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TORSION-FREE AND DIVISIBLE MODULES OVER FINITE-DIMENSIONAL ALGEBRAS

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ABSTRACT. If R is a Dedekind domain, then *div* splits *i.e.*; the maximal divisible submodule of every R-module M is a direct summand of M. We investigate the status of this result for some finite-dimensional hereditary algebras. We use a torsion theory which permits the existence of torsion-free divisible modules for such algebras. Using this torsion theory we prove that the algebras obtained from extended Coxeter-Dynkin diagrams are the only such hereditary algebras for which *div* splits. The field of rational functions plays an essential role. The paper concludes with a new type of infinite-dimensional indecomposable module over a finite-dimensional wild hereditary algebra.

1. **A family of indecomposable torsion-free divisible modules.** A $Kr(n)$-module **V** is a pair of K-vector spaces (V, W) and a K-bilinear map from $K^n \times V$ to W. In this paper K is an algebraically closed field. The bilinear map gives, for a fixed $e \in K^n$, a linear map $T_e\colon V \longrightarrow W$. We write $T_e(v)$ simply as ev. The module **V** is *torsion-free* (respectively, *divisible*) if for every nonzero $e \in K^n$, T_e is one-to-one, (respectively, surjective). Similar definitions have been used in [1], [2], [3] and [7]. We cannot use the torsion theory in [9] because it allows too many torsion-free modules while that in [10] allows no torsion-free divisible modules.

Throughout the paper, $\{e_1, e_2, \ldots, e_n\}$ will be an arbitrary but fixed basis of K^n. Let M be a $K[\zeta]$-module. Make (M, M) a $Kr(n)$-module by setting

$$e_i m = \zeta^{i-1} m. \tag{1}$$

PROPOSITION 1.1. *The $K[\zeta]$-module M is torsion-free if and only if (M, M) is a torsion-free $Kr(n)$-module.*

PROOF. Suppose that M is not a torsion-free $K[\zeta]$-module. Then for some $m \in M$, $m \neq 0$, and some nonzero polynomial $f(\zeta)$, $f(\zeta)m = 0$. Among such annihilators of m, pick one, $p(\zeta)$ of minimal degree. Since $m \neq 0$, $p(\zeta)$ is not a constant. Let α be a root of $p(\zeta)$. Let $p(\zeta) = (\zeta - \alpha)g(\zeta)$. By the choice of $p(\zeta)$, $g(\zeta)m \neq 0$. Let $e = e_2 - \alpha e_1$. Then $T_e g(\zeta)m = 0$. Therefore, (M, M) is not torsion-free.

Suppose $e = \sum_{i=1}^{i=n} \alpha_i e_i \neq 0$, $\alpha_i \in K$, and $T_e m = 0$ for some nonzero $m \in M$. Then $f(\zeta) = \sum_{i=1}^{i=n} \alpha_i \zeta^{i-1} \neq 0$ and $f(\zeta)m = 0$. ∎

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Let α be a nonzero element of K. Let $f_i = e_i$, $i = 1, 2, \ldots, n-1$, and $f_n = \alpha e_n$. Use $\{f_1, \ldots, f_n\}$ in place of $\{e_1, e_2, \ldots, e_n\}$ in (1). In this way we get functors from the category of $K[\zeta]$-modules to the category of $Kr(n)$-modules. These functors have all the properties in Proposition 7.51 of [6]. Denote the $Kr(n)$-module $\big(K(\zeta), K(\zeta)\big)$ by $\mathcal{R}_\alpha$.

PROPOSITION 1.2. *(a) The endomorphism ring of $\mathcal{R}_\alpha$ is isomorphic to the field $K(\zeta)$.*
(b) The module $\mathcal{R}_\alpha$ is indecomposable, torsion-free, and divisible.
(c) Let α and β be nonzero elements of K. Then for $n \geq 3$, $\mathrm{Hom}(\mathcal{R}_\alpha, \mathcal{R}_\beta) \neq 0$ if and only if $\alpha = \beta$.

COROLLARY 1.3. *Suppose that $n \geq 3$. Then there is an infinite family $\mathcal{M}$ of non-isomorphic indecomposable torsion-free divisible $Kr(n)$-modules.*

PROPOSITION 1.4. *No proper nonzero submodule of $\mathcal{R}_\alpha$ is divisible.*

PROOF. If we restrict the operation to e_1 and e_2, then $\mathcal{R}_\alpha = \big(K(\zeta), K(\zeta)\big)$ may be considered as the unique indecomposable torsion-free $Kr(2)$-module, $Q = \big(K(\zeta), K(\zeta)\big)$. Similarly the submodules of $\mathcal{R}_\alpha$ may be considered as submodules of Q. The proposition is true for Q by Proposition 9.8 of [1]. Suppose that (X, Y) is a nonzero divisible submodule of $\mathcal{R}_\alpha$. Then, (X, Y) is a divisible $Kr(2)$-submodule of Q. Hence, $X = Y = K(\zeta)$. So $(X, Y) = \mathcal{R}_\alpha$ as $Kr(n)$-modules. ∎

Restricting the action to $\{e_1, e_2\}$ we get directly or from Proposition 9.8 of [1]:

LEMMA 1.5. *Let (X, Y) be a nonzero proper submodule of $\mathcal{R}_\alpha$. Then $\mathcal{R}_\alpha / (X, Y)$ is not torsion-free.*

Let $(U, Z) = ([1, \zeta], [1, \zeta, \ldots, \zeta^n])$ with $e_i 1 = \zeta^{i-1}$; $e_i \zeta = \zeta^i$, $i = 1, 2, \ldots, n$. So (U, Z) is a submodule of $\mathcal{R}_1 = \mathcal{R}$.

LEMMA 1.6. *The endomorphism ring of (U, Z) is K.*

PROOF. Let $(\phi, \psi)\colon (U, Z) \longrightarrow (U, Z)$ be an endomorphism. Let $\phi(1) = \alpha 1 + \beta\zeta$. Since $e_n 1 = \zeta^{n-1}$ we get that $\psi(\zeta^{n-1}) = \alpha\zeta^{n-1} + \beta\zeta^n$. Now, $e_{n-1}\zeta = \zeta^{n-1}$. Let $u = \phi(\zeta)$. Then $e_{n-1}u = \psi(e_{n-1}\zeta) = \alpha\zeta^{n-1} + \beta\zeta^n$. However, if $\beta \neq 0$, there would be no such element u, because $e_{n-1}f$ has degree less than n for every f in U. So $\beta = 0$ and (ϕ, ψ) is given by multiplication by α in U and Z. ∎

The Ext formula in (48) of [5], (hom in the formula should read Hom) gives,

$$\dim \mathrm{Ext}\big((U, Z), (U, Z)\big) = \dim \mathrm{Hom}\big((U, Z), (U, Z)\big) - 2^2 - (n+1)^2 + 2n(n+1).$$

Since $n \geq 3$ and $\dim \mathrm{Hom}\big((U, Z), (U, Z)\big) \geq 1$ we get that $\dim \mathrm{Ext}\big((U, Z), (U, Z)\big) \neq 0$. Since $Kr(n)$ is hereditary and $(U, Z) \subset \mathcal{R}$ we get that $\mathrm{Ext}\big(\mathcal{R}, (U, Z)\big) \neq 0$.

The following is a projective resolution of (U, Z):

$$0 \longrightarrow (K, L) \overset{(\kappa, \lambda)}{\longrightarrow} (P, Q) \longrightarrow (U, Z) \longrightarrow 0.$$

where $(P, Q) = ([u_1], [z_1, \ldots, z_n]) \oplus ([v_1], [w_1, \ldots, w_n])$, $e_i u_1 = z_i$, $e_i v_1 = w_i$; $(K, L) = (0, [z_2 - w_1, z_3 - w_2, \ldots, z_n - w_{n-1}])$ and (κ, λ) is the inclusion. See [4] or [5, Proposition 0.2] for the projectivity of (P, Q) and (K, L).

If $(\mu, \nu) \in \mathrm{Hom}\big((P, Q), \mathcal{R}\big)$ let $(\kappa, \lambda)^*(\mu, \nu) = (\mu, \nu)(\kappa, \lambda)$.

THEOREM 1.7. *(a)* $\text{Ext}\big((U,Z),\mathcal{R}\big) \neq 0$.

(b) $\text{Ext}(\mathcal{R},\mathcal{R}) \neq 0$.

(c) There is a $Kr(n)$-module M with the property that div M *is not a direct summand of M.*

PROOF. (a) $\text{Ext}\big((U,Z),\mathcal{R}\big) = \text{Hom}\big((K,L),\mathcal{R}\big)/(\kappa,\lambda)^*\Big(\text{Hom}\big((P,Q),\mathcal{R}\big)\Big)$. Let $\psi(z_{i+1}-w_i) = 1$, $i = 1,\ldots,n-1$. Then $(0,\psi) \in \text{Hom}\big((K,L),\mathcal{R}\big)$. However for any $(\mu,\nu) \in \text{Hom}\big((P,Q),\mathcal{R}\big)$ it follows from (1) that $\nu(z_{i+1}-w_i) = \zeta^i f - \zeta^{i-1} g$, where $\mu(u_1) = f$, $\mu(v_1) = g$. Since $n > 2$, $\text{Ext}\big((U,Z),\mathcal{R}\big) = 0$ would contradict $\psi(z_{i+1}-w_i) = 1, i = 1,\ldots,n-1$.

(b) Follows from (a) because $Kr(n)$ is hereditary and $(U,Z) \subset \mathcal{R}$.

(c) Let M be a nonsplit extension of $\mathcal{R}$ by (U,Z) guaranteed by (a). Since (U,Z) is reduced by Proposition 1.4, we have $\mathcal{R}$ as the maximal divisible submodule of M, div M, and is not a direct summand of M. ∎

The status of Theorem 1.7(c) is unknown when *divisible* is used in the sense of [10], see [10, Section 5].

THEOREM 1.8. *There is a purely simple extension, (V,W), of $\mathcal{R}$ by $\mathcal{R}$.*

PROOF. By Theorem 1.7(b), there is a nonsplit extension (V,W) of (V_1,W_1) by (V_2,W_2) where both (V_1,W_1) and (V_2,W_2) are isomorphic to $\mathcal{R}$. Let $(\pi,\rho)\colon (V,W) \longrightarrow (V_2,W_2)$ be the projection. We shall prove by contradiction that (V,W) is purely simple. We shall use (P1): If N is a pure submodule of a torsion-free module M then M/N is torsion-free. Let (X,Y) be an infinite-dimensional pure submodule of (V,W), $(X,Y) \neq (V,W)$. If $(X,Y)\cap(V_1,W_1) \neq 0$, then (P1) and (1) lead to $(V_1,W_1) \subseteq (X,Y)$. By Property (a') of [13, Section F], $(X,Y)/(V_1,W_1)$ is pure in (V_2,W_2). By (P1) and Lemma 1.5 we get that $(X,Y) = (V,W)$ or $(X,Y) = (V_1,W_1)$. Since $K(\zeta)$ is a pure-injective $K[\zeta]$-module, we get by [6, Theorem 7.51] that (V_1,W_1) is a direct summand of (V,W), contradicting Proposition 1.7(b). So we may assume that $(X,Y)\cap(V_1,W_1) = 0$. Therefore, (π,ρ) restricts to an embedding of (X,Y) into (V_2,W_2). Note that a pure submodule of a torsion-free divisible module is divisible. So by Proposition 1.4, $(X,Y) \cong (V_2,W_2)$ via (π,ρ). Hence, (V,W) is a split extension, a contradiction. ∎

The functor in [8] transfers the module in Theorem 1.8 to an arbitrary wild finite-dimensional hereditary algebra. If R is a Dedekind domain or a tame finite-dimensional hereditary algebra, then no extension of an infinitely generated R-module by itself is purely simple, see [7] and [12]. So Theorem 1.8 is a bona fide *wild* theorem.

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