$

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To show Proposition 11, we are going to control all terms on the right-hand side of equation (66). We start by computing and estimating the commutator in equation (65), defining the error term Γ_3.

Lemma 12. *We have*

$$[H_1,\mathcal{B}_3]=\sum_{p,q,r}\hat{V}_N(p-r)(\delta_{0,r}+\varphi_r)\chi_{|p|>N^\alpha}\chi_{|q|\le N^\alpha}a^\dagger_{p+q}a^\dagger_{-p}a_qa_0+h.c.+\mathcal{E}_{[H_1,\mathcal{B}_3]},\tag{67}$$

$$[Q_4,\mathcal{B}_3]=\sum_{r,p,q}\hat{V}_N(p-r)\tilde{\varphi}_r\chi_{|q|\le N^\alpha}(a^\dagger_{p+q}a^\dagger_{-p}a_qa_0+h.c.)+\mathcal{E}_{[Q_4,\mathcal{B}_3]},\tag{68}$$

where

$$\begin{aligned}\pm\mathcal{E}_{[H_1,\mathcal{B}_3]}&\lesssim N^{-3\alpha/2}H_1+N^{-1+5\alpha/2}(\mathcal{N}_++1)^2,\\ \pm\mathcal{E}_{[Q_4,\mathcal{B}_3]}&\lesssim N^{-\alpha/2}Q_4+N^{-1+5\alpha/2}(\mathcal{N}_++1)^2.\end{aligned}\tag{69}$$

Proof. A simple computation shows that

$$\begin{aligned}[H_1,\mathcal{B}_3]&=\sum_{p,q}\big[(p+q)^2+p^2-q^2\big]\tilde{\varphi}_p\chi_{|q|\le N^\alpha}a^\dagger_{p+q}a^\dagger_{-p}a_qa_0+\text{h.c.},\\ &=2\sum_{p,q}p^2\tilde{\varphi}_p\chi_{|q|\le N^\alpha}a^\dagger_{p+q}a^\dagger_{-p}a_qa_0+\text{h.c.}+\mathcal{E}_{[H_1,\mathcal{B}_3]},\end{aligned}$$

with

$$\mathcal{E}_{[H_1,\mathcal{B}_3]}=2\sum_{p,q}p\cdot q\tilde{\varphi}_p\chi_{|q|\le N^\alpha}a^\dagger_{p+q}a^\dagger_{-p}a_qa_0+\text{h.c.}$$

Using the scattering equation (14) yields (67). We now estimate $\mathcal{E}_{[H_1,\mathcal{B}_3]}$. Using $|q|\le N^\alpha$, we find, for any $\delta>0$,

$$\begin{aligned}\pm\mathcal{E}_{[H_1,\mathcal{B}_3]}&=\pm2\sum_{p,q}p\cdot q\tilde{\varphi}_p\chi_{|q|\le N^\alpha}a^\dagger_{p+q}a^\dagger_{-p}(\mathcal{N}_++1)^{-1/2}(\mathcal{N}_++1)^{1/2}a_qa_0+\text{h.c.}\\ &\lesssim\delta\sum_{p,q}p^2a^\dagger_{p+q}a^\dagger_{-p}(\mathcal{N}_++1)^{-1}a_{-p}a_{p+q}+\delta^{-1}\sum_{p,|q|\le N^\alpha}q^2|\tilde{\varphi}_p|^2\,a^\dagger_q(\mathcal{N}_++1)a_q(a^\dagger_0a_0)\\ &\lesssim\delta H_1+\delta^{-1}N^{1+2\alpha}\|\tilde{\varphi}\|_2^2(\mathcal{N}_++1)^2.\end{aligned}$$

Choosing $\delta=N^{-3\alpha/2}$, we conclude that

$$\pm\mathcal{E}_{[H_1,\mathcal{B}_3]}\lesssim N^{-3\alpha/2}H_1+N^{-1+5\alpha/2}(\mathcal{N}_++1)^2.$$

Let us now turn to (68). Recalling (10) and (59), we find

$$\begin{aligned}[Q_4,\mathcal{B}_3]&=\frac{1}{2}\sum_{r,p,q}\sum_{m,n}\hat{V}_N(r)\tilde{\varphi}_m\chi_{|n|\le N^\alpha}\big[a^\dagger_{p+r}a^\dagger_qa_pa_{q+r},a^\dagger_{m+n}a^\dagger_{-m}a_na_0\big]+\text{h.c.}\\ &=\frac{1}{2}\sum_{r,p,q}\sum_{m,n}\hat{V}_N(r)\tilde{\varphi}_m\chi_{|n|\le N^\alpha}\\ &\qquad\times\Big\{a^\dagger_{p+r}a^\dagger_q\big[a_pa_{q+r},a^\dagger_{m+n}a^\dagger_{-m}\big]a_na_0+a^\dagger_{m+n}a^\dagger_{-m}\big[a^\dagger_{p+r}a^\dagger_q,a_na_0\big]a_pa_{q+r}\Big\}.\end{aligned}$$

Using equation (6) and rearranging all terms in normal order, we arrive at

$$[Q_4,\mathcal{B}_3]=\sum_{r,p,q}\hat{V}_N(p-r)\tilde{\varphi}_r\chi_{|q|\le N^\alpha}(a^\dagger_{p+q}a^\dagger_{-p}a_qa_0+\text{h.c.})+\mathcal{E}_{[Q_4,\mathcal{B}_3]},$$

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where

$$\begin{aligned}2\mathcal{E}_{[Q_4,\mathcal{B}_3]} = &-\sum_{p,q,m,r} \hat{V}_N(r)\tilde{\varphi}_m\chi_{|p+r|\le N^\alpha}\, a^\dagger_{m+p+r}a^\dagger_{-m}a^\dagger_q a_{q+r}a_p a_0\\ &-\sum_{p,q,m,r} \hat{V}_N(r)\tilde{\varphi}_m\chi_{|q|\le N^\alpha}\, a^\dagger_{m+q}a^\dagger_{-m}a^\dagger_{p+r} a_{q+r}a_p a_0\\ &+\sum_{p,q,m,r} \hat{V}_N(r)\tilde{\varphi}_m\chi_{|q+r-m|\le N^\alpha}\, a^\dagger_{p+r}a^\dagger_{-m}a^\dagger_q a_{q+r-m}a_p a_0\\ &+\sum_{p,q,m,r} \hat{V}_N(r)\tilde{\varphi}_m\chi_{|p-m|\le N^\alpha}\, a^\dagger_{p+r}a^\dagger_{-m}a^\dagger_q a_{q+r}a_{p-m} a_0\\ &+\sum_{p,q,m,r} \hat{V}_N(r)\tilde{\varphi}_{-q-r}\chi_{|m|\le N^\alpha}\, a^\dagger_{p+r}a^\dagger_q a^\dagger_{-q-r+m} a_p a_m a_0\\ &+\sum_{p,q,m,r} \hat{V}_N(r)\tilde{\varphi}_{-p}\chi_{|p-m|\le N^\alpha}\, a^\dagger_{p+r}a^\dagger_q a^\dagger_{-m} a_{q+r}a_{p-m} a_0 + \text{h.c.} =: \sum_{i=1}^{6}\mathcal{E}_i.\end{aligned}$$

For a parameter $\delta > 0$, we find

$$\begin{aligned}\pm\mathcal{E}_1 &\lesssim \delta\|\hat{V}_N\|_\infty^2\|\chi_{|\cdot|\le N^\alpha}\|_1\mathcal{N}_+^3 + \delta^{-1}\|\tilde{\varphi}\|_2^2\|\chi_{|\cdot|\le N^\alpha}\|_1\mathcal{N}_+^2\mathcal{N}_0\,,\\ &\lesssim N^{-1+5\alpha/2}\mathcal{N}_+^2,\end{aligned}$$

where, in the last step, we chose $\delta = N^{-\alpha/2}$ and used $\mathcal{N}_0 \le N$ (and Lemma 2). To estimate $\mathcal{E}_2,\dots,\mathcal{E}_6$, we switch to position space. For arbitrary $\delta > 0$, we find

$$\begin{aligned}\pm\mathcal{E}_2 &= \pm\int_{\Lambda^2}\mathrm{d}x\mathrm{d}y V_N(x-y)\check{\chi}_{|\cdot|\le N^\alpha}(z-y)a^\dagger_z a^\dagger(\check{\tilde{\varphi}}^z)a^\dagger_x a_x a_y a_0 + \text{h.c.}\\ &\le \delta\|\chi_{|\cdot|\le N^\alpha}\|_2^2\, Q_4 a^\dagger_0 a_0 + \delta^{-1}\int_{\Lambda^2}\mathrm{d}x\mathrm{d}y V_N(x-y)a^\dagger_z a^\dagger(\check{\tilde{\varphi}}^z)a^\dagger_x a_x a(\check{\tilde{\varphi}}^z)a_z\\ &\lesssim \delta N^{1+3\alpha}Q_4 + \delta^{-1}N^{-3-\alpha}\mathcal{N}_+^3\\ &\lesssim N^{-\alpha/2}Q_4 + N^{-2+5\alpha/2}\mathcal{N}_+^3,\end{aligned}$$

where, in the last line, we fixed $\delta = N^{-1-7\alpha/2}$. Similarly, we find

$$\begin{aligned}\pm\mathcal{E}_3 &= \pm\int_{\Lambda^2}\mathrm{d}x\mathrm{d}y V_N(x-y)a^\dagger_x a^\dagger_y a^\dagger(\check{\tilde{\varphi}}^z)a(\check{\chi}^y_{|\cdot|\le N^\alpha})a_x a_0 + \text{h.c.}\\ &\lesssim \delta Q_4 + \delta^{-1}N\|V_N\|_1\|\check{\chi}_{|\cdot|\le N^\alpha}\|_2^2\|\tilde{\varphi}\|_2^2(\mathcal{N}_+ + 1)^3\\ &\lesssim N^{-\alpha/2}Q_4 + N^{-2+5\alpha/2}(\mathcal{N}_+ + 1)^3,\end{aligned}$$

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taking $\delta = N^{-\alpha/2}$. Furthermore, for an arbitrary $\xi \in \mathcal{F}$, we have

$$\begin{aligned}
|\langle \xi, \mathcal{E}_4 \xi\rangle| &= \Big| \int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y)\check{\chi}_{|\cdot|\leq N^\alpha}(x-z)\langle \xi, a_x^\dagger a_z^\dagger a^\dagger(\check{\tilde{\varphi}}^x) a_y a_z a_0 \xi\rangle \Big| \\
&\leq \|\check{\chi}_{|.|\leq N^\alpha}\|_\infty \int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y) \|a_x a_z a(\check{\tilde{\varphi}}^x)\xi\| \|a_y a_z a_0 \xi\| \\
&\lesssim \|\chi_{|.|\leq N^\alpha}\|_1 \|\tilde{\varphi}\|_2 \|V_N\|_1 \left[\langle \xi, \mathcal{N}_+^3 \xi\rangle + \langle \xi, \mathcal{N}_+^2 \mathcal{N}_0 \xi\rangle\right] \\
&\lesssim N^{-1+5\alpha/2}\langle \xi, \mathcal{N}_+^2 \xi\rangle .
\end{aligned}$$

For $\mathcal{E}_5$, we estimate

$$\begin{aligned}
\pm \mathcal{E}_5 &= \pm \int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y)\check{\tilde{\varphi}}(y-z) a_x^\dagger a_y^\dagger a_z^\dagger (\mathcal{N}_+ + 1)^{-1/2}(\mathcal{N}_+ + 1)^{1/2} a_x a(\check{\chi}^z_{|\cdot|\leq N^\alpha}) a_0 + \text{h.c.} \\
&\leq \delta Q_4 + \delta^{-1}\int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y)|\check{\tilde{\varphi}}(z-y)|^2 a_x^\dagger a^\dagger(\check{\chi}^z_{|.|\leq N^\alpha}) a_0^\dagger (\mathcal{N}_+ + 1) a_0 a(\check{\chi}^z_{|.|\leq N^\alpha}) a_x \\
&\lesssim \delta Q_4 + \delta^{-1} N \|V_N\|_1 \|\check{\chi}_{|\cdot|\leq N^\alpha}\|_2^2 \|\tilde{\varphi}\|_2^2 (\mathcal{N}_+ + 1)^3 \\
&\lesssim N^{-\alpha/2} Q_4 + N^{-2+5\alpha/2}\mathcal{N}_+^3 ,
\end{aligned}$$

again choosing $\delta = N^{-\alpha/2}$. By a simple change of variable, it is easy to check that $\mathcal{E}_6 = \mathcal{E}_5$. This concludes the proof of the lemma. □

With the bounds from the last lemma, we can estimate the operator Γ_3 defined in equation (65).

Lemma 13. *We have*

$$\pm \Gamma_3 \lesssim N^{-3\alpha/2} H_1 + N^{-\alpha/2} Q_4 + N^{3\alpha/2-1/2}(\mathcal{N}_+ + 1)^{3/2} + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2 . \tag{70}$$

Proof. With Lemma 12, we find, using the scattering equation (14),

$$\Gamma_3 = \left[H_1 + Q_4, \mathcal{B}_3\right] + Q_3 = \tilde{Q}_{3,1} + \tilde{Q}_{3,2} + Q_3^> + \mathcal{E}_{[H_1,\mathcal{B}_3]} + \mathcal{E}_{[Q_4,\mathcal{B}_3]}$$

with

$$\tilde{Q}_{3,1} = -\sum_{p,q} p^2 \varphi_p \chi_{|p|\leq N^\alpha} \chi_{|q|\leq N^\alpha}\, a^\dagger_{p+q} a^\dagger_{-p} a_q a_0 + \text{h.c.}, \tag{71}$$

$$\tilde{Q}_{3,2} = -\sum_{p,q,r} \hat{V}_N(p-r)\varphi_r \chi_{|r|\leq N^\alpha}\chi_{|q|\leq N^\alpha}\, a^\dagger_{p+q} a^\dagger_{-p} a_q a_0 + \text{h.c.}, \tag{72}$$

$$Q_3^> = \sum_{p,q} \hat{V}_N(p)\chi_{|q|>N^\alpha} a^\dagger_{p+q} a^\dagger_{-p} a_q a_0 + \text{h.c.} \tag{73}$$

It follows easily from Lemma 2 that $\|\chi_{|p|\leq N^\alpha} p^2 \varphi_p\|_2 \lesssim N^{-1+3\alpha/2}$; thus

$$\pm \tilde{Q}_{3,1} \lesssim N^{-1/2+3\alpha/2}(\mathcal{N}_+ + 1)^{3/2}.$$

Denoting $\varphi_p^< = \varphi_p \chi_{|p|\leq N^\alpha}$, we write $\tilde{Q}_{3,2}$ in position space as

$$\begin{aligned}
\pm \tilde{Q}_{3,2} &= \pm \int_{\Lambda^3} \mathrm{d}x\mathrm{d}y\mathrm{d}z\, V_N(x-y)\check{\varphi}^<(x-y)\check{\chi}_{|\cdot|\leq N^\alpha}(x-z) a_x^\dagger a_y^\dagger a_z a_0 + \text{h.c.} \\
&\leq \delta Q_4 + \delta^{-1} N \|V_N^{1/2}\check{\varphi}^<\|_2^2 \|\chi_{|\cdot|\leq N^\alpha}\|_\infty^2 (\mathcal{N}_+ + 1) \\
&\lesssim N^{-\alpha/2} Q_4 + N^{-2+5\alpha/2}(\mathcal{N}_+ + 1)\, ,
\end{aligned}$$

where we chose $\delta = N^{-\alpha/2}$ and used the estimate

$$\int_\Lambda a^\dagger(\check{\chi}^x_{|.|\leq N^\alpha})\, a(\check{\chi}^x_{|.|\leq N^\alpha})\,\mathrm{d}x = d\Gamma(\chi^2_{|p|\leq N^\alpha}) \leq \|\chi_{|.|\leq N^\alpha}\|^2_\infty \mathcal{N}_+ \leq \mathcal{N}_+. \tag{74}$$

Proceeding similarly, we find

$$\begin{aligned}\pm Q_3^> &= \pm \int_{\Lambda^3} \mathrm{d}x\mathrm{d}y\mathrm{d}z\, V_N(x-y)\check{\chi}_{|\cdot|>N^\alpha}(x-z)a^\dagger_x a^\dagger_y a_z a_0 + \text{h.c.}\\ &\lesssim \delta Q_4 + \delta^{-1}N^{1-2\alpha}\|V_N\|_1 H_1 \lesssim N^{-\alpha/2}Q_4 + N^{-3\alpha/2}H_1,\end{aligned}$$

where we took $\delta = N^{-\alpha/2}$ and used that

$$\int_\Lambda a^\dagger(\check{\chi}^x_{|\cdot|>N^\alpha})a(\check{\chi}^x_{|\cdot|>N^\alpha})\mathrm{d}x = \sum_{|p|>N^\alpha} a^\dagger_p a_p \leq N^{-2\alpha}H_1.$$

Combining the bounds for $\tilde{Q}_{3,1}, \tilde{Q}_{3,2}, Q_3^>$ with the estimates for $\mathcal{E}_{[H_1,\mathcal{B}_3]}, \mathcal{E}_{[Q_4,\mathcal{B}_3]}$ from Lemma 12, we obtain the bound (70). □

To obtain similar bounds for the integral, we also need a priori control over the growth of H_1, Q_4.

Lemma 14. *We have*

$$e^{-s\mathcal{B}_3}Q_4 e^{s\mathcal{B}_3} \lesssim Q_4 + \mathcal{N}_+ + 1 + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2, \tag{75}$$

$$e^{-s\mathcal{B}_3}H_1 e^{s\mathcal{B}_3} \lesssim H_1 + Q_4 + \mathcal{N}_+ + 1 + N^{-1+3\alpha}(\mathcal{N}_+ + 1)^2. \tag{76}$$

Proof. For arbitrary $\xi \in \mathcal{F}$, we define $f(s) = \langle \xi, e^{-s\mathcal{B}_3}Q_4 e^{s\mathcal{B}_3}\xi\rangle$ so that

$$f'(s) = \langle \xi, e^{-s\mathcal{B}_3}[Q_4,\mathcal{B}_3]e^{s\mathcal{B}_3}\xi\rangle.$$

From Lemma 12, we find

$$[Q_4,\mathcal{B}_3] = \sum_{r,p,q}\hat{V}_N(p-r)\tilde{\varphi}_r\chi_{|q|\leq N^\alpha}(a^\dagger_{p+q}a^\dagger_{-p}a_q a_0 + \text{h.c.}) + \mathcal{E}_{[Q_4,\mathcal{B}_3]},$$

where

$$\pm\mathcal{E}_{[Q_4,\mathcal{B}_3]} \lesssim N^{-\alpha/2}Q_4 + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2.$$

Switching to position space, we have

$$\begin{aligned}&\sum_{r,p,q}\hat{V}_N(p-r)\tilde{\varphi}_r\chi_{|q|\leq N^\alpha}(a^\dagger_{p+q}a^\dagger_{-p}a_q a_0 + \text{h.c.})\\ &\qquad = \sum_{r,p,q}\hat{V}_N(p-r)\tilde{\varphi}_r\chi_{|q|\leq N^\alpha}(a^\dagger_{p+q}a^\dagger_{-p}a_q a_0 + \text{h.c.})\\ &\qquad = \int_{\Lambda^2}\mathrm{d}x\mathrm{d}y\, V_N(x-y)\check{\tilde{\varphi}}(x-y)a^\dagger_x a^\dagger_y a(\check{\chi}^x_{|\cdot|\leq N^\alpha})a_0 + \text{h.c.}\\ &\qquad \lesssim Q_4 + \int \mathrm{d}x\mathrm{d}y\, V_N(x-y)|\check{\tilde{\varphi}}(x-y)|^2 a^\dagger_0 a^\dagger(\check{\chi}^x_{|.|\leq N^\alpha})a(\check{\chi}^x_{|.|\leq N^\alpha})a_0\\ &\qquad \lesssim Q_4 + N\|V_N\|_1\|\tilde{\varphi}\|^2_\infty\|\chi^2_{|.|\leq N^\alpha}\|_\infty \lesssim Q_4 + \mathcal{N}_+,\end{aligned}$$

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where we used Lemma 2 and argued as in (74). We conclude that

$$\pm\big[Q_4, \mathcal{B}_3\big] \lesssim Q_4 + \mathcal{N}_+ + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2.$$

Therefore, using Lemma 3, we find

$$f'(s) \lesssim f(s) + \langle \xi, \mathcal{N}_+ \xi\rangle + N^{-1+5\alpha/2}\langle \xi, (\mathcal{N}_+ + 1)^2 \xi\rangle.$$

By the Gronwall lemma, we obtain the bound (75).

To prove the estimate (76), we proceed similarly. For $\xi \in \mathcal{F}$, we define $g(s) = \langle \xi, e^{-s\mathcal{B}_3} H_1 e^{s\mathcal{B}_3}\xi\rangle$ for any $|s| \le 1$, which leads to

$$g'(s) = \langle \xi, e^{-\mathcal{B}_3} H_1 e^{s\mathcal{B}_3}\xi\rangle.$$

From Lemma 12, we have

$$\big[H_1, \mathcal{B}_3\big] = \sum_{p,q,r} \hat{V}_N(p-r)(\delta_{0,r} + \varphi_r)\chi_{|p|>N^\alpha}\chi_{|q|\le N^\alpha} a^\dagger_{p+q} a^\dagger_{-p} a_q a_0 + \text{h.c.} + \mathcal{E}_{[H_1,\mathcal{B}_3]},$$

where

$$\pm\mathcal{E}_{[H_1,\mathcal{B}_3]} \le N^{-3\alpha/2} H_1 + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2.$$

Writing $\chi_{|p|>N^\alpha} = 1 - \chi_{|p|\le N^\alpha}$, we decompose

$$\sum_{p,q,r} \hat{V}_N(p-r)(\delta_{0,r} + \varphi_r)\chi_{|p|>N^\alpha}\chi_{|q|\le N^\alpha} a^\dagger_{p+q} a^\dagger_{-p} a_q a_0 + \text{h.c.} = \mathcal{E}_1 + \mathcal{E}_2,$$

where

$$\begin{aligned}\mathcal{E}_1 &= \pm \int_{\Lambda^2} \mathrm{d}x\mathrm{d}y V_N(x-y)(1+\check{\varphi})(x-y) a^\dagger_x a^\dagger_y a(\check{\chi}^x_{|.|\le N^\alpha}) a_0 + \text{h.c.}\\ &\lesssim Q_4 + N\|V_N\|_1 \|(1+\check{\varphi})\|^2_\infty \|\chi_{|\cdot|\le N^\alpha}\|^2_\infty \mathcal{N}_+ \lesssim Q_4 + \mathcal{N}_+\end{aligned}$$

and

$$\begin{aligned}\pm\mathcal{E}_2 &= \pm \sum_{p,q,r} \hat{V}_N(p-r)(\delta_{0,r} + \varphi_r)\chi_{|p|\le N^\alpha}\chi_{|q|\le N^\alpha} a^\dagger_{p+q} a^\dagger_{-p} a_q a_0 + \text{h.c.}\\ &\lesssim \|V_N\|_1 \|(1+\check{\varphi})\|_\infty \Big[\delta \mathcal{N}_+^2 + \delta^{-1} N \|\chi_{|.|\le N^\alpha}\|_2^2 \mathcal{N}_+\Big]\\ &\lesssim N^{-1}\Big[\delta \mathcal{N}_+^2 + \delta^{-1} N^{1+3\alpha}\mathcal{N}_+\Big] \lesssim \mathcal{N}_+ + N^{-1+3\alpha}\mathcal{N}_+^2,\end{aligned}$$

choosing in the last step $\delta = N^{3\alpha}$. Thus, with Lemma 3, we find

$$g'(s) \lesssim f(s) + g(s) + \langle \xi, \mathcal{N}_+ \xi\rangle + N^{-1+3\alpha}\langle \xi, (\mathcal{N}_+ + 1)^2 \xi\rangle.$$

With the estimate (75) and applying Gronwall's lemma, we obtain the desired estimate (76). □

In the next lemma, we control the contribution on the right-hand side of equation (66) arising from the commutator $[Q_3, \mathcal{B}_3]$.

Lemma 15. *We have*

$$\int_0^1 \int_s^1 e^{-t\mathcal{B}_3}[Q_3, B_3] e^{t\mathcal{B}_3} \mathrm{d}t\mathrm{d}s = 2(8\pi \mathfrak{a}_N - \hat{V}(0))\mathcal{N}_+ + \mathcal{E}_{[Q_3,\mathcal{B}_3]},$$

with

$$\pm\mathcal{E}_{[Q_3,\mathcal{B}_3]} \lesssim N^{-2\alpha}H_1 + N^{-\alpha/2}Q_4 + \left[N^{-\alpha/2} + N^{-1+\alpha}\right](\mathcal{N}_+ + 1) + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2.$$

Proof. We compute

$$\begin{aligned}[Q_3,\mathcal{B}_3] &= \sum_{p,q,r,s} \hat{V}_N(p)\tilde{\varphi}_r\chi_{|s|\le N^\alpha}[a^\dagger_{p+q}a^\dagger_{-p}a_q a_0 + a^\dagger_0 a^\dagger_q a_{-p}a_{p+q}, a^\dagger_{r+s}a^\dagger_{-r}a_s a:0 - a^\dagger_0 a^\dagger_s a:-r a_{r+s}]\\ &= \sum_{p,q,r,s} \hat{V}_N(p)\tilde{\varphi}_r\chi_{|s|\le N^\alpha}[a^\dagger_{p+q}a^\dagger_{-p}a_q a_0, a^\dagger_{r+s}a^\dagger_{-r}a_s a:0] + \text{h.c.}\\ &\quad + \sum_{p,q,r,s} \hat{V}_N(p)\tilde{\varphi}_r\chi_{|s|\le N^\alpha}[a^\dagger_0 a^\dagger_s a:-r a_{r+s}, a^\dagger_{p+q}a^\dagger_{-p}a_q a_0] + \text{h.c.} =: \text{(I)} + \text{(II)}.\end{aligned} \tag{77}$$

We start by estimating term (I). With the canonical commutation relations, we obtain

$$\begin{aligned}\text{(I)} = \sum_{p,q,r} \hat{V}_N(p)\Big[&\tilde{\varphi}_r\chi_{|q-r|\le N^\alpha}a^\dagger_{p+q}a^\dagger_{-p}a^\dagger_{-r}a_{q-r} + \tilde{\varphi}_q\chi_{|r|\le N^\alpha}a^\dagger_{p+q}a^\dagger_{-p}a^\dagger_{-q+r}a_r\\ &- \tilde{\varphi}_r\chi_{|q+p|\le N^\alpha}a^\dagger_{r+p+q}a^\dagger_{-r}a^\dagger_{-p}a_q - \tilde{\varphi}_r\chi_{|p|\le N^\alpha}a^\dagger_{r-p}a^\dagger_{-r}a^\dagger_{p+q}a_q\Big]a_0a_0 + \text{h.c.}\\ =:\ &\text{(I)}_a + \text{(I)}_b + \text{(I)}_c + \text{(I)}_d.\end{aligned} \tag{78}$$

To estimate the first term, we rewrite it in position space. We find

$$\begin{aligned}\pm\text{(I)}_a &= \pm\int_{\Lambda^2}\mathrm{d}x\mathrm{d}y\, V_N(x-y)a^\dagger_x a^\dagger_y a^\dagger(\check{\tilde{\varphi}}^x)a(\check{\chi}^x_{|\cdot|\le N^\alpha})a_0a_0 + \text{h.c.}\\ &\le \delta Q_4 + \delta^{-1}\int_{\Lambda^2}\mathrm{d}x\mathrm{d}y\, V_N(x-y)a^\dagger(\check{\chi}^x_{|\cdot|\le N^\alpha})a(\check{\tilde{\varphi}}^x)a^\dagger(\check{\tilde{\varphi}}^x)a(\check{\chi}^x_{|\cdot|\le N^\alpha})a^\dagger_0a^\dagger_0a_0a_0\\ &\lesssim \delta Q_4 + \delta^{-1}N^2\|\tilde{\varphi}\|_2^2\|V_N\|_1\|\chi_{|\cdot|\le N^\alpha}\|_\infty^2(\mathcal{N}_+ + 1)^2\\ &\lesssim \delta Q_4 + \delta^{-1}N^{-\alpha-1}(\mathcal{N}_+ + 1)^2 \le N^{-\alpha/2}Q_4 + N^{-1-\alpha/2}(\mathcal{N}_+ + 1)^2,\end{aligned}$$

where we used that $\mathcal{N}_0 \le N$ and the bound (74) and, in the last step, set $\delta = N^{-\alpha/2}$. The second term in equation (78) is dealt with similarly. We obtain

$$\begin{aligned}\pm\text{(I)}_b &= \pm\int_{\Lambda^3}\mathrm{d}x\mathrm{d}y\mathrm{d}z\, V_N(x-y)\check{\tilde{\varphi}}(x-z)\, a^\dagger_x a^\dagger_y a^\dagger_z a(\check{\chi}^z_{|\cdot|\le N^\alpha})a_0a_0 + \text{h.c.}\\ &\le \delta Q_4 + \delta^{-1}\int_{\Lambda^3}\mathrm{d}x\mathrm{d}y\mathrm{d}z\, V_N(x-y)|\check{\tilde{\varphi}}(x-z)|^2 a^\dagger(\check{\chi}^z_{|\cdot|\le N^\alpha})(\mathcal{N}_+ + 1)a(\check{\chi}^z_{|\cdot|\le N^\alpha})a^\dagger_0a^\dagger_0a_0a_0\\ &\lesssim \delta Q_4 + \delta^{-1}N^2\|\tilde{\varphi}\|_2^2\|V_N\|_1\|\chi_{|\cdot|\le N^\alpha}\|_\infty^2(\mathcal{N}_+ + 1)^2\\ &\lesssim N^{-\alpha/2}Q_4 + N^{-1-\alpha/2}(\mathcal{N}_+ + 1)^2,\end{aligned}$$

choosing again $\delta = N^{-\alpha/2}$. For the third term in equation (78), we bound it, for an arbitrary $\delta > 0$, with Cauchy-Schwarz by

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$$
\begin{aligned}
\pm(\mathrm{I})_c &\lesssim \delta \sum_{p,q,r} |\hat{V}_N(p)| a^\dagger_{r+p+q} a^\dagger_{-r} a^\dagger_{-p} (\mathcal{N}_+ + 1)^{-1} a_{-p} a_{-r} a_{r+p+q} \\
&\quad + \delta^{-1} \sum_{p,q,r} |\hat{V}_N(p)| |\tilde{\varphi}_r|^2 \chi_{|q+p| \le N^\alpha} a^\dagger_q (\mathcal{N}_+ + 1) a_q a^\dagger_0 a^\dagger_0 a_0 a_0 \\
&\lesssim (\delta \|\hat{V}_N\|_\infty + \delta^{-1} N^2 \|\tilde{\varphi}\|_2^2 \|\hat{V}_N * \chi_{|\cdot| \le N^\alpha}\|_\infty)(\mathcal{N}_+ + 1)^2 \\
&\lesssim N^{-1+5\alpha/2} (\mathcal{N}_+ + 1)^2,
\end{aligned}
$$

where, at the end, we took $\delta = N^{-\alpha}$ and used $\|\hat{V}_N * \chi_{|\cdot|\le N^\alpha}\|_\infty \le \|\hat{V}_N\|_\infty \|\chi_{|\cdot|\le N^\alpha}\|_1 \lesssim N^{-1+3\alpha}$. The last term in equation (78) can be bounded, again by Cauchy-Schwarz, by

$$
\pm(\mathrm{I})_d \lesssim \delta(\mathcal{N}_+ + 1)^2 + \delta^{-1} N^2 \|\hat{V}_N\|_\infty^2 \|\tilde{\varphi}\|_2^2 \|\chi_{|.|\le N^\alpha}\|_1 (\mathcal{N}_+ + 1)^2 \lesssim N^{-1+\alpha} (\mathcal{N}_+ + 1)^2,
$$

where we used $\delta = N^\alpha$.

Let us now consider term (II) in equation (77). We write

$$
\begin{aligned}
(\mathrm{II}) &= \sum_{p,q,r,s} \hat{V}_N(p) \tilde{\varphi}_r \chi_{|s| \le N^\alpha} \\
&\qquad \times \Big\{ a^\dagger_0 a^\dagger_s \big[a_{-r} a_{r+s}, a^\dagger_{p+q} a^\dagger_{-p} \big] a_q a_0 + a^\dagger_{p+q} a^\dagger_{-p} \big[a^\dagger_0 a^\dagger_s, a_q a_0 \big] a_{-r} a_{r+s} \Big\} + \mathrm{h.c.} \\
&=: (\mathrm{II})_a + (\mathrm{II})_b.
\end{aligned} \tag{79}
$$

With

$$
a^\dagger_{p+q} a^\dagger_{-p} \big[a^\dagger_0 a^\dagger_s, a_q a_0 \big] a_{-r} a_{r+s} = -\delta_{sq}\, a^\dagger_{p+q} a^\dagger_{-p} a^\dagger_0 a_0 a_{-r} a_{r+s} - a^\dagger_{p+q} a^\dagger_{-p} a_q a^\dagger_s a_{-r} a_{r+s},
$$

we obtain

$$
\begin{aligned}
(\mathrm{II})_b = &- \sum_{p,q,r} \hat{V}_N(p) \tilde{\varphi}_r \chi_{|q| \le N^\alpha}\, a^\dagger_{p+q} a^\dagger_{-p} a^\dagger_0 a_0 a_{-r} a_{r+q} \\
&- \sum_{p,q,r,s} \hat{V}_N(p) \tilde{\varphi}_r \chi_{|s| \le N^\alpha}\, a^\dagger_{p+q} a^\dagger_{-p} a^\dagger_s a_q a_{-r} a_{r+s} \\
&- \sum_{p,r,q} \hat{V}_N(p) \tilde{\varphi}_r \chi_{|q| \le N^\alpha}\, a^\dagger_{p+q} a^\dagger_{-p} a_{-r} a_{r+q} =: (\mathrm{II})_{b1} + (\mathrm{II})_{b2} + (\mathrm{II})_{b3}.
\end{aligned}
$$

We can bound $(\mathrm{II})_{b3}$ by switching to position space. We find

$$
\begin{aligned}
\pm(\mathrm{II})_{b3} &= \pm \int_{\Lambda^4} \mathrm{d}x \mathrm{d}y \mathrm{d}u \mathrm{d}v\, V_N(x-y) \check{\chi}_{|.| \le N^\alpha}(x-u) \check{\tilde{\varphi}}(u-v) a^\dagger_x a^\dagger_y a_u a_v + \mathrm{h.c.} \\
&\lesssim \delta \|\check{\tilde{\varphi}}\|_2^2\, Q_4 + \delta^{-1} \|\check{\chi}_{|.|\le N^\alpha}\|_2^2\, (\mathcal{N}_+ + 1)^2 \lesssim N^{-\alpha/2} Q_4 + N^{-3+5\alpha/2} (\mathcal{N}_+ + 1)^2.
\end{aligned}
$$

Term $(\mathrm{II})_{b1}$ can be bounded analogously, but it contains an additional factor $\mathcal{N}_0 = a^\dagger_0 a_0 \le N$. Thus

$$
\pm(\mathrm{II})_{b1} \le N^{-\alpha/2} Q_4 + N^{-1+5\alpha/2} (\mathcal{N}_+ + 1)^2.
$$

Term $(\mathrm{II})_{b1}$ can also be bounded in position space. We obtain

$$
\begin{aligned}
\pm(\mathrm{II})_{b2} &= \pm \int_{\Lambda^3} \mathrm{d}x \mathrm{d}y \mathrm{d}u\, V_N(x-y)\, a^\dagger_x a^\dagger_y a^\dagger(\check{\chi}^u_{|.|\le N^\alpha}) a_x a(\check{\tilde{\varphi}}^u) a_u + \mathrm{h.c.} \\
&\le \delta \|\check{\chi}_{|.|\le N^\alpha}\|_2^2\, Q_4 \mathcal{N}_+ + \delta^{-1} \|V_N\|_1 \|\check{\tilde{\varphi}}\|_2^2 (\mathcal{N}_+ + 1)^3 \lesssim N^{-\alpha/2} Q_4 + N^{-1+5\alpha/2} (\mathcal{N}_+ + 1)^2,
\end{aligned}
$$

where we chose $\delta = N^{-1-7\alpha/2}$.

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Let us now consider term $(\mathrm{II})_a$, defined on the right-hand side of equation (79). With

$$\begin{aligned} &a_0^\dagger a_s^\dagger \big[a_{-r}a_{r+s}, a_{p+q}^\dagger a_{-p}^\dagger\big] a_q a_0 \\ &\quad = a_0^\dagger a_s^\dagger \Big\{\delta_{-r,p+q} a_{-p}^\dagger a_{r+s} + \delta_{r,p} a_{p+q}^\dagger a_{r+s} + \delta_{r+s,p+q} a_{-r} a_{-p}^\dagger + \delta_{r+s,-p} a_{-r} a_{p+q}^\dagger\Big\} a_q a_0, \end{aligned}$$

we obtain, rearranging the terms in normal order (with appropriate changes of variables),

$$\begin{aligned} (\mathrm{II})_a &= \sum_{p,q} \big(\hat V_N(p) + \hat V_N(p-q)\big)\tilde\varphi_p \chi_{|q|\le N^\alpha} a_0^\dagger a_q^\dagger a_q a_0 \\ &\quad + 4 \sum_{p,q,s} \hat V_N(p)\tilde\varphi_{p+q}\chi_{|s|\le N^\alpha}\, a_0^\dagger a_s^\dagger a_{-p}^\dagger a_{-p-q+s} a_q a_0 + \text{h.c.} \\ &=: 2\sum_{p,q} \big(\hat V_N(p) + \hat V_N(p-q)\big)\tilde\varphi_p \chi_{|q|\le N^\alpha} a_0^\dagger a_q^\dagger a_q a_0 + (\mathrm{II})_{a1}, \end{aligned} \tag{80}$$

where we can bound, with $\mathcal{N}_0 = a_0^\dagger a_0 \le N$,

$$\pm(\mathrm{II})_{a1} \lesssim N\|\hat V_N\|_\infty \Big[\delta\|\tilde\varphi\|_2^2 + \delta^{-1}\|\chi_{|.|\le N^\alpha}\|_2^2\Big]\mathcal{N}_+^2 \lesssim N^{-1+\alpha}(\mathcal{N}_+ + 1)^2,$$

choosing $\delta = N^{1-\alpha}$. Collecting all the estimates we have proved so far, we conclude from equation (77) that

$$\big[\mathcal{Q}_3, \mathcal{B}_3\big] = 2\sum_{p,q}\big(\hat V_N(p) + \hat V_N(p-q)\big)\tilde\varphi_p\chi_{|q|\le N^\alpha} a_0^\dagger a_q^\dagger a_q a_0 + \mathcal{E}_1,$$

where

$$\pm\mathcal{E}_1 \lesssim N^{-\alpha/2}\mathcal{Q}_4 + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2. \tag{81}$$

At the expense of adding an additional small error to the right-hand side of the estimate (81), in the main term, we can replace $a_0^\dagger a_0 = N - \mathcal{N}_+$ by a factor of N, since

$$\pm\sum_{p,q}(\hat V_N(p) + \hat V_N(p-q))\tilde\varphi_p\chi_{|q|\le N^\alpha} a_q^\dagger \mathcal{N}_+ a_q \lesssim N^{-1}\|\tilde\varphi\|_1(\mathcal{N}_+ + 1)^2 \lesssim N^{-1}(\mathcal{N}_+ + 1)^2.$$

Moreover, from

$$\begin{aligned} &\pm N\sum_{p,q}\big(\hat V_N(p-q) - \hat V_N(p)\big)\tilde\varphi_p\chi_{|q|\le N^\alpha} a_q^\dagger a_q \\ &\qquad \lesssim \sum_{p,q}|q/N|\|\nabla\hat V\|_\infty|\tilde\varphi_p|\chi_{|q|\le N^\alpha} a_q^\dagger a_q \lesssim \|\tilde\varphi\|_1 N^{-1+\alpha}\mathcal{N}_+ \lesssim N^{-1+\alpha}\mathcal{N}_+, \end{aligned} \tag{82}$$

we arrive at

$$\big[\mathcal{Q}_3, \mathcal{B}_3\big] = 4N\sum_p \hat V_N(p)\tilde\varphi_p \sum_{|q|\le N^\alpha} a_q^\dagger a_q + \mathcal{E}_2,$$

where

$$\pm\mathcal{E}_2 \lesssim N^{-\alpha/2}\mathcal{Q}_4 + N^{-1+\alpha}\mathcal{N}_+ + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2.$$

Conjugating with $e^{t\mathcal{B}_3}$ and integrating over t and s, we obtain, with the help of Lemma 10 and Lemma 14 (and the observation that the first estimate in Lemma 10 also holds, if we replace $\mathcal{N}_+$ on the left-hand

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side by $\sum_{|p|\le N^\alpha} a_p^\dagger a_p$),

$$\int_0^1\int_s^1 e^{-t\mathcal{B}_3}[Q_3,\mathcal{B}_3]e^{t\mathcal{B}_3}\mathrm{d}t\mathrm{d}s = 2N\sum_q \hat{V}_N(q)\tilde{\varphi}_q \sum_{|p|\le N^\alpha} a_p^\dagger a_p + \mathcal{E}_3,$$

where

$$\pm\mathcal{E}_3 \lesssim N^{-\alpha/2}(Q_4+\mathcal{N}_+ +1) + N^{-1+\alpha}\mathcal{N}_+ + N^{-1+5\alpha/2}(\mathcal{N}_+ +1)^2.$$

The claim now follows from equation (19) and the observation that

$$\mathcal{N}_+ - \sum_{|p|\le N^\alpha} a_p^\dagger a_p = \sum_{|p|\ge N^\alpha} a_p^\dagger a_p \le N^{-2\alpha}H_1. \qquad \square$$

Finally, we consider the conjugation of the operator $\tilde{Q}_2$, defined in equation (25).

Lemma 16. *We have*

$$e^{-\mathcal{B}_3}\tilde{Q}_2 e^{\mathcal{B}_3} = \tilde{Q}_2 + \mathcal{E}_{\tilde{Q}_2} \tag{83}$$

with

$$\pm\mathcal{E}_{\tilde{Q}_2} \lesssim N^{-1/2+\alpha}(\mathcal{N}_+ +1)^{3/2} + N^{-3/2+\alpha}(\mathcal{N}_+ +1)^{5/2}.$$

Proof. We have

$$\begin{aligned} e^{-\mathcal{B}_3}\tilde{Q}_2 e^{\mathcal{B}_3} - \tilde{Q}_2 &= \int_0^1 e^{-s\mathcal{B}_3}[\tilde{Q}_2,\mathcal{B}_3]e^{s\mathcal{B}_3}\mathrm{d}s \\ &= \frac{4\pi\mathfrak{a}_N}{N}\sum_{|r|\le N^\alpha}\int_0^1 e^{-s\mathcal{B}_3}[a_r^\dagger a_{-r}^\dagger a_0a_0 + \text{h.c.},\mathcal{B}_3]e^{s\mathcal{B}_3}\mathrm{d}s. \end{aligned}$$

We compute the commutator

$$[a_r^\dagger a_{-r}^\dagger a_0a_0 + \text{h.c.}, a_{p+q}^\dagger a_{-p}^\dagger a_q a_0 - \text{h.c.}] \tag{84}$$

$$\begin{aligned} &= [a_r^\dagger a_{-r}^\dagger a_0a_0, a_{p+q}^\dagger a_{-p}^\dagger a_q a_0] + [a_0^\dagger a_0^\dagger a_r a_{-r}, a_{p+q}^\dagger a_{-p}^\dagger a_q a_0] + \text{h.c.} \\ &= 2\Big\{a_0^\dagger a_0^\dagger(\delta_{p+q,r}a_{-p}^\dagger a_{-p-q} + \delta_{p,r}a_{p+q}^\dagger a_p)a_q a_0 \\ &\qquad - \delta_{r,q}a_{p+q}^\dagger a_{-p}^\dagger a_{-q}^\dagger a_0a_0a_0 - a_{p+q}^\dagger a_{-p}^\dagger a_0^\dagger a_q a_r a_{-r}\Big\} + \text{h.c.} \end{aligned} \tag{85}$$

Hence, we obtain

$$\begin{aligned} &\frac{4\pi\mathfrak{a}_N}{N}\sum_{|r|\le N^\alpha}\Big[a_r^\dagger a_{-r}^\dagger a_0a_0 + \text{h.c.},\mathcal{B}_3\Big] \\ &\qquad = \frac{8\pi\mathfrak{a}_N}{N}\sum_{p,q:|p+q|\le N^\alpha}\tilde{\varphi}_p\chi_{|q|\le N^\alpha}a_0^\dagger a_0^\dagger a_{-p}^\dagger a_{-p-q}a_q a_0 \\ &\qquad\quad - \frac{4\pi\mathfrak{a}_N}{N}\sum_{p,|q|\le N^\alpha}\tilde{\varphi}_p\chi_{|q|\le N^\alpha}a_{p+q}^\dagger a_{-p}^\dagger a_{-q}^\dagger a_0a_0a_0 \\ &\qquad\quad - \frac{4\pi\mathfrak{a}_N}{N}\sum_{p,q,|r|\le N^\alpha}\tilde{\varphi}_p\chi_{|q|\le N^\alpha}a_{p+q}^\dagger a_{-p}^\dagger a_0^\dagger a_q a_r a_{-r} =: \text{(I)} + \text{(II)} + \text{(III)}. \end{aligned}$$

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With Cauchy-Schwarz and using the bounds from Lemma 2, we can bound

$$\begin{aligned} \pm(\mathrm{I}) &\lesssim N^{-1/2-\alpha/2}(\mathcal{N}_+ + 1)^{3/2}, \\ \pm(\mathrm{II}) &\lesssim N^{-1/2+\alpha}(\mathcal{N}_+ + 1)^{3/2}, \\ \pm(\mathrm{III}) &\lesssim N^{-3/2+\alpha}(\mathcal{N}_+ + 1)^{5/2}. \end{aligned}$$

The claim now follows with Lemma 10. □

We are now ready to conclude the proof of Proposition 11.

Proof of Proposition 11. Recall from equation (66) that

$$\begin{aligned} e^{-\mathcal{B}_3}e^{-\mathcal{B}_2}H_N e^{\mathcal{B}_2}e^{\mathcal{B}_3} &= 4\pi\mathfrak{a}_N(N-1) + \sum_{|p|\le N^\alpha}\frac{(4\pi\mathfrak{a}_N)^2}{p^2} + H_1 + Q_4 + e^{-\mathcal{B}_3}(\tilde{H}_2 + \tilde{Q}_2 + \mathcal{E}_{\mathcal{B}_2})e^{\mathcal{B}_3} \\ &\quad + \int_0^1 e^{-t\mathcal{B}_3}\Gamma_3 e^{t\mathcal{B}_3}\mathrm{d}t + \int_0^1\int_s^1 e^{-t\mathcal{B}_3}[Q_3,\mathcal{B}_3]e^{t\mathcal{B}_3}\mathrm{d}t\mathrm{d}s, \end{aligned}$$

where

$$\pm\mathcal{E}_{\mathcal{B}_2} \le N^{-\alpha/2}Q_4 + \left[N^{-\alpha/2} + N^{-1+5\alpha/2}\right](\mathcal{N}_+ + 1) + N^{-1+\alpha}\mathcal{N}_+^2 + N^{-2}H_1.$$

With Lemma 10 and Lemma 14, this also implies that

$$\begin{aligned} \pm e^{-\mathcal{B}_3}\mathcal{E}_{\mathcal{B}_2}e^{\mathcal{B}_3} &\lesssim N^{-\alpha/2}Q_4 + N^{-2}H_1 + \left[N^{-\alpha/2} + N^{-1+5\alpha/2}\right](\mathcal{N}_+ + 1) \\ &\quad + N^{-1+2\alpha}(\mathcal{N}_+ + 1)^2. \end{aligned}$$

Applying the first bound in Lemma 10 to the operator $\tilde{H}_2 = (2\hat{V}(0) - 8\pi\mathfrak{a}_N)\mathcal{N}_+$, defined in equation (24), we obtain

$$\pm\left[e^{-\mathcal{B}_3}\tilde{H}_2 e^{\mathcal{B}_3} - \tilde{H}_2\right] \lesssim N^{-\alpha/2}(\mathcal{N}_+ + 1).$$

Combining Lemma 13 with Lemma 14, we obtain

$$\begin{aligned} &\int_0^1 e^{-t\mathcal{B}_3}\Gamma_3 e^{t\mathcal{B}_3}\mathrm{d}t \\ &\qquad \lesssim N^{-3\alpha/2}H_1 + N^{-\alpha/2}Q_4 + N^{-\alpha/2}(\mathcal{N}_+ + 1) \\ &\qquad\quad + N^{3\alpha/2-1/2}(\mathcal{N}_+ + 1)^{3/2} + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2. \end{aligned}$$

Together with the bounds in Lemmas 15 and 16, and with the observation that

$$8\pi\mathfrak{a}_N\sum_{|p|>N^\alpha}a_p^\dagger a_p \le N^{-2\alpha}H_1,$$

we conclude the proof of Proposition 11. □

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5. Diagonalisation of a quadratic Hamiltonian

From Proposition 11, we find

$$\begin{aligned} &e^{-\mathcal{B}_3} e^{-\mathcal{B}_2} H_N e^{\mathcal{B}_2} e^{\mathcal{B}_3} \\ &\quad = 4\pi \mathfrak{a}_N (N-1) + \frac{1}{4} \sum_{|p|\le N^\alpha} \frac{(8\pi\mathfrak{a}_N)^2}{p^2} + \sum_{|p|>N^\alpha} p^2 a_p^\dagger a_p + Q_4 \\ &\qquad + \sum_{|p|\le N^\alpha} (p^2 + 8\pi\mathfrak{a}_N) a_p^\dagger \frac{a_0 a_0^\dagger}{N} a_p + \frac{1}{2} \sum_{|p|\le N^\alpha} 8\pi\mathfrak{a}_N [a_p^\dagger a_{-p}^\dagger \frac{a_0 a_0}{N} + \text{h.c.}] + \mathcal{E}, \end{aligned} \tag{86}$$

with an error $\mathcal{E}$ satisfying the bound (64). Here we used the observation that, on the sector $\{\mathcal{N} = N\}$, we can write

$$\begin{aligned} \sum_{|p|\le N^\alpha} (p^2 + 8\pi\mathfrak{a}_N) a_p^\dagger a_p &= \sum_{|p|\le N^\alpha} (p^2 + 8\pi\mathfrak{a}_N) a_p^\dagger \frac{a_0^\dagger a_0 + \mathcal{N}_+ + 1}{N} a_p \\ &= \sum_{|p|\le N^\alpha} (p^2 + 8\pi\mathfrak{a}_N) a_p^\dagger \frac{a_0 a_0^\dagger}{N} a_p + \frac{1}{N} \sum_{|p|\le N^\alpha} (p^2 + 8\pi\mathfrak{a}_N) a_p^\dagger \mathcal{N}_+ a_p \end{aligned}$$

where the term

$$\frac{1}{N} \sum_{|p|\le N^\alpha} (p^2 + 8\pi\mathfrak{a}_N) a_p^\dagger \mathcal{N}_+ a_p \lesssim N^{2\alpha-1} (\mathcal{N}_+ + 1)^2$$

can be absorbed on the right-hand side of the estimate (64).

In this section, we will diagonalise the operator on the last line of equation (86). Inspired by Bogoliubov theory (on states with $a_0, a_0^\dagger \sim \sqrt{N}$, this operator is approximately quadratic), we define, for $|p| \le N^\alpha$, the coefficients

$$\tau_p = -\frac{1}{4} \log\left[1 + \frac{16\pi\mathfrak{a}_N}{p^2}\right]$$

so that

$$\tanh(2\tau_p) = -\frac{8\pi\mathfrak{a}_N}{p^2 + 8\pi\mathfrak{a}_N}.$$

We also introduce the notation $\gamma_p = \cosh \tau_p$ and $\nu_p = \sinh \tau_p$.

Lemma 17. *We have the pointwise bound $\gamma_p \lesssim 1$ and $\tau_p, \nu_p \lesssim \chi_{|p|\le N^\alpha}/p^2$. Moreover,*

$$\|\tau\|_\infty \le \|\tau\|_2 \lesssim 1, \qquad \|\nu\|_\infty \le \|\nu\|_2 \lesssim 1, \qquad \|\gamma - 1\|_\infty \le \|\gamma - 1\|_2 \lesssim 1$$

and

$$\|\check{\tau}\|_\infty \le \|\tau\|_1 \lesssim N^\alpha, \qquad \|\check{\nu}\|_\infty \le \|\nu\|_1 \lesssim N^\alpha.$$

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With these coefficients, we can write

$$\sum_{|p|\le N^\alpha}(p^2+8\pi\mathfrak{a}_N)a_p^\dagger\frac{a_0a_0^\dagger}{N}a_p+\frac{1}{2}\sum_{|p|\le N^\alpha}8\pi\mathfrak{a}_N\,[a_p^\dagger a_{-p}^\dagger\frac{a_0a_0}{N}+\text{h.c.}]$$
$$=\sum_{|p|\le N^\alpha}\sqrt{|p|^4+16\pi\mathfrak{a}_Np^2}\,\big(\gamma_pa_p^\dagger\frac{a_0}{\sqrt{N}}+\nu_p\frac{a_0^\dagger}{\sqrt{N}}a_{-p}\big)\big(\gamma_p\frac{a_0^\dagger}{\sqrt{N}}a_p+\nu_pa_{-p}^\dagger\frac{a_0}{\sqrt{N}}\big)$$
$$-\frac{1}{2}\sum_{|p|\le N^\alpha}\big[p^2+8\pi\mathfrak{a}_N-\sqrt{|p|^4+16\pi\mathfrak{a}_Np^2}\big]+\delta$$

with an error δ satisfying

$$\pm\delta\lesssim N^{\alpha-1}(\mathcal{N}_++1).$$

Here, we used the relations

$$\gamma_p^2+\nu_p^2=\cosh(2\tau_p)=\frac{1}{\sqrt{1-\tanh^2(2\tau_p)}}=\frac{p^2+8\pi\mathfrak{a}_N}{\sqrt{|p|^4+16\pi\mathfrak{a}_Np^2}},$$
$$2\gamma_p\nu_p=\sinh(2\tau_p)=\frac{\tanh(2\tau_p)}{\sqrt{1-\tanh^2(2\tau_p)}}=\frac{8\pi\mathfrak{a}_N}{\sqrt{|p|^4+16\pi\mathfrak{a}_Np^2}},$$
$$\nu_p^2=\frac{1}{2}\big[\cosh(2\tau_p)-1\big]=\frac{1}{2}\frac{p^2+8\pi\mathfrak{a}_N-\sqrt{|p|^4+16\pi\mathfrak{a}_Np^2}}{\sqrt{|p|^4+16\pi\mathfrak{a}_Np^2}}$$

and the commutator

$$\Big[\frac{a_0^\dagger}{\sqrt{N}}a_{-p},a_{-p}^\dagger\frac{a_0}{\sqrt{N}}\Big]=\frac{1}{N}a_0^\dagger a_0-\frac{1}{N}a_{-p}^\dagger a_{-p}=1-\frac{1}{N}(\mathcal{N}_++a_{-p}^\dagger a_{-p}).$$

The contribution proportional to N^{-1} on the right-hand side of the last equation produces (using Lemma 17) the error δ. Inserting in equation (86), we conclude that

$$\begin{aligned}&e^{-\mathcal{B}_3}e^{-\mathcal{B}_2}H_Ne^{\mathcal{B}_2}e^{\mathcal{B}_3}\\&=4\pi\mathfrak{a}_N(N-1)-\frac{1}{2}\sum_{|p|\le N^\alpha}\Big[p^2+8\pi\mathfrak{a}_N-\sqrt{|p|^4+16\pi\mathfrak{a}_Np^2}-\frac{(8\pi\mathfrak{a}_N)^2}{2p^2}\Big]\\&\quad+\sum_{|p|\le N^\alpha}\sqrt{|p|^4+16\pi\mathfrak{a}_Np^2}\,\big(\gamma_pb_p^\dagger+\nu_pb_{-p}\big)\big(\gamma_pb_p+\nu_pb_{-p}^\dagger\big)+\sum_{|p|>N^\alpha}p^2a_p^\dagger a_p+\mathcal{Q}_4+\mathcal{E},\end{aligned}\tag{87}$$

where $\mathcal{E}$ still satisfies the estimate (64) and where we introduced the modified creation and annihilation operators

$$b_p=\frac{a_0^\dagger}{\sqrt{N}}a_p,\quad b_p^\dagger=a_p^\dagger\frac{a_0}{\sqrt{N}}\tag{88}$$

satisfying the commutation relations

$$[b_p,b_q]=[b_p^\dagger,b_q^\dagger]=0,\qquad[b_p,b_q^\dagger]=\delta_{p,q}\big(1-\frac{\mathcal{N}_+}{N}\big)-\frac{1}{N}a_q^\dagger a_p\tag{89}$$

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and $[a_p^\dagger a_r, b_q^\dagger] = \delta_{r,q} b_p^\dagger$, $[a_p^\dagger a_r, b_q] = -\delta_{p,q} b_r$. On states with few excitations $a_0, a_0^\dagger \simeq \sqrt{N}$, we have $b_p^\dagger \simeq a_p^\dagger$, $b_p \simeq a_p$. According to Bogoliubov theory, we can therefore expect that the operators $(\gamma_p b_p^\dagger . + \nu_p b_{-p})$ and $(\gamma_p b_p + \nu_p b_{-p}^\dagger)$ can be rotated back to $b_p^\dagger$ and, respectively, b_p through conjugation of the Hamiltonian with the unitary transformation generated by the antisymmetric operator

$$\mathcal{B}_4 = \frac{1}{2} \sum_{|p| \le N^\alpha} \tau_p \big(b_p^\dagger b_{-p}^\dagger - b_p b_{-p} \big) = \frac{1}{2} \sum_{|p| \le N^\alpha} \tau_p \big(a_p^\dagger a_{-p}^\dagger \frac{a_0 a_0}{N} - \text{h.c.} \big).$$

Notice that $\mathcal{B}_4$ has the same form as the operator $\mathcal{B}_2$ defined in equation (13) (with a different choice of the coefficients, of course; here it is more convenient to keep the factor N^{-1} out of τ_p). To control the action of $\mathcal{B}_4$, we will need rough a priori bounds on the growth of the number and the energy of the excitations.

Lemma 18. *For every $k \in \mathbb{N}$, we have* [1]

$$e^{-\mathcal{B}_4} (\mathcal{N}_+ + 1)^k e^{\mathcal{B}_4} \lesssim (\mathcal{N}_+ + 1)^k. \tag{90}$$

Moreover,

$$e^{-\mathcal{B}_4} H_1 e^{\mathcal{B}_4} \lesssim H_1 + N^\alpha, \tag{91}$$

$$e^{-\mathcal{B}_4} Q_4 e^{\mathcal{B}_4} \lesssim Q_4 + N^{-1+2\alpha} + N^{-1} (\mathcal{N}_+ + 1)^2. \tag{92}$$

Proof. The proof of the bound (90) is standard (based on Gronwall's lemma and the bounds in Lemma 17). To prove equation (91), we define $g(s) = e^{-s\mathcal{B}_4} H_1 e^{s\mathcal{B}_4}$ and compute (using the commutation relations after equation (89))

$$g'(s) = e^{-s\mathcal{B}_4} [H_1, \mathcal{B}_4] e^{s\mathcal{B}_4} = \sum_{|p| \le N^\alpha} p^2 \tau_p e^{-s\mathcal{B}_4} \big[b_p^\dagger b_{-p}^\dagger + \text{h.c.} \big] e^{s\mathcal{B}_4} \lesssim e^{-s\mathcal{B}_4} H_1 e^{s\mathcal{B}_4} + \sum_{|p| \le N^\alpha} p^2 \tau_p^2.$$

From Lemma 17, we have $\sum_p p^2 \tau_p^2 \lesssim N^\alpha$; with Gronwall's lemma, we obtain the bound (91).

To show the bound (92), we set $h(s) = e^{-s\mathcal{B}_4} Q_4 e^{s\mathcal{B}_4}$, and then

$$h'(s) = e^{-s\mathcal{B}_4} [Q_4, \mathcal{B}_4] e^{s\mathcal{B}_4}. \tag{93}$$

Proceeding as in (38), we find (we use here the convention that $\tau_q = 0$, for $|q| > N^\alpha$)

$$[Q_4, \mathcal{B}_4] = \frac{1}{2} \sum_q (\hat{V}_N * \tau)(q) b_q^\dagger b_{-q}^\dagger + \sum_{p,q,s} \hat{V}_N(q-p) \tau_s b_{p+s-q}^\dagger b_q^\dagger a_{-s}^\dagger a_p + \text{h.c.}$$

Switching to position space, we write

$$[Q_4, \mathcal{B}_4] = \frac{1}{2} \int_{\Lambda^2} \mathrm{d}x \mathrm{d}y V_N(x-y) \check{\tau}(x-y) b_x^\dagger b_y^\dagger + \int_{\Lambda^3} \mathrm{d}x \mathrm{d}y \mathrm{d}z V_N(x-y) \check{\tau}(x-z) b_x^\dagger b_y^\dagger a_z^\dagger a_y + \text{h.c.}$$

With the bounds from Lemma 17, we conclude

$$\pm [Q_4, \mathcal{B}_4] \lesssim Q_4 + N^{-1+2\alpha} + N^{-1} (\mathcal{N}_+ + 1)^2.$$

Inserting this in equation (93), applying the bound (90) and then Gronwall's lemma, we obtain the desired bound (92). ◻

[1] The estimate for Q_4 will only be used in the next section to show upper bounds on the eigenvalues of H_N; for the lower bounds, we will only use the fact that $Q_4 \ge 0$.

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We are now ready to state the main result of this section, which shows that conjugation with $e^{\mathcal{B}_4}$ diagonalises the quadratic part of the Hamilton operator.

Proposition 19. *We have*

$$\begin{aligned} &e^{-\mathcal{B}_4}e^{-\mathcal{B}_3}e^{-\mathcal{B}_2}H_N e^{\mathcal{B}_2}e^{\mathcal{B}_3}e^{\mathcal{B}_4} \\ &\quad= 4\pi\mathfrak{a}_N(N-1) + \frac{1}{2}\sum_p\left[\sqrt{p^4+16\pi\mathfrak{a}_N p^2} - p^2 - 8\pi\mathfrak{a}_N + \frac{(8\pi\mathfrak{a}_N)^2}{2p^2}\right] \\ &\qquad+ \sum_p \sqrt{p^4+16\pi\mathfrak{a}_N p^2}\, a_p^\dagger a_p + e^{-\mathcal{B}_4}Q_4 e^{\mathcal{B}_4} + \mathcal{E}_{\mathcal{B}_4}, \end{aligned} \tag{94}$$

where

$$\begin{aligned} \pm\mathcal{E}_{\mathcal{B}_4} \lesssim\ & N^{-3\alpha/2}(H_1 + N^\alpha) + N^{-\alpha/2}e^{-\mathcal{B}_4}Q_4 e^{\mathcal{B}_4} + N^{-\alpha/2}(\mathcal{N}_+ + 1) \\ &+ N^{(3\alpha-1)/2}(\mathcal{N}_+ + 1)^{3/2} + N^{-1+5\alpha/2}(\mathcal{N}_+ + 1)^2. \end{aligned} \tag{95}$$

Proof. For $s \in [0;1]$, we define

$$E(s) = \sum_{|p|\le N^\alpha} \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2}\,\left(\gamma_p^s b_p^\dagger + \nu_p^s b_{-p}\right)\left(\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger\right)$$

with the operators $b_p, b_p^\dagger$ defined in equation (88) and with the notation $\gamma_p^s = \cosh(s\tau_p)$ and $\nu_p^s = \sinh(s\tau_p)$. In particular, for $s = 1$, this is exactly the operator appearing on the third line in equation (87). For ψ in the sector $\{\mathcal{N} = N\}$, we define $f_\psi : [0;1] \to \mathbb{R}$ by

$$f_\psi(s) = \langle\psi, e^{-s\mathcal{B}_4}E(s)e^{s\mathcal{B}_4}\psi\rangle.$$

The idea is that the generalised Bogoliubov transformation $e^{s\mathcal{B}_4}$ approximately cancels (on states with few excitations) the symplectic rotations determined by the coefficients γ_p^s, ν_p^s (it would precisely cancel them if the operators $b_p^\dagger, b_p$ satisfied canonical commutation relations); hence, on states with few excitations, we expect f_ψ to be approximately constant in s. More precisely, we claim that

$$\left|f_\psi'(s)\right| \lesssim N^{2\alpha-1}\langle\psi, (\mathcal{N}_+ + 1)^2\psi\rangle. \tag{96}$$

Assuming for a moment that the bound (96) holds true, we could conclude, integrating over $s \in [0;1]$, that

$$e^{-\mathcal{B}_4}E(1)e^{\mathcal{B}_4} = E(0) + \delta$$

with

$$\pm\delta \lesssim N^{2\alpha-1}(\mathcal{N}_+ + 1)^2.$$

With the bounds from Lemma 18 (and noticing that the action of $\mathcal{B}_4$ on the high-momenta part of the kinetic energy is trivial), this would imply that

$$\begin{aligned} &e^{-\mathcal{B}_4}e^{-\mathcal{B}_3}e^{-\mathcal{B}_2}H_N e^{\mathcal{B}_2}e^{\mathcal{B}_3}e^{\mathcal{B}_4} \\ &\quad= 4\pi\mathfrak{a}_N(N-1) - \frac{1}{2}\sum_{|p|\le N^\alpha}\left[p^2 + 8\pi\mathfrak{a}_N - \sqrt{|p|^4+16\pi\mathfrak{a}_N p^2} - \frac{(8\pi\mathfrak{a}_N)^2}{2p^2}\right] \\ &\qquad+ \sum_{|p|\le N^\alpha}\sqrt{|p|^4+16\pi\mathfrak{a}_N p^2}\, a_p^\dagger\frac{a_0 a_0^\dagger}{N}a_p + \sum_{|p|>N^\alpha}p^2 a_p^\dagger a_p + e^{-\mathcal{B}_4}Q_4 e^{\mathcal{B}_4} + \mathcal{E}, \end{aligned} \tag{97}$$

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where $\mathcal{E}$ satisfies the bound (95). Writing $a_0 a_0^\dagger = a_0^\dagger a_0 + 1 = N - \mathcal{N}_+ + 1$, we could then replace

$$\sum_{|p|\le N^\alpha} \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2}\, a_p^\dagger \frac{a_0 a_0^\dagger}{N} a_p = \sum_{|p|\le N^\alpha} \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2}\, a_p^\dagger a_p + \delta$$

with

$$\pm\delta \le \frac{1}{N} \sum_{|p|\le N^\alpha} \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2}\, a_p^\dagger (\mathcal{N}_+ + 1) a_p \lesssim N^{-1+2\alpha} (\mathcal{N}_+ + 1)^2.$$

Furthermore, since

$$\left| p^2 + 8\pi\mathfrak{a}_N - \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2} - \frac{(8\pi\mathfrak{a}_N)^2}{2p^2} \right| \lesssim (1+p^2)^{-2},$$

we could write

$$\sum_{|p|\le N^\alpha} \left[p^2 + 8\pi\mathfrak{a}_N - \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2} - \frac{(8\pi\mathfrak{a}_N)^2}{2p^2} \right] \\ = \sum_p \left[p^2 + 8\pi\mathfrak{a}_N - \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2} - \frac{(8\pi\mathfrak{a}_N)^2}{2p^2} \right] + \mathcal{O}(N^{-\alpha}).$$

Similarly, from $\left| p^2 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2} \right| \lesssim 1$, we could bound

$$\pm \sum_{|p|>N^\alpha} \left[p^2 - \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2} \right] a_p^\dagger a_p \lesssim N^{-2\alpha} H_1.$$

Inserting all these estimates in equation (97), we would end up with (94) and (95).

It remains to show the bound (96). To this end, we observe that

$$f'_\psi(s) = \frac{d}{ds} \langle \psi, e^{s\mathcal{B}_4} E(s) e^{-s\mathcal{B}_4} \psi \rangle = \langle \psi, e^{s\mathcal{B}_4} \{ [\mathcal{B}_4, E(s)] + \frac{\partial E(s)}{\partial s} \} e^{-s\mathcal{B}_4} \psi \rangle. \tag{98}$$

We have, denoting $\varepsilon_p = \sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2}$,

$$[\mathcal{B}_4, E(s)] = \frac{1}{2} \sum_{|p|\le N^\alpha} \varepsilon_p \sum_{|q|\le N^\alpha} \tau_q \{ [b_q^\dagger b_{-q}^\dagger, (\gamma_p^s b_p^\dagger + \nu_p^s b_{-p})] (\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger) \\ + (\gamma_p^s b_p^\dagger + \nu_p^s b_{-p}) [b_q^\dagger b_{-q}^\dagger, (\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger)] \} + \text{h.c.}$$

A long but straightforward computation, based on the commutation relations (89), leads to

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$$\begin{aligned}
&\big[\mathcal{B}_4, E(s)\big] \\
&= -\frac{1}{2}\sum_{|p|\leq N^\alpha} \varepsilon_p \tau_p \nu_p^s b_p^\dagger \Big(1-\frac{\mathcal{N}_+}{N}\Big)(\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger) - \frac{1}{2}\sum_{|p|\leq N^\alpha} \varepsilon_p \tau_p \nu_p^s \Big(1-\frac{\mathcal{N}_+}{N}\Big) b_p^\dagger (\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger) \\
&\quad - \frac{1}{2}\sum_{|p|\leq N^\alpha} \varepsilon_p \tau_p \gamma_p^s (\gamma_p^s b_p^\dagger + \nu_p^s b_{-p}) b_{-p}^\dagger \Big(1-\frac{\mathcal{N}_+}{N}\Big) - \frac{1}{2}\sum_{|p|\leq N^\alpha} \varepsilon_p \tau_p \gamma_p^s (\gamma_p^s b_p^\dagger + \nu_p^s b_{-p}) \Big(1-\frac{\mathcal{N}_+}{N}\Big) b_{-p}^\dagger \\
&\quad + \frac{1}{2N}\sum_{|p|,|q|\leq N^\alpha} \varepsilon_p \tau_q \nu_p^s \big[b_q^\dagger a_{-q}^\dagger a_{-p} + a_q^\dagger a_{-p} b_{-q}^\dagger\big](\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger) \\
&\quad + \frac{1}{2N}\sum_{|p|,|q|\leq N^\alpha} \varepsilon_p \tau_q \gamma_p^s (\gamma_p^s b_p^\dagger + \nu_p^s b_{-p}) \big[b_q^\dagger a_{-q}^\dagger a_p + a_q^\dagger a_p b_{-q}^\dagger\big] + \text{h.c.}
\end{aligned} \tag{99}$$

To compute the explicit time derivative of the observable $E(s)$, on the other hand, we notice that $d\gamma_p^s/ds = \tau_p \nu_p^s$ and $d\nu_p^s/ds = \tau_p \gamma_p^s$. Thus, we obtain

$$\frac{\partial E(s)}{\partial s} = \sum_{|p|\leq N^\alpha} \varepsilon_p \tau_p \big(\nu_p^s b_p^\dagger + \gamma_p^s b_{-p}\big)\big(\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger\big) + \sum_{|p|\leq N^\alpha} \varepsilon_p \tau_p (\gamma_p^s b_p^\dagger + \nu_p^s b_{-p})(\nu_p^s b_p + \gamma_p^s b_{-p}^\dagger).$$

Combining the last equation with equation (99), we observe that (as expected) all large contributions cancel. We find

$$\begin{aligned}
\big[\mathcal{B}_4, E(s)\big] + \frac{\partial E(s)}{\partial s} &= \frac{1}{N}\sum_{|p|\leq N^\alpha} \varepsilon_p \tau_p \nu_p^s \mathcal{N}_+ b_p^\dagger (\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger) \\
&\quad + \frac{1}{N}\sum_{|p|\leq N^\alpha} \varepsilon_p \tau_p \gamma_p^s \, (\gamma_p^s b_p^\dagger + \nu_p^s b_{-p}) \mathcal{N}_+ b_{-p}^\dagger \\
&\quad + \frac{1}{N}\sum_{|p|,|q|\leq N^\alpha} \varepsilon_p \tau_q \nu_p^s \, b_q^\dagger a_{-q}^\dagger a_{-p} (\gamma_p^s b_p + \nu_p^s b_{-p}^\dagger) \\
&\quad + \frac{1}{N}\sum_{|p|,|q|\leq N^\alpha} \varepsilon_p \tau_q \gamma_p^s \, (\gamma_p^s b_p^\dagger + \nu_p^s b_{-p}) b_q^\dagger a_{-q}^\dagger a_p + \text{h.c.}
\end{aligned}$$

Using the bounds in Lemma 17, with the estimate $\varepsilon_p \lesssim p^2$ and the restrictions $|p|, |q| \leq N^\alpha$, we arrive at

$$\pm\Big\{\big[\mathcal{B}_4, E(s)\big] + \frac{\partial E(s)}{\partial s}\Big\} \lesssim N^{2\alpha-1}(\mathcal{N}_+ + 1)^2.$$

From equation (98), this implies that

$$|f_\psi'(s)| \lesssim N^{2\alpha-1}\langle \psi, e^{-s\mathcal{B}_4}(\mathcal{N}_+ + 1)^2 e^{s\mathcal{B}_4}\psi\rangle.$$

Applying Lemma 18, we obtain the desired bound (96). □

6. Optimal BEC and proof of Theorem 1

Let us denote

$$\widetilde{H}_N = H_N - 4\pi\mathfrak{a}_N(N-1) + \frac{1}{2}\sum_p \left[\sqrt{|p|^4 + 16\pi\mathfrak{a}_N p^2} - p^2 - 8\pi\mathfrak{a}_N + \frac{(8\pi\mathfrak{a}_N)^2}{2p^2}\right], \tag{100}$$

and

$$E_\infty = \sum_p \sqrt{|p|^4 + p^2 16\pi \mathfrak{a}_N}\, a_p^\dagger a_p. \tag{101}$$

Moreover, let $\mathcal{U} = e^{\mathcal{B}_2} e^{\mathcal{B}_3} e^{\mathcal{B}_4}$. Observe that $\mathcal{U}$ is a unitary operator. From Proposition 19, we have

$$\mathcal{U}^\dagger \widetilde{H}_N \mathcal{U} = E_\infty + e^{-\mathcal{B}_4} Q_4 e^{\mathcal{B}_4} + \mathcal{E}_{\mathcal{B}_4}, \tag{102}$$

where $\mathcal{E}_{\mathcal{B}_4}$ satisfies the bound (95). To prove that the error term $\mathcal{E}_{\mathcal{B}_4}$ is small, we show first that low-energy states exhibit complete Bose-Einstein condensation.

Proposition 20 (Optimal BEC). *On* $\{\mathcal{N} = N\}$, *we have*

$$H_N \geq 4\pi \mathfrak{a}_N N + C^{-1}\mathcal{N}_+ - C, \tag{103}$$

for some constant $C > 0$ *independent of N.*

Proof. To take care of the terms on the second line of the bound (95), we use localisation in the number of particles, a tool developed in [17] and, in the present setting, in [7]. Here, we make use of the results of [18, 19, 24], which imply that, if $\psi_N \in L^2_s(\Lambda^N)$ is a normalised sequence of approximate ground states of the Hamilton operator H_N satisfying

$$\left| \frac{1}{N}\langle \psi_N, H_N \psi_N\rangle - 4\pi \mathfrak{a}_N \right| \to 0$$

as $N \to \infty$, then ψ_N exhibit condensation, in the sense that

$$\lim_{N\to\infty} \frac{1}{N}\langle \psi_N, \mathcal{N}_+ \psi_N\rangle = 0. \tag{104}$$

Now let $f, g : \mathbb{R} \to [0,1]$ be smooth functions such that $f(s)^2 + g(s)^2 = 1$ for all $s \in \mathbb{R}$, and $f(s) = 1$ for $s \leq 1/2$, $f(s) = 0$ for $s \geq 1$. For $M_0 \geq 1$, we define $f_{M_0}(\mathcal{N}_+) = f(\mathcal{N}_+/M_0)$ and $g_{M_0}(\mathcal{N}_+) = g(\mathcal{N}_+/M_0)$. Then we have

$$H_N = f_{M_0} H_N f_{M_0} + g_{M_0} H_N g_{M_0} + \mathcal{E}_{M_0}, \tag{105}$$

with

$$\mathcal{E}_{M_0} = \frac{1}{2}\big([f_{M_0},[f_{M_0},H_N]] + [g_{M_0},[g_{M_0},H_N]]\big). \tag{106}$$

In view of equation (9), we can write (with $h = f, g$)

$$\begin{aligned}
[h_{M_0},[h_{M_0},H_N]] = &\left[h(\mathcal{N}_+/M_0) - h((\mathcal{N}_+-2)/M_0)\right]^2 \sum_{p\neq 0} \hat{V}_N(p) a_p^\dagger a_{-p}^\dagger a_0 a_0 + \text{h.c.}, \\
&+ \left[h(\mathcal{N}_+/M_0) - h((\mathcal{N}_+-1)/M_0)\right]^2 \sum_{q,r,q+r\neq 0} \hat{V}_N(r) a_{q+r}^\dagger a_{-r}^\dagger a_q a_0 + \text{h.c.}
\end{aligned}$$

This easily implies that

$$\pm \mathcal{E}_{M_0} \lesssim M_0^{-2}(Q_4 + N)\mathbb{1}^{\{M_0/3 \leq \mathcal{N}_+ \leq 2M_0\}}. \tag{107}$$

We choose $M_0 = \varepsilon N$, for some $\varepsilon > 0$ independent of N, to be fixed later. We introduce the notation $\mathcal{N}_+^{\mathcal{U}} = \mathcal{U}^\dagger \mathcal{N}_+ \mathcal{U}$. We use equation (102) with the bound (95) for the error term $\mathcal{E}_4$; we pick

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$\alpha = -\log \ell / \log N$ so that $N^\alpha = 1/\ell$, for some $\ell > 0$ independent of N, to be specified below. For N large enough, we obtain from Proposition 19

$$\begin{aligned} f_{M_0}(\mathcal{N}_+) H_N f_{M_0}(\mathcal{N}_+) &= \mathcal{U} f_{M_0}(\mathcal{N}_+^{\mathcal{U}}) \mathcal{U}^\dagger H_N \mathcal{U} f_{M_0}(\mathcal{N}_+^{\mathcal{U}}) \mathcal{U}^\dagger \\ &\geq \mathcal{U} f_{M_0}(\mathcal{N}_+^{\mathcal{U}}) \Big(4\pi \mathfrak{a}_N N + H_1 - C + e^{-\mathcal{B}_4} Q_4 e^{\mathcal{B}_4} + \mathcal{E}_4 \Big) f_{M_0}(\mathcal{N}_+^{\mathcal{U}}) \mathcal{U}^\dagger \\ &\geq \mathcal{U} f_{M_0}(\mathcal{N}_+^{\mathcal{U}}) \big(4\pi \mathfrak{a}_N N + (1 - C\ell^{1/2} - C\ell^{-5/2} N^{-1} - C\ell^{-7/2}\varepsilon)(H_1 + 1) \\ &\qquad\qquad + (1 - C\ell^{1/2}) e^{-\mathcal{B}_4} Q_4 e^{\mathcal{B}_4} \big) f_{M_0}(\mathcal{N}_+^{\mathcal{U}}) \mathcal{U}^\dagger \\ &\geq f_{M_0}(\mathcal{N}_+)^2 (4\pi \mathfrak{a}_N N - C + C^{-1} \mathcal{N}_+), \end{aligned} \tag{108}$$

choosing first $\ell > 0$ small enough and then $\varepsilon > 0$ sufficiently small. Here, we used $(\mathcal{N}_+ + 1)^j \lesssim (\mathcal{N}_+^{\mathcal{U}} + 1)^j$ (as follows from Lemma 3, Lemma 10 and Lemma 18), to estimate the error terms on the second line of (95). Moreover, we used the bounds $\mathcal{N}_+, \mathcal{N}_+^{\mathcal{U}} \lesssim H_1$.

On the other hand, following an argument from [7, Prop. 6.1], we find

$$g_{M_0}(\mathcal{N}_+)(H_N - 4\pi \mathfrak{a}_N N) g_{M_0}(\mathcal{N}_+) \geq C^{-1} \mathcal{N}_+ g_{M_0}(\mathcal{N}_+)^2. \tag{109}$$

Indeed, otherwise, we could find a normalised sequence Ψ_N, supported on $\{\mathcal{N}_+ > \varepsilon N\}$, satisfying

$$\left| \frac{1}{N} \langle \psi_N, H_N \psi_N \rangle_{\Psi_N} - 4\pi \mathfrak{a}_N \right| \to 0$$

as $N \to \infty$, in contradiction with (104).

Finally, we deal with the error term $\mathcal{E}_{M_0 = \varepsilon N}$. For ψ_N with $\langle \psi_N, H_N \psi_N \rangle \leq CN$, we immediately find, from the bound (107), that

$$\langle \psi_N, \mathcal{E}_{M_0 = \varepsilon N} \psi_N \rangle \lesssim \varepsilon^{-2} N^{-1}.$$

Since the bound (103) holds trivially on states with $\langle \psi_N, H_N \psi_N \rangle \geq CN$, this, together with the estimates (108) and (109), concludes the proof of Proposition 20. □

With Proposition 19 and Proposition 20, we are now ready to show Theorem 1, determining the low-energy spectrum of the operator Hamilton operator H_N.

Proof of Theorem 1. We continue to use the notation $\widetilde{H}_N$ and E_∞ introduced in equations (100) and (101). Moreover, we denote by $\lambda_1(\widetilde{H}_N) \leq \lambda_2(\widetilde{H}_N) \leq \ldots$ and $\lambda_1(E_\infty) \leq \lambda_2(E_\infty) \leq \ldots$ the ordered eigenvalues of $\widetilde{H}_N$ and, respectively, E_∞. We now choose $L \in \mathbb{N}$, with $\lambda_L(\widetilde{H}_N) \leq \Theta$ for some $1 \leq \Theta \leq N^{1/17}$. Then we claim that

$$\lambda_L(\widetilde{H}_N) = \lambda_L(E_\infty) + \mathcal{O}(\Theta N^{-1/17}). \tag{110}$$

Since $\lambda_0(E_\infty) = 0$, the estimate (110) shows that the ground state energy E_N of H_N satisfies the estimate (5). It is then easy to check, using (110), that the excitations of $H_N - E_N$ satisfy the claim in (4).

To prove equation (110), we show first a lower bound and then a matching upper bound. We again use Proposition 19, but this time we choose the exponents $\alpha = 2/17$.

Lower bound on $\lambda_L(\widetilde{H}_N)$. We again use the localisation identity given by equation (105), but this time we take $M_0 = N^{1/2+1/34}$. Let Y denote the subspace generated by the first L eigenfunctions of $\widetilde{H}_N$, and let us denote $Z = \mathcal{U}^\dagger Y$, which is of dimension L. From the decomposition (105), we have

$$\lambda_L(\widetilde{H}_N) \geq P_Y \Big(f_{M_0}(\mathcal{N}_+) \widetilde{H}_N f_{M_0}(\mathcal{N}_+) + g_{M_0}(\mathcal{N}_+) \widetilde{H}_N g_{M_0}(\mathcal{N}_+) + \mathcal{E}_{M_0} \Big) P_Y. \tag{111}$$

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From Proposition 20, we have

$$g_{M_0}(\mathcal{N}_+)\widetilde{H}_N g_{M_0}(\mathcal{N}_+) \geq C g_{M_0}^2(\mathcal{N}_+)(C^{-1}M_0 - C) \geq 0$$

for N large enough (recall the choice $M_0 = N^{1/2+1/34}$). Here, we used that g_{M_0} is supported on $\mathcal{N}_+ > M_0/3$. Moreover, with the bound (107), we find

$$P_Y \mathcal{E}_{M_0} P_Y \geq -CM_0^{-2} P_Y(Q_4 + N)P_Y \geq -CM_0^{-2} P_Y(H_N + N)P_Y \geq -CM_0^2 N \geq -CN^{-1/17}$$

because (from the upper bound), we know that $H_N \leq CN$ on Y. From the estimate (111), we obtain

$$\lambda_L(\widetilde{H}_N) \geq P_Y f_{M_0}(\mathcal{N}_+)\widetilde{H}_N f_{M_0}(\mathcal{N}_+)P_Y - CN^{-1/17}.$$

We now use Proposition 19 to estimate

$$\begin{aligned} &P_Y f_{M_0}(\mathcal{N}_+)\,\widetilde{H}_N\, f_{M_0}(\mathcal{N}_+)P_Y \\ &\qquad \geq \mathcal{U}P_Z f_{M_0}(\mathcal{N}_+^{\mathcal{U}})\,\mathbb{1}^{\{\mathcal{N}_+\leq N\}}\big(e^{\mathcal{B}_4}(E_\infty + \mathcal{E}_4)e^{-\mathcal{B}_4} + Q_4\big)\mathbb{1}^{\{\mathcal{N}_+\leq N\}}\, f_{M_0}(\mathcal{N}_+^{\mathcal{U}})P_Z\mathcal{U}^\dagger. \end{aligned}$$

Using that $\mathcal{N}_+ \lesssim H_1 \leq E_\infty$ and the choices $M_0 = N^{1/2+1/34}$, $\alpha = 2/17$, we find

$$\begin{aligned} &P_Y f_{M_0}(\mathcal{N}_+)\,\widetilde{H}_N\, f_{M_0}(\mathcal{N}_+)P_Y \\ &\qquad \geq \Big(1 - CN^{-1/17}\Big)\mathcal{U}P_Z f_{M_0}(\mathcal{N}_+^{\mathcal{U}})e^{\mathcal{B}_4}\, E_\infty\, e^{-\mathcal{B}_4} f_{M_0}(\mathcal{N}_+^{\mathcal{U}})P_Z\mathcal{U}^\dagger. \end{aligned}$$

Now it turns out that for N large enough, $\dim f_{M_0}(\mathcal{N}_+^{\mathcal{U}})P_Z = L$ because

$$\max_{\xi\in P_Z} \frac{\left\|\sqrt{1 - f_{M_0}(\mathcal{N}_+^{\mathcal{U}})^2}\,\xi\right\|^2}{\|\xi\|^2} \leq CM_0^{-1} \max_{\xi\in P_Y} \frac{\|\mathcal{N}_+^{1/2}\xi\|^2}{\|\xi\|^2} \leq C\lambda_L(\widetilde{H}_N)M_0^{-1} \underset{N\to\infty}{\longrightarrow} 0;$$

see, for instance, [17, Prop. 6.1 ii)]. Thus

$$\begin{aligned} \lambda_L(\widetilde{H}_N) &\geq \max_{\xi\in Y} \frac{\langle e^{-\mathcal{B}_4} f_{M_0}(\mathcal{N}_+^{\mathcal{U}})P_Z\mathcal{U}^\dagger\xi, E_\infty e^{-\mathcal{B}_4} f_{M_0}(\mathcal{N}_+^{\mathcal{U}})P_Z\mathcal{U}^\dagger\xi\rangle}{\|\xi\|^2} - C\Theta N^{-1/17} \\ &\geq \max_{\xi\in e^{-\mathcal{B}_4} f_{M_0}(\mathcal{N}_+^{\mathcal{U}})Z} \frac{\langle\xi, E_\infty\xi\rangle}{\|\xi\|^2}(1 - C\lambda_L(\widetilde{H}_N)M_0^{-1}) - C\Theta N^{-1/17} \\ &\geq \min_{\dim X = L}\max_{\xi\in X} \frac{\langle\xi, E_\infty\xi\rangle}{\|\xi\|^2}(1 - C\lambda_L(\widetilde{H}_N)M_0^{-1}) - C\Theta N^{-1/17} \\ &\geq \lambda_L(E_\infty)(1 - C\Theta N^{-1/2}) - C\Theta N^{-1/17}, \end{aligned}$$

which implies $\lambda_L(\widetilde{H}_N) \geq \lambda_L(E_\infty) + \mathcal{O}(\Theta N^{-1/17})$.

Upper bound on $\lambda_L(\widetilde{H}_N)$. Let Z denote the subspace generated by the first L eigenfunctions of E_∞ and P_Z be the orthogonal projection onto Z. The normalised eigenfunctions of E_∞ have the form

$$\xi = \prod_{j=1}^{k} \frac{a^\dagger(p_j)^{n_j}}{\sqrt{n_j!}}\Omega \tag{112}$$

for some $k \geq 1$, $p_j \in \Lambda_+^*$, $n_j \geq 1$ and where Ω is the vacuum. Note that

$$P_Z\mathcal{N}_+P_Z \lesssim P_Z H_1 P_Z \leq P_Z E_\infty P_Z \leq \lambda_L(E_\infty) \leq C\Theta \leq CN^{1/17},$$

where we used the lower bound that we proved above. Since $[P_Z, \mathcal{N}_+] = 0$, this bound can also be applied to powers of $\mathcal{N}_+$. Note also that $e^{\mathcal{B}_4} E_\infty e^{-\mathcal{B}_4}$ almost commutes with $\mathcal{N}_+$ in the sense that

$$\mathbb{1}^{\{\mathcal{N}_+ \le N\}} e^{\mathcal{B}_4} E_\infty e^{-\mathcal{B}_4} \mathbb{1}^{\{\mathcal{N}_+ \le N\}} - e^{\mathcal{B}_4} E_\infty e^{-\mathcal{B}_4} \le \frac{1}{2} \sum_p 8\pi \mathfrak{a}_N \chi_{|p| \le N^\alpha} [\mathbb{1}^{\{\mathcal{N}_+ > N\}} a^\dagger_p a^\dagger_{-p} + a_p a_{-p} \mathbb{1}^{\{\mathcal{N}_+ > N\}}]$$
$$\le C N^{3\alpha/2 - 1} (\mathcal{N}_+ + 1)^2.$$

Hence, we have

$$P_Z e^{-\mathcal{B}_4} \mathbb{1}^{\{\mathcal{N}_+ \le N\}} e^{\mathcal{B}_4} E_\infty e^{-\mathcal{B}_4} \mathbb{1}^{\{\mathcal{N}_+ \le N\}} e^{\mathcal{B}_4} P_Z \le P_Z \Big(E_\infty + C N^{3\alpha/2-1} (\mathcal{N}_+ + 1)^2 \Big) P_Z$$
$$\le \lambda_L(E_\infty) + C\Theta N^{-1/17}.$$

Together with Proposition 19, we find

$$\lambda_L(E_\infty) + C\Theta N^{-1/17} \ge P_Z e^{-\mathcal{B}_4} \mathbb{1}^{\{\mathcal{N}_+ \le N\}} \Big(\mathcal{U}^\dagger \widetilde{H}_N \mathcal{U} - e^{\mathcal{B}_4} \mathcal{E}_4 e^{-\mathcal{B}_4} - Q_4 \Big) \mathbb{1}^{\{\mathcal{N}_+ \le N\}} e^{\mathcal{B}_4} P_Z. \tag{113}$$

Again because

$$\max_{\xi \in e^{\mathcal{B}_4} P_Z} \frac{\|\mathbb{1}^{\{\mathcal{N}_+ > N\}} \xi\|^2}{\|\xi\|^2} \le C N^{-1} \max_{\xi \in P_Z} \frac{\|e^{-\mathcal{B}_4} \mathcal{N}_+^{1/2} \xi\|^2}{\|\xi\|^2} \le C \Theta_0 N^{-1} \underset{N\to\infty}{\longrightarrow} 0,$$

we have $\dim \mathbb{1}^{\{\mathcal{N}_+ \le N\}} e^{\mathcal{B}_4} P_Z = L$ for N large enough. With Lemma 18, we obtain

$$P_Z e^{-\mathcal{B}_4} \mathbb{1}^{\{\mathcal{N}_+ \le N\}} \Big(e^{\mathcal{B}_4} \mathcal{E}_4 e^{-\mathcal{B}_4} + Q_4 \Big) \mathbb{1}^{\{\mathcal{N}_+ \le N\}} e^{\mathcal{B}_4} P_Z \lesssim N^{-1/17} \Theta + P_Z Q_4 P_Z.$$

To estimate $P_Z Q_4 P_Z$, we use an argument from [6, Lemma 6.1]: from $P_Z E_\infty P_Z \le \Theta$, we must have $a_p \xi = 0$ for all $|p| > \Theta^{1/2}$ and $\xi \in Z$. This implies that

$$\langle \xi, Q_4 \xi \rangle \le \sum_{p,q,r} \hat{V}_N(r) \chi_{|r| \le \Theta^{1/2}} \|a_{p+r} a_q \xi\| \|a_p a_{q+r} \xi\|$$
$$\le C \Theta^{3/2} N^{-1} \|(\mathcal{N}_+ + 1)\xi\|^2 \le C \Theta^{7/2} N^{-1} \|\xi\|^2 \le C N^{-1/17} \|\xi\|^2,$$

for all $\xi \in Z$. Applying the min-max principle, we conclude from the estimate (113) that

$$\lambda_L(E_\infty) \ge \max_{\xi \in \mathcal{U} \mathbb{1}^{\{\mathcal{N}_+ \le N\}} e^{\mathcal{B}_4} Z} \frac{\langle \xi, \widetilde{H}_N \xi \rangle}{\|\xi\|^2} - C\Theta N^{-1/17}$$
$$\ge \min_{\dim X = L} \max_{\xi \in X} \frac{\langle \xi, \widetilde{H}_N \xi \rangle}{\|\xi\|^2} - C\Theta N^{-1/17}$$
$$\ge \lambda_L(\widetilde{H}_N) - C\Theta N^{-1/17}.$$

□

Acknowledgements. We thank P.T. Nam for insightful discussions.

Conflict of Interest. None.

Financial Support. B.S. gratefully acknowledges support from the European Research Council through the ERC Advanced Grant CLaQS. Additionally, B.S. acknowledges partial support from the NCCR SwissMAP and the Swiss National Science Foundation through the grant 'Dynamical and energetic properties of Bose-Einstein condensates'.

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https://doi.org/10.1017/fms.2022.78 Published online by Cambridge University Press