CORRESPONDENCE.

ON THE VALUE OF REVERSIONARY ANNUITIES PAYABLE HALF YEARLY OR QUARTERLY

To the Editor of the Assurance Magazine.

SIR,—In an extract from the writings of the late Dr. Thomas Young, contained in the last Number of the *Assurance Magazine*, it is stated—"if it be required to ascertain the value of a reversionary annuity payable half yearly or quarterly, the calculation becomes a little paradoxical," because the same addition is made to both terms of the formula A—AB. Now, since the addition for payment consists—1st, of the value of interest if he survive the year, viz., $A\left(\frac{\overline{p-1}+\overline{p-2}+.+2+1}{p}i\right)\frac{1}{p}$, or $A\frac{p-1}{2p}i$ (1), where p represents the number of intervals and A an annuity; 2nd, of the chance of losing the payments and interest, viz.:—

If he die in the second interval, $A_2\frac{1}{p^2}\left(1+\frac{i}{2p}\right)$,

supposing death to occur in the middle: A_n to signify the present value of £1 if he die in the nth interval, $\frac{1}{p}$ the chance of his so dying.

If he die in the third interval, $A_3\frac{1}{p^2}\left(1+\frac{i}{2p}+\frac{i}{p}+1+\frac{i}{2p}\right)$

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" pth " $A_p\frac{1}{p^2}\left(1+\frac{i}{2p}+\frac{\overline{p-2}\,i}{p}+1+\frac{i}{2p}+\frac{\overline{p-3}\,i}{p}+..+1+\frac{i}{2p}\right)$

$$\text{or, }\left(\frac{1}{p^2}+\frac{i}{2p^3}\right)(A_2+2A_3+3A_4\ldots+\overline{p-1}A_p)(\alpha)+\frac{1}{p^2}\left(A_3\frac{i}{p}+A_4\Sigma\frac{2i}{p}+\ldots+A_p\Sigma\frac{\overline{p-2}i}{p}\right)$$

$$\text{or, }\quad\ldots\ldots\quad+\frac{1}{p^2}\Sigma A_{p-k+2}\left(\frac{p-k+1}{p}\right)i\left(\frac{p-k}{2}\right)(\beta)$$

from $k=p$ to $k=2$.

If the values of A_p increase in arithmetical progression, difference S.

$$(\alpha)=\frac{1}{p^2}\left(1+\frac{i}{2p}\right)\left\{A_2\overline{1+2+3+\ \ldots\ +\overline{p-1}}\left(=\frac{p.\overline{p-1}}{2}\right)\right.$$
$$\left.+S\,\overline{2+2.3+3.4+4.5.+\overline{p-2}.\overline{p-1}}\left(=\frac{p.\overline{p-1}.\overline{p-2}}{3}\right)\right\}.$$

$$\beta=\frac{i}{2p^3}\left\{A_2\overline{1+2^2+3^2+\ldots+\overline{p-2}^2}+S\,\overline{+2^2+2.3^2+\ldots+\overline{p-3}.\overline{p-2}^2}\right.$$
$$\left.+A_2\overline{1+2+3\ldots+p-2}+S\,\overline{+2+\ldots+\overline{p-3}.\overline{p-2}}\right\},$$

$$=\frac{i}{2p^3}\left\{A_2\frac{p.\overline{p-1}.\overline{p-2}}{3}+S\overline{p-1}.\overline{p-2}.\overline{p-3}.+\frac{1}{4}\overline{p-1}.\overline{p-2}.\overline{p-3}.\overline{p-4}\right)\right\}.$$

But let S vanish, or A_n be constant, then the series become

$$A_n\left(\frac{p-1}{2p}+\frac{i}{12p^2}\,\overline{2p-1}.\overline{p-1}.\right)\quad.\quad(2)$$

Now the difference for the single and joint lives is, from (1) (A—AB) $\frac{\overline{p-1}i}{2p}$, from (2)$-\frac{(A-AB)i}{1+\frac{i}{2}}\left(\frac{p-1}{2p}+\text{terms affected by }i\right)$, which, being nearly equal and having different signs, destroy one another.

Let the annuity be payable p times in a year, commencing at the end of the pth interval in which A may have died; and if the chance of dying in either of the p intervals be the same, it is evident that for every chance of death taking place so as to receive $p-1$ payments before the end of the year, there is another equal chance of only receiving at the end of the year £$\frac{1}{p}$, or an annuity with $p-1$. payments deferred; and the increase in the value of an annuity, in consequence of the first payment being anticipated p intervals, is nearly equal to the loss if it be deferred in like manner.* Let the annuity be payable half yearly, then there are two chances, either of receiving the first payment of the annuity at the end of half a year in which death has taken place—*i. e.*, an annuity of £$\frac{1}{2}$ anticipated half a year, which is equivalent to an ordinary half yearly annuity $=A-AB+\frac{\mathfrak{AB}}{\frac{1}{4}}$; or, of receiving an annuity of £$\frac{1}{2}$ deferred half a year$=A-AB-\frac{\mathfrak{AB}}{\frac{1}{4}}$;and these two are supposed to have equal chances.

In a paper by Mr. Ivory, read before the Institute of Actuaries and printed in the *Journal*, vol. iv., page 299, the formula given for this annuity is

$$a_{m_1}-a_{m_1m}-\frac{A_{m_{(1)}m_1}}{4},$$

which is correct on the assumption that the annuity forgone during the joint lives is considered as payable annually, and implies that the first payment takes place half a year, on an average, after death; but it involves the conclusion that a mode of payment which would be selected by the annuitant for his convenience, and apparently to his advantage, should be a saving of expense. The fallacy arises from not observing that when p is the interval of payment, death is supposed to occur at the interval $\frac{1}{2}p$, and that by analogy to this result $am_1-a_{m_1m}$ should be reduced by $\frac{A_{m_{(1)}m}}{2}$ to render the first annual instalment payable one year, on an average, after death: but if this be not stipulated for, and the formula $a_{m_1}-a_{m_1m}$ be retained in the case of annual payments, the first half yearly payment should be considered as commencing at the end of a quarter of a year; and if it be by agreement deferred a quarter of a year more, the proper deduction is $\frac{Am_{(1)}m_1}{4}$.

* For let r be the period anticipated, A (an annuity to commence at the end of $2r)+.5$ is its value$=(1+A_p)\times t$, a fraction generally nearly equal to unity; then A_pt=annuity deferred for period of $r=(1+A_p)t-1$ nearly.$=A+.5-1=A-.5$.

If, in addition to the general advantage of exactness in such questions, there be any other reason for considering the present subject, it is, that if these reversionary annuities (*e.g.*, on a wife's life after her husband's death) should become a desirable and common form of investment, more frequent than annual payments would generally be selected.

I shall be much obliged by your insertion of the above in your very valuable *Journal*, and am,

Your obedient Servant,

THOMAS CARR.

London, 9*th June*, 1857.

MR. SCRATCHLEY ON POST OBITS.

To the Editor of the Assurance Magazine.

SIR,—I have no wish to add to the criticisms on Mr. Scratchley, already so ably done in the last Number of your *Magazine;* but, if you do not think that the subject has already occupied too much of your space, I would beg to supply an omission of the reviewers'.

I do not think they have given sufficient prominence to the simplicity of the formulæ applicable to the case—where, as in the ordinary way of negotiating such a transaction, the elements to be considered are the actual premiums and annuity values (without regard to their mathematical relation) that may be charged by the responsible Offices from whom the life benefits are purchased.

This is easiest shown, by solving the inquiry of what annuity should be given to the borrower for a specific reversion to be assured.

Let the sum be £1, and the first payment of annuity to the borrower to be made (as most usual) immediately. With an investment of £1, the lender purchases an annuity, which would be payable during joint lives only in a case like Mr. Scratchley's, stipulating that the first payment be made "down"; and the Office purchase price for such an annuity of £1 being a, the annuity purchased is $\frac{1}{a}$: he retains from this the annual premium for assurance of £$1=\pi$, and a year's interest, r, discounted for a year$=vr$, paying the difference to the borrower, whose annuity is therefore

$$\frac{1}{a}-(\pi+rv)=\frac{1}{a}+v-(1+\pi);$$

and its arithmetical complement, or

$$\frac{1}{1\div a+v-(1+\pi)},$$

is the post obit to be secured for annuity of £1.

Should it be preferred to effect the assurance by single premium A, this, of course, has to be subtracted from the £1 invested, and the annuity paid to the borrower will be

$$\frac{1-\text{A}}{a}-1+v;$$