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RATE OF APPROXIMATION OF FUNCTIONS OF BOUNDED VARIATION BY MODIFIED LUPAS OPERATORS

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This paper discusses the rate of approximation of functions of bounded variation using the Modified Lupas operator. We obtain an approximation theorem and our estimate is essentially the best possible.

1. INTRODUCTION

Let f be a function defined on $(0, \infty)$ with bounded variation in each finite interval and $f(x) = O(x^r)$ $x \to \infty$. We denote $\{f\}$ by $BV_{\mathrm{loc},r}(0, \infty)$. The Modified Lupas operator M_n applied to f is

$$M_n(f, x) = (n-1)\sum_{k=0}^{\infty} P_{nk}(x)\int_0^{\infty} P_{nk}(t)f(t)\,dt \tag{1.1}$$

where
$$P_{nk}(t) = \binom{n+k-1}{k}\frac{t^k}{(1+t)^{n+k}}.$$

It is also written as

$$\int_0^{\infty} H_n(x, t)f(t)\,dt \tag{1.2}$$

where
$$H_n(x, t) = (n-1)\sum_{k=0}^{\infty} P_{nk}(x)P_{nk}(t).$$

This kernel is positive and

$$\int_0^{\infty} H_n(x, t)\,dt = 1, \tag{1.3}$$

therefore $M_n(f, x)$ is linear positive [1].

In this paper, the main result is the following:

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THEOREM. *Let f be a function belonging to $BV_{\mathrm{loc},r}(0,\infty)$. Then for every $x \in (0,\infty)$ and n sufficiently large, we have*

(1.4)
$$\left|M_n(f,x) - \frac{1}{2}(f(x+) - f(x-))\right| \leqslant \frac{5(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-\frac{x}{\sqrt{k}}}^{x+\frac{x}{\sqrt{k}}} (g_x)$$
$$+ \frac{50}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2} |f(x+) - f(x-)| + \frac{(1+x)^r}{x^4} O\left(\frac{1}{n^2}\right),$$

where
$$g_x(t) = \begin{cases} f(t) - f(x+) & x < t < \infty, \\ 0 & t = x, \\ f(t) - f(x-) & 0 \leqslant t < x. \end{cases}$$

The proof of the theorem will use the Bojanic method [2] and some results of probability theory.

2. Lemmas

LEMMA 1. *If $\{\xi_k\}$ $(k \geqslant 1)$ are independent random variables with the same distribution functions and $0 < D\xi_k < \infty$, $\beta_3 = E(\xi_r - \xi_i)^3 < \infty$, then*

(2.1)
$$\max_y \left| P\left(\frac{1}{b_1\sqrt{n}} \sum_{k=1}^{n} (\xi_k - a_1) \leqslant y\right) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2} dt \right| < \frac{c}{\sqrt{n}} \frac{\beta_3}{b_1^3}$$

where $a_1 = E(\xi_1)$ (expectation of ξ_1), $b_1^2 = D\xi_1 = E(\xi_1 - E\xi_1)^2$ and $1/\sqrt{2\pi} \leqslant c <$ 0.82 (see [3]).

LEMMA 2. [3] *If $\{\xi_i\}$ are independent random variables with the same geometric distribution functions $P(\xi_i = k) = x^k(1-x)$, $i = 1, 2, \ldots$, then $E\xi_i = x/(1-x)$, $D\xi_i = x/(1-x)^2$, $\eta_n = \sum_{i=1}^{n} \xi_i$ with*

(2.2)
$$P(\eta_n = k) = \binom{n+k-1}{k} x^k (1-x)^n.$$

LEMMA 3. *For every $x \in (0, +\infty)$, $k \in \mathbb{N}$, we have*

(2.3)
$$P_{nk}(x) \leqslant \frac{33}{\sqrt{n}} \left(\frac{1+x}{x}\right)^{3/2}.$$

PROOF: Since

$$P_{nk}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}} \qquad x \in (0,\infty),$$

we take $t = x/(1+x)$; then $t \in (0,1)$ and

$$P_{nk}(x) = \binom{n+k-1}{k} t^k (1-t)^n. \tag{2.4}$$

Using Lemma 2 we have

$$\binom{n+k-1}{k} t^k (1-t)^n = P(k-1 < \eta_n \leqslant k) = P(w(k-1,n) < w(\eta_n, n) \leqslant w(k,n))$$

where $w(k,n) = (x(1-t) - nt)/\sqrt{nt}$. Using Lemma 1 we have $a_1 = t/(1-t)$, $b_1 = \sqrt{t}/(1-t)$. Hence

$$\left| \binom{n+k-1}{k} t^k (1-t)^n - \frac{1}{\sqrt{2\pi}} \int_I e^{-t^2/2} dt \right| < 2 \frac{\beta_3}{\sqrt{n}\left(\sqrt{t}/(1-t)\right)^3} \tag{2.5}$$

where $I = [w(k-1,n),\, w(k,n)]$,

$$\beta_3 = E\left(\xi_k - \frac{t}{1-t}\right)^3 = \sum_{k=0}^{\infty} \left(k - \frac{t}{1-t}\right)^3 t^k (1-t)$$

$$\leqslant \sum_{k=0}^{\infty} \left[k^3 + 3k^2 \frac{t}{1-t} + 3k\left(\frac{t}{1-t}\right)^2 + \left(\frac{t}{1-t}\right)^3\right] t^k (1-t).$$

Since

$$\sum_{k=0}^{\infty} t^k (1-t) = 1, \qquad \sum_{k=0}^{\infty} k t^k (1-t) = \frac{t}{1-t},$$

$$\sum_{k=0}^{\infty} k^2 t^k (1-t) = \frac{t(1+t)}{1-t^2}, \qquad \sum_{k=0}^{\infty} k^3 t^k (1-t) = \frac{t^3 + 4t^2 + t}{(1-t)^3},$$

therefore $\beta_3 \leqslant 16/(1-t)^3$.

But the second term on the left side of (2.5) is not greater than $(1-t)/\sqrt{2\pi nt}$. Hence we have

$$\binom{n+k-1}{k} t^k (1-t)^n \leqslant \frac{33}{\sqrt{n} t^{3/2}};$$

therefore

$$P_{nk}(x) \leqslant \frac{33}{\sqrt{n}} \left(\frac{1+x}{x}\right)^{3/2}, \quad \text{and (2.3) is proved.}$$

□

Lemma 4. *For every $k \geqslant 0$ we have*

$$\sum_{j=0}^{k} P_{n-1,j}(x) = (n-1) \int_x^{\infty} P_{nk}(t) dt.$$

PROOF: This can easily be proved by differentiating both the left-hand and right-hand sides. ∎

LEMMA 5. *If n is sufficiently large, for every $k \geqslant 0$ we have*

$$\left|\sum_{j=0}^{k} P_{n-1,j}(x) - \sum_{j=0}^{k} P_{nj}(x)\right| \leqslant \frac{33}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}.$$

PROOF: From (2.2) we have $\sum_{j=0}^{k} P_{nj}(x) = P(\eta_n \leqslant k)$. Again using Lemma 1 we have

$$\left|\sum_{j=0}^{k} P_{nj}(x) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{w(k,n)} e^{-u^2/2}du\right| < \frac{16}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2};$$

hence

$$\begin{aligned}\left|\sum_{j=0}^{k} P_{n-1,j}(x) - \sum_{j=0}^{k} P_{nj}(x)\right| \\ < \left|\frac{1}{\sqrt{2\pi}}\int_{w(k,n)}^{w(k,n-1)} e^{-u^2/2}du\right| + \frac{32}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2} \leqslant \frac{33}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}.\end{aligned}$$

∎

LEMMA 6. *If n is sufficiently large and $x \in (0, \infty)$, then*

$$\frac{x(1+x)}{n} \leqslant M_n\Big((t-x)^2, x\Big) \leqslant \frac{3x(1+x)}{n}; \tag{2.8}$$

$$M_n\Big((t-x)^4, x\Big) = O\left(\frac{1}{n^2}\right). \tag{2.9}$$

PROOF: Let

$$T_{nm} = (n-r+1)\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^{\infty} P_{n-r,k+r}(t)(t-x)^m dt. \tag{2.10}$$

Then

$$\begin{aligned}T_{n2} &= (n-r+1)\sum_{k=0}^{\infty} P_{n+r,k}(x)\int_0^{\infty} P_{n+r,k}(x)\int_0^{\infty} P_{n-r,k+r}(t)(t-x)^2 dt \\ &= \frac{2(n-1)x(1+x)}{(n-r-2)(n-r-3)} + \frac{(r+1)(r+2)(1+2x)^2}{(n-r-2)(n-r-3)}.\end{aligned}$$

Putting $r = 0$ we have

$$
(2.11)\qquad
\begin{aligned}
T_{nm} &= (n+1)\sum_{k=0}^{\infty} P_{nk}(x)\int_0^{\infty} P_{nk}(t)(t-x)^m dt = O\left(\frac{1}{n^{[(m+1)/2]}}\right),\\
T_{n2} &= \frac{2(n-1)x(1+x)}{(n-2)(n-3)} + \frac{2(1+2x)^2}{(n-2)(n-3)}.
\end{aligned}
$$

From this, (2.8) and (2.9) can be proved (see [4]). □

LEMMA 7. *In n is sufficiently large and $x \in (0, \infty)$, then for $0 \leqslant y < x$, we have*

$$
(2.12)\qquad \int_0^y H_n(x, t)dt \leqslant \frac{3x(1+x)}{n(x-y)^2};
$$

for $x < z < \infty$, we have

$$
(2.13)\qquad \int_z^{\infty} H_n(x, t)dt \leqslant \frac{3x(1+x)}{n(z-x)^2}.
$$

PROOF: Since $0 \leqslant y < x$, for $t \in [0, y]$ we have $(x-t)/(x-y) \geqslant 1$. From

$$
M_n(f, x) = \int_0^{\infty} H_n(x, t)f(t)dt
$$

we have

$$
\frac{x(1+x)}{n} \leqslant \int_0^{\infty} H_n(x, t)(t-x)^2 dt \leqslant \frac{3x(1+x)}{n};
$$

therefore

$$
\begin{aligned}
\int_0^y H_n(x, t)dt &\leqslant \int_0^y \left(\frac{x-t}{x-y}\right)^2 H_n(x, t)dt\\
&\leqslant \frac{1}{(x-y)^2}\int_0^{\infty} (x-t)^2 H_n(x, t)dt \leqslant \frac{3x(1+x)}{n(x-y)^2},
\end{aligned}
$$

proving (2.12). The proof of (2.13) is similar. □

3. PROOF OF THE THEOREM

Now

$$
f(t) = \frac{f(x+)+f(x-)}{2} + g_x(t) + \frac{f(x+)-f(x-)}{2}\,\mathrm{sign}\,(t-x);
$$

hence

$$
(3.1)\qquad
\begin{aligned}
&\left|M_n(f, x) - \frac{1}{2}(f(x+)+f(x-))\right|\\
&\qquad \leqslant |M_n(g_x, x)| + \frac{1}{2}\,|f(x+)-f(x-)|\,|M_n(\mathrm{sign}(t-x), x)|.
\end{aligned}
$$

Thus to estimate $|M_n(f, x) - (f(x+) - f(x-))/2|$ we need an estimate for $M_n(g_x, x)$ and $M_n(\mathrm{sign}(t-x), x)$.

To estimate $M_n(\mathrm{sign}(t-x), x)$, we first decompose it into two parts as follows:

$$M_n(\mathrm{sign}(t-x), x) = \int_0^\infty \mathrm{sign}(t-x) H_n(x, t)\,dt = \int_x^\infty H_n(x, t)\,dt - \int_0^x H_n(x, t)\,dt \overset{\mathrm{def}}{=} A_n(x) - B_n(x).$$

Using Lemma 4, we have

$$\begin{aligned} A_n(x) &= \int_x^\infty H_n(x, t)\,dt = \int_x^\infty (n-1)\sum_{k=0}^\infty P_{nk}(x)P_{nk}(t)\,dt \\ &= (n-1)\sum_{k=0}^\infty P_{nk}(x)\int_x^\infty P_{nk}(t)\,dt \\ &= \sum_{k=0}^\infty P_{nk}(x)\sum_{j=0}^k P_{n-1,j}(x). \end{aligned}$$

By Lemma 5, it follows that

$$\begin{aligned} &\left|A_n(x) - \sum_{k=0}^\infty P_{nk}(x)\sum_{j=0}^k P_{nj}(x)\right| \\ &= \left|\sum_{k=0}^\infty P_{nk}(x)\sum_{j=0}^k P_{nj}(x) - \sum_{k=0}^\infty P_{nk}(x)\sum_{j=0}^k P_{n-1,j}(x)\right| \\ &\leqslant \sum_{k=0}^\infty P_{nk}(x)\frac{33}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2} \leqslant \frac{33}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}. \end{aligned} \tag{3.2}$$

Let

$$\begin{aligned} S &= \sum_{k=0}^\infty \left(P_{nk}(x)\sum_{j=0}^k P_{nj}(x)\right) \\ &= P_{n0}P_{n0} + P_{n1}(P_{n0} + P_{n1}) + \cdots + P_{nn}(P_{n0} + \cdots + P_{nn}) + \cdots. \end{aligned}$$

Since

$$\begin{aligned} 1 &= (P_{n0} + P_{n1} + \cdots + P_{nm} + \cdots)(P_{n0} + P_{n1} + \cdots + P_{nm} + \cdots) \\ &= P_{n0}(P_{n0} + P_{n1} + \cdots + P_{nm} + \cdots) + P_{n1}(P_{n0} + \cdots) + P_{nm}(P_{n0} + \cdots) + \cdots, \end{aligned}$$

we have

$$1 - S = P_{n0}(P_{n1} + P_{n2} + \cdots) + P_{n1}(P_{n2} + P_{n3} + \cdots) + P_{nm}(P_{n,m+1} + P_{n,m+2} + \cdots) + \cdots$$
$$= P_{n1}P_{n0} + P_{n2}(P_{n0} + P_{n1}) + \cdots + P_{nm}(P_{n0} + P_{n1} + \cdots + P_{n,m-1}) + \cdots,$$

and
$$2S - 1 = P_{n0}^2 + P_{n1}^2 + \cdots + P_{nm}^2 + \cdots.$$

Using Lemma 3, we obtain

$$\left|S - \frac{1}{2}\right| = \frac{1}{2}(P_{n0}^2 + P_{n1}^2 + \cdots) + \frac{1}{2}\sum_{k=0}^{\infty} P_{nk}^2 \leqslant \frac{1}{2}\sum_{k=0}^{\infty} P_{nk}\left[\frac{33}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}\right] \leqslant \frac{33}{2\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}. \tag{3.3}$$

By (3.2), (3.3) and $B_n(x) = 1 - A_n(x)$, we have

$$|A_n(x) - B_n(x)| = |2A_n(x) - 1| \leqslant \frac{100}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}.$$

Hence

$$|M_n(\operatorname{sign}(t-x), x)| \leqslant \frac{100}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}. \tag{3.4}$$

The estimate of $M_n(g_x, x)$ is similar to [4]. We first decompose $[0, +\infty)$ into three parts, as follows:

$$I_1 = \left[0, x - \frac{x}{\sqrt{n}}\right], \quad I_2 = \left[x - \frac{x}{\sqrt{n}}, x + \frac{x}{\sqrt{n}}\right], \quad I_3 = \left[x + \frac{x}{\sqrt{n}}, \infty\right).$$

Then

$$M_n(g_x, x) = \int_0^\infty g_x(t)H_n(x,t)\,dt = \left(\int_{I_1} + \int_{I_2} + \int_{I_3}\right) g_x(t)H_n(x,t)\,dt \overset{\text{def}}{=} \Delta_{1,n}(f,x) + \Delta_{2,n}(f,x) + \Delta_{3,n}(f,x).$$

First, we estimate $\Delta_{2,n}(f,x)$. For $t \in I_2$, we have

$$|g_x(t)| = |g_x(t) - g_x(x)| \leqslant \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} (g_x)$$

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and so

$$\text{(3.5)}\qquad |\Delta_{2,n}(f,x)| \leqslant \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x)\int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} H_n(x,t)dt \leqslant \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x).$$

Secondly, we estimate $\Delta_{1,n}(f,x)$. Let $\lambda_n(x,t)=\int_0^t H_n(x,u)du$. We have

$$\begin{aligned}|\Delta_{1,n}(f,x)| &= \left|\int_0^{x-x/\sqrt{n}} g_x(t)H_n(x,t)dt\right| = \left|\int_0^y g_x(t)H_n(x,t)dt\right| \\ &= \left|\int_0^y g_x(t)d_t\lambda_n(x,t)\right| \\ &= \left|g_x(y+)\lambda_n(x,y) - \int_0^y \lambda_n(x,t)d_t g_x(t)\right| \\ &\leqslant \bigvee_{y+}^{x}(g_x)\lambda_n(x,y) + \int_0^y \lambda_n(x,t)d_t\left(-\bigvee_t^x(g_x)\right),\end{aligned}$$

where

$$y = x - \frac{x}{\sqrt{n}}.$$

By Lemma 7, we have

$$|\Delta_{1,n}(f,x)| \leqslant \bigvee_{y+}^{x}(g_x)\frac{3x(1+x)}{n(x-y)^2} + \frac{3x(1+x)}{n}\int_0^y \frac{1}{(x-t)^2}d_t\left(-\bigvee_t^x(g_x)\right)$$

where n is sufficiently large.

Since

$$\int_0^y \frac{1}{(x-t)^3}d_t\left(-\bigvee_t^x(g_x)\right) = -\frac{\mathsf{V}_{y+}^{x}(g_x)}{(x-y)^2} + \frac{\mathsf{V}_0^x(g_x)}{x^2} + 2\int_0^y \bigvee_t^x(g_x)\frac{dt}{(x-t)^3},$$

we have

$$\begin{aligned}|\Delta_{1,n}(f,x)| &\leqslant \frac{3x(1+x)}{n}\left[\bigvee_{y+}^{x}(g_x)\Big/(x-y)^2 + \int_0^y \frac{1}{(x-t)^2}d_t\left(-\bigvee_t^x(g_x)\right)\right] \\ &\leqslant \frac{3x(1+x)}{n}\left[\bigvee_0^x(g_x)\Big/x^2 + 2\int_0^{x-x/\sqrt{n}}\bigvee_t^x(g_x)\frac{dt}{(x-t)^3}\right].\end{aligned}$$

Furthermore, since

$$\int_0^{x-\frac{x}{\sqrt{n}}}\bigvee_t^x(g_x)\frac{dt}{(x-t)^2} + \int_1^n \bigvee_{x-x/\sqrt{T}}^{x}(g_x)\frac{1}{x^3/T^{3/2}}\frac{1}{2}x\frac{1}{T^{3/2}}dT \leqslant \frac{1}{2x^2}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x),$$

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it follows that

$$(3.6) \qquad |\Delta_{1,n}(f, x)| \leqslant \frac{3(1+x)}{nx}\left[\bigvee_0^x (g_x) + \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^x (g_x)\right].$$

Last, we estimate $\Delta_{3,n}(f, x)$. Since

$$\begin{aligned} |\Delta_{3,n}(f, x)| &= \left|\int_{x+x/\sqrt{n}}^\infty g_x(t)H_n(x, t)dt\right| \\ &= \left|\left[\int_{x+x/\sqrt{n}}^{2x} + \int_{2x}^\infty\right] g_x(t)H_n(x, t)dt\right| \stackrel{\text{def}}{=} R_{1,n} + R_{2,n}, \end{aligned}$$

using similar methods as above, we have

$$(3.7) \qquad |R_{1,n}| \leqslant \frac{4(1+x)}{nx}\sum_{k=1}^n \bigvee_x^{x+x/\sqrt{k}} (g_x).$$

Since $g_x(t) = 0(t^r)$ $(t \to \infty)$, there exists $M > 0$ such that

$$(3.8) \qquad \begin{aligned} |R_{2n}| &\leqslant M\int_{2x}^\infty t^r H_n(x, t)dt \leqslant M\int_{|t-x|\geqslant x} t^r H_n(x, t)dt \\ &\leqslant \frac{M}{x^4}\sum_{k=0}^\infty (n-1)\int_0^\infty P_{nk}(x)P_{nk}(t)t^r(t-x)^4 dt. \end{aligned}$$

From

$$\begin{aligned} &\frac{(n-1)P_{nk}(t)P_{nk}(x)t^r}{(n-r-1)P_{n+r,k}(x)P_{n-r,k+r}(t)} \\ &\qquad = (1+x)^r\frac{(k+1)\cdots(k+r)(n-1)n\cdots(n+r-1)}{(k+n)\cdots(k+n+r-1)(n-r-1)\cdots(n-1)} \end{aligned}$$

it follows that, for every k and $n > r+1$, we have

$$\frac{(k+1)\cdots(k+r)}{(k+n)\cdots(k+n+r-1)} < 1, \qquad \lim_n \frac{(n-1)\cdots(n+r-1)}{(n-r-1)\cdots(n-1)} = 1.$$

Hence if n is sufficiently large, then

$$(n-1)P_{nk}(x)P_{nk}(t)t^r \leqslant (1+x)^r(n-r-1)P_{n+r,k}(x)P_{n-r,k+r}(t).$$

Replacing above in (3.6), we have

$$\begin{aligned}|R_{2n}| &\leqslant \frac{M}{x^4}\sum_{k=0}^{\infty}\int_0^{\infty}(1+x)^r(n-r+1)P_{n+r,k}(x)P_{n-r,k+r}(t)(t-x)^4\,dt \\ &\leqslant \frac{M}{x^4}(1+x)^r(n-r+1)\sum_{k=0}^{\infty}P_{n+r,k}(x)\int_0^{\infty}P_{n-r,k+r}(t)(t-x)^4\,dt \\ &= \frac{M(1+x)^r}{x^4}T_{n4}.\end{aligned}$$

Using (2.11), we obtain

$$|R_{2n}| \leqslant \frac{(1+x)^r}{x^4}O\left(\frac{1}{n^2}\right). \tag{3.9}$$

From (3.5), (3.6), (3.7) and (3.9), it follows that

$$\begin{aligned}|M_n(g_x, x)| \leqslant{}& \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) + \frac{3(1+x)}{nx}\left[\bigvee_0^x(g_x) + \sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x}(g_x)\right] \\ &+ \frac{4(1+x)}{nx}\sum_{k=1}^{n}\bigvee_{x}^{x+x/\sqrt{k}}(g_x) + \frac{(1+x)^r}{x^4}O\left(\frac{1}{n^2}\right),\end{aligned} \tag{3.10}$$

where n is sufficiently large.

Our theorem now follows from (3.1), (3.4), (3.10); that is

$$\begin{aligned}&\left|M_n(f, x) - \frac{1}{2}[f(x+) + f(x-)]\right| \\ &\leqslant \frac{50}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}|f(x+) - f(x-)| \\ &\quad + \frac{(1+x)^r}{x^4}O\left(\frac{1}{n^2}\right) \qquad + \frac{4(1+x)}{nx}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \\ &\leqslant \frac{50}{\sqrt{n}}\left(\frac{1+x}{x}\right)^{3/2}|f(x+) - f(x-)| \\ &\quad + \frac{5(1+x)}{nx}\sum_{k=1}^{n}\bigvee_{x-x\sqrt{k}}^{x+x/\sqrt{k}}(g_x) + \frac{(1+x)^r}{x^4}O\left(\frac{1}{n^2}\right).\end{aligned}$$

4. Remark

We shall prove that our estimate is essentially the best possible. Consider the function $f(t) = |t - x|$ $(0 < x < \infty)$ on $[0, \infty)$. It obviously belongs to $BV_{\mathrm{loc},r}(0, \infty)$. Since

$$M_n(f, x) = \int_0^\infty H_n(x, t)f(t)dt = \int_0^\infty (n-1)\left[\sum_{k=0}^\infty P_{nk}(x)P_{nk}(t)\right] f(t)dt,$$

by Lemma 6, for any small $\delta > 0$ and n sufficiently large, we have

$$\begin{aligned} M_n(|t-x|, x) &= (n-1)\sum_{k=0}^\infty P_{nk}(x)\left[\int_{x-\delta}^{x+\delta} + \int_{|t-x|>\delta}\right] |t-x|\, P_{nk}(t)dt \\ &\leqslant \delta + \frac{1}{\delta}M_n\left((t-x)^2, x\right) \leqslant \delta + \frac{1}{n\delta}3x(1+x); \end{aligned} \tag{4.1}$$

and

$$\begin{aligned} M_n(|t-x|, x) &\geqslant \int_{x-\delta}^{x+\delta} |t-x|\, H_n(x, t)dt \geqslant \frac{1}{\delta}\int_{x-\delta}^{x+\delta} (t-x)^2 H_n(x, t)dt \\ &\geqslant \frac{x(1+x)}{n\delta} - \frac{1}{\delta}\int_{|t-x|>\delta} (t-x)^2 H_n(x, t)dt. \end{aligned}$$

Furtherfore, using Lemma 6, we have

$$\int_{|t-x|>\delta} (t-x)^2 H_n(x, t)dt \leqslant \frac{1}{\delta^2}M_n\left((t-x)^4, x\right) \leqslant \frac{c_1}{\delta^2 n^2}.$$

Hence

$$M_n(|t-x|, x) \geqslant \frac{x(1+x)}{n\delta} - \frac{c_1}{n^2\delta^2}. \tag{4.2}$$

Choose $\delta = 2\sqrt{c_1/nx(1+x)}$. We obtain from (4.1) and (4.2) that

$$\frac{3[x(1+x)]^{3/2}}{8\sqrt{c_1 n}} \leqslant M_n(|t-x|, x) \leqslant \frac{2[c_1 + (x(1+x))^2]}{\sqrt{c_1 n x}}. \tag{4.3}$$

For $f(t) = |t-x|$, putting $t = x$, we have

$$\begin{aligned} &\left|M_n(f, x) - \frac{1}{2}(f(x+) + f(x-))\right| \\ &\quad = |M_n(|t-x|, x)| \leqslant \frac{3(1+x)}{nx}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (f) + \frac{(1+x)}{x^4}O\left(\frac{1}{n^2}\right). \end{aligned}$$

Since

$$\bigvee_{x-\beta}^{x+\alpha}(f) = \alpha + \beta,$$

it follows that

$$M_n(|t-x|, x) \leqslant \frac{4(1+x)}{nx} \sum_{k=1}^{n} \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(f) \tag{4.4}$$
$$= \frac{8(1+x)}{n} \sum_{k=1}^{n} \frac{1}{\sqrt{k}} \leqslant \frac{8(1+x)}{\sqrt{n}}.$$

By comparing (4.3) and (4.4), we see that (1.2) cannot be asymptotically improved for $BV_{\mathrm{loc},r}(0, \infty)$; that is, our estimate is essentially the best possible.

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