1

Semigroups and Generators

1.1 Motivation from Partial Differential Equations

Consider the following initial value problem on $\mathbb{R}^d$:

$$\frac{\partial u(t,x)}{\partial t} = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}^2 u(t,x) + \sum_{i=1}^{d} b_i(x)\partial_i u(t,x) - c(x)u(t,x),$$
$$u(0,\cdot) = f(\cdot), \tag{1.1.1}$$

where $f \in C_c^2(\mathbb{R}^d)$. Here we have used the simplifying notation ∂_i for $\frac{\partial}{\partial x_i}$ and ∂_{ij} for $\frac{\partial^2}{\partial x_i \partial x_j}$. We will assume that c, b_i and $a_{j,k}$ are bounded smooth (i.e., infinitely differentiable) functions on $\mathbb{R}^d$ for each $i, j, k = 1, \ldots, d$ with $c \geq 0$, and that the matrix-valued function $a = (a_{ij})$ is uniformly elliptic in that there exists $K > 0$ so that

$$\inf_{x \in \mathbb{R}^d} a(x)\xi \cdot \xi \geq K|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d,$$

with $(a_{ij}(x))$ being a symmetric matrix for each $x \in \mathbb{R}^d$.

The point of these conditions is to ensure that (1.1.1) has a unique solution.[1] We make no claims that they are, in any sense, optimal. We rewrite (1.1.1) as an abstract ordinary differential equation:

$$\frac{du_t}{dt} = Au_t,$$
$$u_0 = f, \tag{1.1.2}$$

[1] See, e.g., Chapter 6, section 5 of Engel and Nagel [31].

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where A is the linear operator

$$(Ag)(x) = \sum_{i,j=1}^{d} a_{ij}(x)\partial_{ij}^2 g(x) + \sum_{i=1}^{d} b_i(x)\partial_i g(x) - c(x)g(x),$$

acting on the linear space of all functions $g\colon \mathbb{R}^d \to \mathbb{R}$ that are at least twice differentiable.

The original PDE (1.1.1) acted on functions of both time and space and the solution u is a function from $[0, \infty) \times \mathbb{R}^d \to \mathbb{R}$. In (1.1.2), we have hidden the spatial dependence within the structure of the operator A and our solution is now a family of functions $u_t\colon E \to E$, where E is a suitable space of functions defined on $\mathbb{R}^d$. It is tempting to integrate (1.1.2) naively, write the solution as

$$u_t = e^{tA} f, \tag{1.1.3}$$

and seek to interpret e^{tA} as a linear operator in E. From the discussion above, it seems that E should be a space of twice-differentiable functions, but such spaces do not have a rich structure from a functional analytic perspective. Our goal will be to try to make sense of e^{tA} when E is a Banach space, such as $C_0(\mathbb{R}^d)$, or $L^p(\mathbb{R}^d)$ (for $p \geq 1$). If we are able to do this, then writing $T_t := e^{tA}$, we should surely have that for all $s, t \geq 0$,

$$T_{s+t} = T_s T_t \text{ and } T_0 = I.$$

We would also expect to be able to recapture A from the mapping $t \to T_t$ by $A = \frac{d}{dt}\Big|_{t=0} T_t$. Note that if $E = L^2(\mathbb{R}^d)$, then we are dealing with operators on a Hilbert space, and if we impose conditions on the b_i's and a_{jk}'s such that A is self-adjoint,[2] then we should be able to use spectral theory to write

$$A = \int_{\sigma(A)} \lambda dE(\lambda),\ T_t = \int_{\sigma(A)} e^{t\lambda} dE(\lambda),$$

where $\sigma(A)$ is the spectrum of A. We now seek to turn these musings into a rigorous mathematical theory.

1.2 Definition of a Semigroup and Examples

Most of the material given below is standard. There are many good books on semigroup theory and we have followed Davies [27] very closely.

[2] This is non-trivial as A is not a bounded operator, see below.

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Let E be a real or complex Banach space and $\mathcal{L}(E)$ be the algebra of all bounded linear operators on E. A C_0-*semigroup*[3] on E is a family of bounded, linear operators $(T_t, t \geq 0)$ on E for which

(S1) $T_{s+t} = T_s T_t$ for all $s, t \geq 0$,
(S2) $T_0 = I$,
(S3) the mapping $t \to T_t \psi$ from $[0, \infty)$ to E is continuous for all $\psi \in E$.

We briefly comment on these. (S1) and (S2) are exactly what we expect when we try to solve differential equations in Banach space – we saw this in section 1.1. We need (S3) since without it, as we will see, it is very difficult to do any analysis. For many classes of examples that we consider, $(T_t, t \geq 0)$ will be a C_0-semigroup such that T_t is a contraction[4] for all $t > 0$. In this case, we say that $(T_t, t \geq 0)$ is a *contraction semigroup*. Finally, if only (S1) and (S2) (but not (S3)) are satisfied we will call $(T_t, t \geq 0)$ an *algebraic operator semigroup*, or *AO semigroup*, for short.

The condition (S3) can be simply expressed as telling us that the mapping $t \to T_t$ is strongly continuous in $\mathcal{L}(E)$. We now show that we can replace it with the seemingly weaker condition:

(S3)$'$ The mapping $t \to T_t \psi$ from $\mathbb{R}^+$ to E is continuous at $t = 0$ for all $\psi \in E$.

Proposition 1.2.1 *If $(T_t, t \geq 0)$ is a family of bounded linear operators on E satisfying (S1) and (S2), then it satisfies (S3) if and only if it satisfies (S3)$'$.*

Before we prove Proposition 1.2.1, we will establish a lemma that we will need for its proof, and which will also be useful for us later on. For this we need the *principle of uniform boundedness*, which states that if $(B_i, i \in \mathcal{I})$ is a family of operators in $\mathcal{L}(E)$ such that the sets $\{||B_i \psi||, i \in \mathcal{I}\}$ are bounded for each $\psi \in E$, then the set $\{||B_i||, i \in \mathcal{I}\}$ is also bounded.[5]

Lemma 1.2.2

1. *If $(T_t, t \geq 0)$ is a family of bounded linear operators on E so that $t \to T_t \psi$ is continuous for all $\psi \in E$, then for all $h > 0$,*

$$c_h = \sup\{||T_t||, 0 \leq t \leq h\} < \infty.$$

2. *If $(T_t, t \geq 0)$ is a family of bounded linear operators on E so that $t \to T_t \psi$ is continuous at zero for all $\psi \in E$, then there exists $h > 0$ so that $c_h < \infty$.*

[3] The reason for the nomenclature "C_0" is historical. The founders of the subject introduced a hierarchy of semigroups of type "C_j" (see Hille [45]). Only C_0 remains in general use today.
[4] A bounded linear operator X in E is a contraction if $||X|| \leq 1$, or equivalently $||X\psi|| \leq ||\psi||$ for all $\psi \in E$.
[5] This is proved in elementary texts on functional analysis, e.g., Simon [90], pp. 398–9.

Proof. 1. From the given continuity assumption, it follows that the set $\{||T_t\psi||, 0 \leq t \leq h\}$ is bounded, for all $\psi \in E$, and then the result follows from the principle of uniform boundedness.

2. Assume that the desired conclusion is false. Then $c_h = \infty$ for all $h > 0$. So taking $h = 1, 1/2, \ldots, 1/n, \ldots$, we can always find $0 < t_n < 1/n$ so that $||T_{t_n}|| > n$. But then by the uniform boundedness theorem, there exists $\psi \in E$ so that $\{||T_{t_n}\psi||, n \in \mathbb{N}\}$ is unbounded. But $\lim_{n\to\infty} T_{t_n}\psi = T_0\psi$, and this yields the required contradiction. □

Note that in Lemma 1.2.2 (2), $c(h') \leq c(h) < \infty$ for all $0 \leq h' \leq h$, and if we assume that $T_0 = I$ therein, then $c(h') \geq 1$.

Proof of Proposition 1.2.1. Sufficiency is obvious. For necessity, let $t > 0, \psi \in E$ be arbitrary. Then for all $h > 0$, by (S1) and (S2)

$$||T_{t+h}\psi - T_t\psi|| \leq ||T_t||.||T_h\psi - \psi|| \to 0 \text{ as } h \to 0.$$

Then $t \to T_t\psi$ is right continuous from $[0, \infty)$ to E. To show left continuity, let h be as in Lemma 1.2.2 (2) so that $c_h < \infty$, and in view of the discussion after the proof of the last lemma, we take $h < t$. Since $t \to T_t\psi$ is continuous at zero, given any $\epsilon > 0$, there exists $\delta > 0$ so that if $0 < s < \delta$, then $||T_s\psi - \psi|| < \epsilon/[c_h^2(||T_{t-h}|| + 1)]$. Now choose $\delta' = \min\{\delta, h\}$. Then for all $0 < s < \delta'$,

$$\begin{aligned}
||T_t\psi - T_{t-s}\psi|| &\leq ||T_{t-s}||.||T_s\psi - \psi|| \\
&\leq ||T_{t-\delta'}||.||T_{\delta'-s}||.||T_s\psi - \psi|| \\
&< ||T_{t-\delta'}||c_{\delta'}\frac{\epsilon}{(c_h^2(||T_{t-h}|| + 1)} \\
&\leq \frac{||T_{t-h}||}{||T_{t-h}|| + 1}\frac{||T_{h-\delta'}||}{c_h}\frac{c_{\delta'}}{c_h}\epsilon \\
&< \frac{c_{h-\delta'}}{c_h}\epsilon \leq \epsilon
\end{aligned}$$

(where we have repeatedly used (S1)), and the proof is complete. □

Example 1.2.3 Take $E = \mathbb{C}$. Fix $a \in \mathbb{C}$ and define for all $t \geq 0$,

$$T_t z = e^{ta}z,$$

for each z in $\mathbb{C}$. Then $(T_t, t \geq 0)$ is a C_0-semigroup and you can check that

- if $\Re(a) < 0$, then $(T_t, t \geq 0)$ is a contraction semigroup,
- if $\Re(a) = 0$, then T_t is an isometry for all $t > 0$,
- if $\Re(a) > 0$, then $\lim_{t\to\infty} ||T_t|| = \infty$.

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Example 1.2.4 If A is a bounded operator on E, define

$$T_t = e^{tA} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n.$$

You can check in Problem 1.1 that the series is norm convergent (uniformly on finite intervals) and that $(T_t, t \geq 0)$ is a C_0-semigroup. Note that

$$||e^{tA}|| \leq \sum_{n=0}^{\infty} \frac{t^n}{n!} ||A||^n = e^{t||A||}.$$

Example 1.2.5 (The Translation Semigroup) Here we take $E = C_0(\mathbb{R})$ or $L^p(\mathbb{R})$ for $1 \leq p < \infty$ and define

$$(T_t f)(x) = f(x + t).$$

Verifying (S1) and (S2) is trivial. We will establish (S3) in Chapter 3, when this example will be embedded within a more general class, or you can prove it directly in Problem 1.4.

Example 1.2.6 (Probabilistic Representations of Semigroups) Let $(\Omega, \mathcal{F}, P)$ be a probability space and $(X(t), t \geq 0)$ be a stochastic process taking values in $\mathbb{R}^d$. We obtain linear operators on $B_b(\mathbb{R}^d)$ by averaging over those paths of the process that start at some fixed point. To be precise, we define for each $f \in B_b(\mathbb{R}^d)$, $x \in \mathbb{R}^d$, $t \geq 0$:

$$(T_t f)(x) = \mathbb{E}(f(X(t))|X(0) = x).$$

Then we may ask when does $(T_t, t \geq 0)$ become a semigroup, perhaps on a nice closed subspace of $B_b(\mathbb{R}^d)$, such as $C_0(\mathbb{R}^d)$? We will see in Chapter 7 that this is intimately related to the *Markov property*.

Example 1.2.7 (Semidynamical Systems) Let M be a locally compact space and $(\tau_t, t \geq 0)$ be a semigroup of transformations of M so that $\tau_{s+t} = \tau_s \tau_t$ for all $s, t \geq 0$, τ_0 is the identity mapping and $t \to \tau_t x$ is continuous for all $x \in M$. Then we get a semigroup on $C_0(M)$ by the prescription

$$T_t f = f \circ \tau_t.$$

If $(\tau_t, t \geq 0)$ extends to a group $(\tau_t, t \in \mathbb{R})$, then we have reversible dynamics, since the system can be run both backwards and forwards in time. If this is not the case, we have irreversible dynamics. In physics, the latter is associated with entropy production, which yields the “arrow of time”. These themes will be developed in Chapter 8.

For later work, the following general norm-estimate on C_0-semigroups will be invaluable.

Theorem 1.2.8 *For any C_0-semigroup $(T_t, t \geq 0)$, there exists $M \geq 1$ and $a \in \mathbb{R}$ so that for all $t \geq 0$,*

$$||T_t|| \leq Me^{at}. \tag{1.2.4}$$

Proof. First fix $n \in \mathbb{N}$, and define $c_n := \sup\{||T_t||, 0 \leq t \leq 1/n\} < \infty$ by Lemma 1.2.2(1). Since $||T_0|| = 1$, we must have $c_n \geq 1$. By repeated use of (S1), we deduce that $||T_t|| \leq c$, where $c := (c_n)^n$, for all $0 \leq t \leq 1$. Now for arbitrary $t \geq 0$, let $[t]$ denote the integer part of t. Again by repeated use of (S1), we have

$$\begin{aligned} ||T_t|| &\leq c^{[t]}c \\ &\leq c^{t+1} \\ &\leq Me^{at}, \end{aligned}$$

where $M := c$ and $a := \log(c)$. □

From the form of M and a obtained in the proof of Theorem 1.2.8, we see that if $(T_t, t \geq 0)$ is a contraction semigroup, then $M = 1$ and $a = 0$. But in some cases it is possible that we may also find that there exists $b > 0$ so that $||T_t|| \leq e^{-bt}$, for all $t \geq 0$, due to some additional information. Indeed (see, e.g., Engel and Nagel [32] p. 5), one may introduce the *growth type* $a_0 \geq -\infty$ of the semigroup:

$$a_0 := \inf\{a \in \mathbb{R}; \text{there exists } M_a \geq 1 \text{ so that } ||T_t|| \leq M_a e^{at} \text{ for all } t \geq 0\},$$

but we will not pursue this direction further herein.

For readers who like abstract mathematics, an (algebraic) *semigroup* is a set S that is equipped with a binary operation $*$, which is associative, i.e., $(a*b)*c = a*(b*c)$ for all $a, b, c \in S$. The semigroup is said to be a *monoid* if there is a neutral element e, so that $a * e = e * a = a$ for all $a \in S$. We have a *topological semigroup* (or monoid) if the set S is equipped with a topology such that the mapping $(a, b) \to a * b$ from $S \times S$ to S is continuous. Finally a *representation* of a topological monoid S is a mapping $\pi : S \to \mathcal{L}(E)$, where E is a real or complex Banach space so that $\pi(a * b) = \pi(a)\pi(b)$ for all $a, b \in S$, $\pi(e) = I$ and for all $\psi \in E$, the mapping $a \to \pi(a)\psi$ is continuous from S to E. Now $[0, \infty)$ is easily seen to be a topological monoid, under addition, when equipped with the usual topology inherited from $\mathbb{R}$, and what we have called a C_0-semigroup is simply a representation of $[0, \infty)$.

1.3 Unbounded Operators and Generators

1.3.1 Unbounded Operators and Density of Generators

Our next goal is to seek insight into the infinitesimal behaviour of semigroups. We will seek to answer the following question, motivated by our work in section 1.1: Is there always a linear operator A in E such that $A\psi = \left.\frac{dT_t\psi}{dt}\right|_{t=0}$, and if so, what properties does it have? We will see that the answer to the first part of the problem is affirmative. But in general, we expect A to be an unbounded operator,[6] i.e., an operator that is only defined on a linear manifold D that is a proper subset of E; indeed this was exactly the case with the second-order differential operator that we considered in section 1.1. We use the term "unbounded" for such operators, since if $||A\psi|| \leq K||\psi||$ for all $\psi \in D$, and if D is dense in E, then it is easy to see that A may be extended to a bounded operator on the whole of E (see Problem 1.8(a)). The operators that we deal with will usually have the property that D is dense, but no such bounded extension as just discussed, can exist. Let us make a formal definition of linear operator that includes both the bounded and unbounded cases.

Let D be a linear manifold in a complex Banach space E. We say that $X : D \to E$ is a linear operator with domain D if

$$X(cf + g) = cXf + Xg$$

for all $f, g \in D$ and all $c \in \mathbb{C}$. We sometimes use the notation $\mathrm{Dom}(X)$ or D_X for the space D. The operator X is said to be *densely defined* if D is dense in E. We say that a linear operator X_1 having domain D_1 is an *extension* of X if $D \subseteq D_1$ and $X_1 f = Xf$ for all $f \in D$. In this case, we also say that X is a *restriction* of X_1 to D, and we write $X \subseteq X_1$.

Example 1.3.9 Let $E = C([0, 1])$ and consider the linear operator $(Xf)(x) = f'(x)$ for $0 \leq x \leq 1$, with domain $D = C^\infty([0, 1])$. Then X is densely defined. It has an extension to the space $D_1 = C^1([0, 1])$. Consider the sequence (g_n) in D where $g_n(x) = e^{-nx}$ for all $x \in [0, 1]$. Then $||Xg_n|| = n$, and so X is clearly unbounded.

Now let $(T_t, t \geq 0)$ be a C_0-semigroup acting in E. We define[7]

$$D_A = \left\{ \psi \in E; \exists \phi_\psi \in E \text{ such that } \lim_{t \to 0} \left|\left| \frac{T_t\psi - \psi}{t} - \phi_\psi \right|\right| = 0 \right\}.$$

[6] A good reference for unbounded operators, at least in Hilbert spaces, is Chapter VIII of Reed and Simon [76]. For Banach spaces, there is a great deal of useful information to be found in Yosida [101].

[7] The limit being taken here and in (1.3.5) below is one–sided.

It is easy to verify that D_A is a linear space and thus we may define a linear operator A in E, with domain D_A, by the prescription

$$A\psi = \phi_\psi,$$

so that, for each $\psi \in D_A$,

$$A\psi = \lim_{t\to 0} \frac{T_t\psi - \psi}{t}. \tag{1.3.5}$$

We call A the *infinitesimal generator*, or for simplicity, just the *generator*, of the semigroup $(T_t, t \geq 0)$. We also use the notation $T_t = e^{tA}$, to indicate that A is the infinitesimal generator of a C_0-semigroup $(T_t, t \geq 0)$. We have already seen examples, such as Example 1.2.4 above, where this notation agrees with familiar use of the exponential mapping. We will show below that D_A is dense in E. For now, in the general case, we may only assert that $0 \in D_A$.

Example 1.3.10 It is easy to see that the generators in Examples 1.1 and 1.2 are z with domain $\mathbb{C}$, and A with domain E (respectively). For Example 1.2.5 the generator is the differentiation operator $Xf = f'$, with domain

$$D_X := \{f \in C_0^1(\mathbb{R});\ f' \in C_0(\mathbb{R})\}.$$

To see this, observe that for $f \in D_X$, given any $\epsilon > 0$, there exists $\delta > 0$ so that for all $|h| < \delta$

$$\sup_{x\in\mathbb{R}} \left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| < \epsilon,$$

and the result follows. (Hint: First take $f \in C_c^\infty(\mathbb{R})$).

In order to explore properties of A we will need to use Banach space integrals. Let $0 \leq a < b < \infty$. We wish to integrate *continuous* functions Φ from $[a, b]$ to E. As we are assuming continuity, we can define $\int_a^b \Phi(s)ds$ to be an E-valued Riemann integral, rather than using more sophisticated techniques. To be precise, $\int_a^b \Phi(s)ds$ is the unique vector in E so that for any $\epsilon > 0$ there exists a partition $a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b$ so that

$$\left\| \int_a^b \Phi(s)ds - \sum_{j=1}^{n+1} \Phi(u_j)(t_j - t_{j-1}) \right\| < \epsilon,$$

where $t_{j-1} < u_j < t_j$, for all $j = 1, \ldots, n+1$. The following basic properties will be used extensively in the sequel. They are all established by straightforward manipulations using partitions, or by variations on known arguments for the case $E = \mathbb{R}$ (see Problem 1.2).

Proposition 1.3.11 (Properties of the Riemann Integral) *Let* $\Phi\colon [0,\infty] \to E$ *be continuous.*

(RI1) *For all* $c > 0$, $\int_a^b \Phi(s+c)ds = \int_{a+c}^{b+c} \Phi(s)ds$.

(RI2) *For all* $a < c < b$, $\int_a^b \Phi(s)ds = \int_a^c \Phi(s)ds + \int_c^b \Phi(s)ds$.

(RI3) $\left|\left|\int_a^b \Phi(s)ds\right|\right| \leq \int_a^b ||\Phi(s)||ds$.

(RI4) *For all* $t \geq 0$, $\lim_{h\to 0} \frac{1}{h}\int_t^{t+h} \Phi(s)ds = \Phi(t)$.

In addition, it is a straightforward exercise to check that for any $X \in \mathcal{L}(E)$

$$X\int_a^b \Phi(s)ds = \int_a^b X\Phi(s)ds. \tag{1.3.6}$$

We will mostly want to consider the case where $[a,b] = [0,t]$ and $\Phi(s) = T_s\psi$ for some fixed vector $\psi \in E$ and C_0-semigroup $(T_t, t \geq 0)$, so the desired continuity follows from (S3). From now on we will use the notation

$$\psi(t) := \int_0^t T_s\psi ds, \tag{1.3.7}$$

and we will frequently take $X = T_s$ for some $s > 0$ in (1.3.6) to obtain

$$T_s\psi(t) = \int_0^t T_{s+u}\psi du. \tag{1.3.8}$$

The following technical lemma will be very useful for us. In particular, it tells us that D_A contains much more than just the zero vector.

Lemma 1.3.12 *For each* $t \geq 0$, $\psi \in E$, $\psi(t) \in D_A$ *and*

$$A\psi(t) = T_t\psi - \psi.$$

Proof. Using (1.3.8), (RI1), (RI2) and (RI4), we find that for each $t \geq 0$,

$$\begin{aligned}
\lim_{h\to 0}\frac{1}{h}\big[T_h\psi(t) - \psi(t)\big] &= \lim_{h\to 0}\left(\frac{1}{h}\int_0^t T_{h+u}\psi\, du - \frac{1}{h}\int_0^t T_u\psi\, du\right)\\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_h^{t+h} T_u\psi\, du - \frac{1}{h}\int_0^t T_u\psi\, du\right)\\
&= \lim_{h\to 0}\left(\frac{1}{h}\int_t^{t+h} T_u\psi\, du - \frac{1}{h}\int_0^h T_u\psi\, du\right)\\
&= T_t\psi - \psi,
\end{aligned}$$

and the result follows. □

We now show that the generator of a C_0-semigroup is always densely defined.

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Theorem 1.3.13

(1) *D_A is dense in E.*
(2) *$T_t D_A \subseteq D_A$ for each $t \geq 0$.*
(3) *$T_t A\psi = AT_t\psi$ for each $t \geq 0$, $\psi \in D_A$.*

Proof. (1) By Lemma 1.3.12, $\psi(t) \in D_A$ for each $t \geq 0$, $\psi \in E$, but by (RI4), $\lim_{t\to 0}(\psi(t)/t) = \psi$; hence D_A is dense in E as required.

For (2) and (3), suppose that $\psi \in D_A$ and $t \geq 0$; then, by the definition of A and the continuity of T_t, we have

$$\left[\lim_{h\to 0} \frac{1}{h}(T_h - I)\right] T_t\psi = \lim_{h\to 0} \frac{1}{h}(T_{t+h} - T_t)\psi$$
$$= T_t \left[\lim_{h\to 0} \frac{1}{h}(T_h - I)\right]\psi \; = \; T_t A\psi.$$

From this it is clear that $T_t\psi \in D_A$ whenever $\psi \in D_A$, and so (2) is satisfied. We then obtain (3) when we take the limit. □

1.3.2 Differential Equations in Banach Space

Let D be a dense linear manifold in E, I be an interval in $\mathbb{R}$ and $t \to \psi(t)$ be a mapping from I to D. Let $t \in I$ be such that there exists $\delta > 0$ so that $(t - \delta, t + \delta) \subseteq I$. We say that the mapping ψ is (strongly) differentiable at t if there exists $\psi'(t) \in E$ so that

$$\lim_{h\to 0} \left|\left| \frac{\psi(t+h) - \psi(t)}{h} - \psi'(t) \right|\right| = 0.$$

We then call $\psi'(t)$ the (strong) derivative of ψ at t, and (with the usual abuse of notation) we write $\frac{d\psi}{dt} := \psi'(t)$. If $O \subseteq I$ is open we say that ψ is differentiable on O if it is differentiable (in the above sense) at every point in O. By a standard argument, we can see that if ψ is differentiable on O, then it is continuous there. In principle, we could try to solve differential equations in Banach space that take the general form

$$\frac{d\psi}{dt} = F(\psi, t),$$

where $F\colon D \times O \to E$ is a suitably regular mapping. We are only going to pursue this theme in the special case where $I = [0, \infty)$, $O = (0, \infty)$, $D = D_A$ and $F(\psi, t) = A\psi$, where A is the generator of a C_0-semigroup. For more general investigations in this area, see, e.g., Deimling [28].

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Lemma 1.3.14

1. *If $f \in D_A$ then for all $t \geq 0$,*

$$T_t f - f = \int_0^t T_s A f ds. \tag{1.3.9}$$

2. *The mapping $\psi(t) = T_t f$ is a solution of the initial value problem (ivp)*

$$\frac{d\psi}{dt} = A\psi \ , \ \psi(0) = f.$$

Proof. 1. Let $\phi \in E'$ and define $F : [0, \infty) \to \mathbb{C}$ by

$$F(t) = \left\langle T_t f - f - \int_0^t T_s A f, \phi \right\rangle,$$

where we recall that $\langle \cdot, \cdot \rangle$ denotes the dual pairing between E and E'. Then the right derivative $D_+ F(t) = \langle A T_t f - T_t A f, \phi \rangle = 0$ for all $t > 0$, by Theorem 1.3.13(3) and (RI4). Since $F(0) = 0$ and F is continuous, it follows by a variation on the mean value theorem (see Lemma 1.4.4 on p. 24 of Davies [27] for details) that $F(t) = 0$ for all $t > 0$. Since ϕ is arbitrary, the result follows.

2. From (1), (RI2) and Theorem 1.3.13 (3), we have

$$\frac{1}{h}(T_{t+h} f - T_t f) = \frac{1}{h} \int_t^{t+h} A T_s f ds,$$

and the result follows when we pass to the limit using (RI4). □

In relation to Lemma 1.3.14, we would like to be able to show that $u(t, \cdot) := T_t f$ is the *unique solution* to our ivp. We will do this in the next theorem.

Theorem 1.3.15 *If A is the generator of $(T_t, t \geq 0)$ and $\psi : [0, \infty) \to D_A$ is such that $\psi_0 = f$ and $\frac{d\psi}{dt} = A\psi$, then $\psi(t) = T_t f$ for all $t \geq 0$.*

Proof. Let $\phi \in E'$ and fix $t > 0$. For $0 \leq s \leq t$, define

$$F(s) := \langle T_t \psi(t - s), \phi \rangle.$$

Then the right derivative

$$\begin{aligned} D_+ F(s) &= \lim_{h \to 0} \left\langle \frac{1}{h}(T_{s+h}\psi(t - s - h) - T_s \psi(t - s)), \phi \right\rangle \\ &= \lim_{h \to 0} \left\langle \frac{1}{h}(T_{s+h} - T_s)\psi(t - s - h), \phi \right\rangle \\ &\quad + \lim_{h \to 0} \left\langle \frac{1}{h}(T_s(\psi(t - s - h) - \psi(t - s)), \phi \right\rangle \end{aligned}$$

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$$= \langle AT_s\psi(t-s), \phi\rangle - \langle T_s A\psi(t-s), \phi\rangle$$
$$= 0,$$

by Theorem 1.3.13 (3). F is continuous, so as in the proof of Lemma 1.3.14, we have $F(t) = F(0)$ for all $t > 0$, i.e., $\langle T_t\psi_0, \phi\rangle = \langle\psi(t), \phi\rangle$, and so $\psi(t) = T_t\psi_0$. This shows that any solution of the ivp is generated from the initial value by a C_0-semigroup with generator A.

To complete the proof, suppose that $(T_t, t \geq 0)$ and $(S_t, t \geq 0)$ are two distinct C_0-semigroups having the same generator A. Then $\psi(t) = T_t\psi_0$ and $\xi(t) = S_t\psi_0$ both satisfy the conditions of the theorem. Then we conclude from the above discussion that $S_t\psi_0 = T_t\psi_0$ for all $\psi_0 \in D_A$. But D_A is dense in E and hence $S_t = T_t$ for all $t \geq 0$, and the result follows. □

1.3.3 Generators as Closed Operators

Unbounded operators cannot be continuous, as they would then be bounded. The closest we can get to continuity is the property of being closed, which we now describe. Let X be a linear operator in E with domain D_X. Its *graph* is the set $G_X \subseteq E \times E$ defined by

$$G_X = \{(\psi, X\psi); \psi \in D_X\}.$$

We say that the operator X is *closed* if G_X is closed in $E \times E$. You can check that this is equivalent to the requirement that, for every sequence $(\psi_n, n \in \mathbb{N})$ in D_X which converges to $\psi \in E$, and for which $(X\psi_n, n \in \mathbb{N})$ converges to $\phi \in E$, $\psi \in D_X$ and $\phi = X\psi$. If X is closed and $D_X = E$, then the *closed graph theorem* states that X is bounded.

If X is a closed linear operator, then it is easy to check that its domain D_X is itself a Banach space with respect to the *graph norm* $|||\cdot|||$ where

$$|||\psi||| = ||\psi|| + ||X\psi||$$

for each $\psi \in D_X$ (see Problem 1.9).

It is not difficult to construct examples of linear operators that are densely defined, but not closed, or closed but not densely defined. We are fortunate that generators of C_0-semigroups satisfy both of these properties.

Theorem 1.3.16 *If A is the generator of a C_0-semigroup, then A is closed.*

Proof. Let $(\psi_n, n \in \mathbb{N})$ be a sequence in E such that $\psi_n \in D_A$ for all $n \in \mathbb{N}$, $\lim_{n\to\infty}\psi_n = \psi \in E$ and $\lim_{n\to\infty} A\psi_n = \phi \in E$. We must prove that $\psi \in D_A$ and $\phi = A\psi$.

First observe that, for each $t \geq 0$, by continuity, equation (1.3.9) and Theorem 1.3.13 (3),

$$\begin{aligned} T_t\psi - \psi &= \lim_{n\to\infty}(T_t\psi_n - \psi_n) \\ &= \lim_{n\to\infty}\int_0^t T_s A\psi_n\, ds \\ &= \int_0^t T_s\phi\, ds, \end{aligned} \tag{1.3.10}$$

where the passage to the limit in the last line is justified by the fact that

$$\begin{aligned} \left|\left|\int_0^t T_s A\psi_n\, ds - \int_0^t T_s\phi\, ds\right|\right| &\leq \int_0^t ||T_s(A\psi_n - \phi)||ds \\ &\leq tM||(A\psi_n - \phi)|| \to 0, \text{as } n \to \infty, \end{aligned}$$

with $M := \sup\{||T_s||, 0 \leq s \leq t\} < \infty$ by Lemma 1.2.2. Now, by (RI4) applied to (1.3.10), we have

$$\lim_{t\to 0}\frac{1}{t}(T_t\psi - \psi) = \phi,$$

from which the required result follows. □

1.3.4 Closures and Cores

This section may be omitted at first reading.

In many situations, a linear operator only fails to be closed because its domain is too small. To accommodate this we say that a linear operator X in E is *closable* if it has a closed extension $\tilde{X}$. Hence X is closable if and only if there exists a closed operator $\tilde{X}$ for which $\overline{G_X} \subseteq G_{\tilde{X}}$. Note that there is no reason why $\tilde{X}$ should be unique, and we define the *closure* $\overline{X}$ of a closable operator X to be its smallest closed extension (i.e., its domain is the intersection of the domains of all of its closed extensions), so that $\overline{X}$ is the closure of X if and only if the following hold:

1. $\overline{X}$ is a closed extension of X;
2. if X_1 is any other closed extension of X, then $D_{\overline{X}} \subseteq D_{X_1}$.

The next theorem gives a useful practical criterion for establishing closability.

Theorem 1.3.17 *A linear operator X in E with domain D_X is closable if and only if for every sequence $(\psi_n, n \in \mathbb{N})$ in D_X which converges to 0 and for which $(X\psi_n, n \in \mathbb{N})$ converges to some $\phi \in E$, we always have $\phi = 0$.*

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Proof. If X is closable, then the result is immediate from the definition. Conversely, let (x, y_1) and (x, y_2) be two points in $\overline{G_X}$. Our first task is to show that we always have $y_1 = y_2$. Let $(x_n^{(1)}, n \in \mathbb{N})$ and $(x_n^{(2)}, n \in \mathbb{N})$ be two sequences in D_X that converge to x; then $(x_n^{(1)} - x_n^{(2)}, n \in \mathbb{N})$ converges to 0 and $(Xx_n^{(1)} - Xx_n^{(2)}, n \in \mathbb{N})$ converges to $y_1 - y_2$. Hence $y_1 = y_2$ by the criterion.

From now on, we write $y = y_1 = y_2$ and define $X_1 x = y$. Then X_1 is a well-defined linear operator with

$$D_{X_1} = \{x \in E; \text{ there exists } y \in E \text{ such that } (x, y) \in \overline{G_X}\}.$$

Clearly X_1 extends X and by construction we have $G_{X_1} = \overline{G_X}$, so that X_1 is closed, as required. □

It is clear that the operator X_1 constructed in the proof of Theorem 1.3.17 is the closure of X. Indeed, from the proof of Theorem 1.3.17, we see that a linear operator X is closable if and only if it has an extension X_1 for which

$$G_{X_1} = \overline{G_X}.$$

Having dealt with the case where the domain is too small, we should also consider the case where we know that an operator X is closed, but the domain is too large or complicated for us to work in it with ease. In that case it is very useful to have a core available.

Let X be a closed linear operator in E with domain D_X. A linear subspace C of D_X is said to be a *core* for X if

$$\overline{X|_C} = X,$$

i.e., given any $\psi \in D_X$, there exists a sequence $(\psi_n, n \in \mathbb{N})$ in C such that $\lim_{n\to\infty} \psi_n = \psi$ and $\lim_{n\to\infty} X\psi_n = X\psi$.

Now we return to the study of C_0-semigroups $(T_t, t \geq 0)$. The next result is extremely useful in applications.

Theorem 1.3.18 *If $D \subseteq D_A$ is such that*

1. *D is dense in E,*
2. *$T_t(D) \subseteq D$ for all $t \geq 0$,*

then D is a core for A.

Proof. Let $\overline{D}$ be the closure of D in D_A with respect to the graph norm $|||.|||$ (where we recall that $|||\psi||| = ||\psi|| + ||A\psi||$ for each $\psi \in D_A$).

Let $\psi \in D_A$; then by hypothesis (1), we know there exists $(\psi_n, n \in \mathbb{N})$ in D such that $\lim_{n\to\infty} \psi_n = \psi$. Define $\psi(t) = \int_0^t T_s\psi\, ds$ and $\psi_n(t) = \int_0^t T_s\psi_n\, ds$

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for each $n \in \mathbb{N}$ and $t \geq 0$. Approximating $\psi_n(t)$ by Riemann sums, and using hypothesis (2), we deduce that $\psi_n(t) \in \overline{D}$. Now using (1.3.9), Lemma 1.3.12, Lemma 1.2.2 and (1.2.4) we find that there exists $C_t > 0$ so that

$$\begin{aligned}
&\lim_{n\to\infty} |||\psi(t) - \psi_n(t)||| \\
&\quad = \lim_{n\to\infty} ||\psi(t) - \psi_n(t)|| + \lim_{n\to\infty} ||A\psi(t) - A\psi_n(t)|| \\
&\quad \leq \lim_{n\to\infty} \int_0^t ||T_s(\psi - \psi_n)||ds + \lim_{n\to\infty} ||(T_t\psi - T_t\psi_n|| + \lim_{n\to\infty} ||\psi - \psi_n)|| \\
&\quad \leq (tC_t + Me^{at} + 1) \lim_{n\to\infty} ||\psi - \psi_n)|| = 0,
\end{aligned}$$

and so $\psi(t) \in \overline{D}$ for each $t \geq 0$.

Now using (1.3.9) again and also (RI4), we obtain

$$\begin{aligned}
&\lim_{t\to 0} \left|\left|\left| \frac{1}{t}\psi(t) - \psi \right|\right|\right| \\
&\quad = \lim_{t\to 0} \left|\left| \frac{1}{t}\int_0^t T_s\psi\, ds - \psi \right|\right| + \lim_{t\to 0} \left|\left| \frac{1}{t}A\psi(t) - A\psi \right|\right| \\
&\quad = \lim_{t\to 0} \left|\left| \frac{1}{t}\int_0^t T_s\psi\, ds - \psi \right|\right| + \lim_{t\to 0} \left|\left| \frac{1}{t}(T_t\psi - \psi) - A\psi \right|\right| = 0.
\end{aligned}$$

From this we can easily deduce that $D_A \subseteq \overline{D}$, from which it is clear that D is a core for A, as required. □

Example 1.3.19 If we return to the translation semigroup discussed in Examples 1.3 and 1.7, then it is very easy to check the hypotheses of Theorem 1.3.18 and show that $C_c^\infty(\mathbb{R})$ is a core for the generator.

1.4 Norm-Continuous Semigroups

Let $(T_t, t \geq 0)$ be a family of linear operators in $\mathcal{L}(E)$ that satisfy (S1) and (S2), but instead of (S3), we have that for all $t \geq 0$, $T_s \to T_t$ in the norm topology in $\mathcal{L}(E)$ as $s \to t$, i.e.,

$$\lim_{s\to t} ||T_t - T_s|| = 0.$$

If we imitate the proof of Proposition 1.2.1, we find that the above convergence is equivalent to requiring $\lim_{s\to 0} ||T_s - I|| = 0$. Semigroups that satisfy this condition are said to be *norm continuous*. It is clear that every norm-continuous semigroup is a C_0-semigroup. Note that for a norm-continuous semigroup $(T_t, t \geq 0)$, the mapping $t \to T_t$ is continuous from $[0, \infty)$ to $\mathcal{L}(E)$, and so the (Riemann) integral $\int_0^t T_s ds$, is well defined in the sense discussed in section 1.1.

Example 1.4.20 If we return to Example 1.2.4, then using an $\epsilon/3$–argument, one can easily show that $e^{tA} := \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ is norm continuous for $A \in \mathcal{L}(E)$ (see Problem 1.1(b)). The next result shows that this class of examples comprises the entirety of the norm-continuous semigroups.

Theorem 1.4.21 *A C_0-semigroup $(T_t, t \geq 0)$ is norm continuous if and only if its generator A is bounded.*

Proof. By the discussion in Example 1.4.20, we need only prove necessity. So let $(T_t, t \geq 0)$ be norm continuous. By (RI4) we can and will choose h sufficiently small so that

$$\left|\left| I - \frac{1}{h}\int_0^h T_t dt \right|\right| < 1.$$

Now for such a value of h, define $W = \int_0^h T_t dt$. Then W is bounded and invertible (and its bounded inverse is given in terms of a Neumann series[8]). Define $V \in \mathcal{L}(E)$ by $V := W^{-1}(T_h - I)$. We will show that V is the generator of the semigroup, and then we are done. First observe that for $t > 0$, by (1.3.8) and (RI2) we have

$$\begin{aligned} W(T_t - I) &= \int_t^{t+h} T_s ds - \int_0^h T_s ds \\ &= \int_h^{t+h} T_s ds - \int_0^t T_s ds \\ &= (T_h - I)\int_0^t T_s ds. \end{aligned}$$

Then by (1.3.6)

$$T_t - I = V\int_0^t T_s ds = \int_0^t V T_s ds,$$

and so

$$\frac{T_t - I}{t} = \frac{1}{t}\int_0^t V T_s ds.$$

[8] To be precise,

$$W^{-1} = \frac{1}{h}\sum_{n=0}^{\infty}\left(I - \frac{1}{h}W\right)^n.$$

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From here, we can easily deduce that the mapping $t \to T_t$ is norm-differentiable and that

$$V = \lim_{t \to 0} \frac{T_t - I}{t}$$

is the generator. □

1.5 The Resolvent of a Semigroup

1.5.1 The Resolvent of a Closed Operator

Let X be a linear operator in E with domain D_X. Its *resolvent set* is

$$\rho(X) := \{\lambda \in \mathbb{C};\, \lambda I - X \text{ is a bijection from } D_X \text{ to } E\}.$$

The *spectrum* of X is the set $\sigma(X) = \rho(X)^{\mathrm{c}}$. Note that every eigenvalue of X is an element of $\sigma(X)$. If $\lambda \in \rho(X)$, the linear operator $R_\lambda(X) = (\lambda I - X)^{-1}$ is called the *resolvent* of T. For simplicity, we will sometimes write $R_\lambda := R_\lambda(X)$, when there can be no doubt as to the identity of X.

We remark that $\lambda \in \rho(X)$ if and only if for all $g \in E$ there exists a unique $f \in D_X$ so that

$$(\lambda I - X)f = g.$$

If we take X to be a partial differential operator, as in section 1.1, we see that resolvents (when they exist) are the operators that generate unique solutions to *elliptic* equations. For the next result we recall from section 1.3.3 that the domain of a closed linear operator is a Banach space, when it is equipped with the graph norm.

Proposition 1.5.22 *If X is a closed linear operator in E with domain D_X and resolvent set $\rho(X)$, then, for all $\lambda \in \rho(X)$, $R_\lambda(X)$ is a bounded operator from E into D_X (where the latter space is equipped with the graph norm).*

Proof. We will need the inverse mapping theorem, which states that a continuous bijection between two Banach spaces always has a continuous inverse (see, e.g., Reed and Simon [76], p. 83). For each $\lambda \in \rho(X)$, $\psi \in D_X$,

$$||(\lambda I - X)\psi|| \leq |\lambda|\,||\psi|| + ||X\psi|| \leq \max\{1, |\lambda|\}\,|||\psi|||.$$

So $\lambda I - X$ is bounded and hence continuous from D_X to E. The result then follows by the inverse mapping theorem. □

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It follows from Proposition 1.5.22 that if X is both densely defined and closed, then $R_\lambda(X)$ extends to an operator in $\mathcal{L}(E)$, which we continue to denote[9] as $R_\lambda(X)$. Indeed we have shown that for all $\psi \in D_X$ there exists $K > 0$ so that

$$||R_\lambda(X)\psi|| \leq |||R_\lambda(X)\psi||| \leq K||\psi||,$$

and the result follows by the density of D_X in E. It is a trivial, but useful, consequence of the definition of resolvent set that if $\lambda \in \rho(X)$, then for every $x \in D_X$ there exists $y \in E$ so that $x = R_\lambda(X)y$.

The next result summarises some key properties of resolvents:

Proposition 1.5.23 *Let X be a closed linear operator acting in E.*

1. *The resolvent set $\rho(X)$ is open.*
2. *For all $\lambda, \mu \in \rho(X)$,*
$$R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu. \tag{1.5.11}$$
3. *For all $\lambda, \mu \in \rho(X)$,*
$$R_\lambda R_\mu = R_\mu R_\lambda.$$
4. *For all $\lambda \in \rho(X)$, $f \in D_X$,*
$$R_\lambda X f = X R_\lambda f.$$

Proof. 1. We will never use this directly, so we omit the proof and direct the reader to the literature (see, e.g., Lemma 8.1.3 in Davies [27] pp. 212–3).

2. For all $f \in E$, we have
$$\begin{aligned}(\lambda I - X)&[R_\lambda - R_\mu - (\mu - \lambda)R_\lambda R_\mu]f\\ &= f - (\lambda I - \lambda I + \mu I - X)R_\mu f\\ &= 0,\end{aligned}$$
but $\lambda I - X$ is invertible and so we must have $R_\lambda - R_\mu - (\mu - \lambda)R_\lambda R_\mu = 0$, as is required.
3. This follows immediately from (2).
4. First note that $XR_\lambda f$ is meaningful, as R_λ maps E to D_X. The result follows from writing
$$\begin{aligned}XR_\lambda f &= -(\lambda I - X)(\lambda I - X)^{-1} f + \lambda R_\lambda f\\ &= -(\lambda I - X)^{-1}(\lambda I - X) f + \lambda R_\lambda f\\ &= R_\lambda X f.\end{aligned}$$
□

[9] As is standard, we use the same notation, $R_\lambda(X)$, to denote the resolvent acting from E to E, and from D_X (equipped with the graph norm) to E; it should always be clear which of these we will mean from the context that we are in.

Although we will mostly be concerned with the case where X is an unbounded operator, the following result is instructive.

Proposition 1.5.24 *If X is a bounded linear operator in E and $\lambda \in \mathbb{C}$ with $|\lambda| > ||X||$, then $\lambda \in \rho_X$.*

Proof. It is well known from elementary Banach space theory that if $c \in \mathbb{C}$ such that $|c|.||X|| < 1$, then $I - cX$ is invertible. Hence, if $\lambda \in \mathbb{C}$ with $|\lambda| > ||X||$, then $\lambda^{-1}(\lambda I - X)$ is invertible, and the result follows. □

An immediate consequence of the last result is that the spectrum $\sigma(X) \subseteq B_{||X||}(0) \subset \mathbb{C}$.

1.5.2 Properties of the Resolvent of a Semigroup

If $(T_t, t \geq 0)$ is a C_0-semigroup having generator A, then since A is closed and densely defined, $R_\lambda(A) = (\lambda I - A)^{-1}$ is a well-defined bounded linear operator in E for $\lambda \in \rho(A)$. We call it the *resolvent of the semigroup*. Of course, there is no a priori reason why $\rho(A)$ should be non-empty. The following key theorem will put that doubt to rest. Before we state it, we recall the key estimate (1.2.4) $||T_t|| \leq Me^{at}$ for some $M > 1, a \in \mathbb{R}$, for all $t \geq 0$. Using this estimate, it is not difficult to see that for any $h \in C((0, \infty))$ which satisfies $\int_0^\infty h(t)e^{at}dt < \infty$, we may define a bounded linear operator on E by the prescription

$$\left(\int_0^\infty h(t)T_t dt\right)\psi = \lim_{T\to\infty}\int_0^T h(t)T_t\psi dt, \tag{1.5.12}$$

for all $\psi \in E$.

Theorem 1.5.25 *Let A be the generator of a C_0-semigroup $(T_t, t \geq 0)$ satisfying $||T_t|| \leq Me^{at}$ for all $t \geq 0$. The following hold:*

1. *$\{\lambda \in \mathbb{C}; \Re(\lambda) > a\} \subseteq \rho(A)$,*
2. *for all $\Re(\lambda) > a$,*

$$R_\lambda(A) = \int_0^\infty e^{-\lambda t}T_t dt, \tag{1.5.13}$$

3. *for all $\Re(\lambda) > a$,*

$$||R_\lambda(A)|| \leq \frac{M}{\Re(\lambda) - a}. \tag{1.5.14}$$

Proof. For each $\Re(\lambda) > a$, we define a linear operator $S_\lambda(A)$ on E by the Fourier–Laplace transform on the right-hand side of (1.5.13). Our goal is to

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prove that this really is the resolvent. Note first of all that $S_\lambda(A)$ is a bounded operator on E of the form (1.5.12); indeed, for each $\psi \in E, t \geq 0$, on using (RI3) and (1.2.4) we have,

$$\begin{aligned} ||S_\lambda(A)\psi|| &\leq \int_0^\infty e^{-\Re(\lambda)t}||T_t\psi||dt \leq ||\psi||M\int_0^\infty e^{(a-\Re(\lambda))t}dt \\ &= \frac{M}{\Re(\lambda)-a}||\psi||. \end{aligned}$$

Hence we have $||S_\lambda(A)|| \leq \dfrac{M}{\Re(\lambda)-a}$.

Now define $\psi_\lambda = S_\lambda(A)\psi$ for each $\psi \in E$. Then by (1.3.8), change of variable and (RI4), we have

$$\begin{aligned} &\lim_{h\to 0}\frac{1}{h}(T_h\psi_\lambda - \psi_\lambda) \\ &= \lim_{h\to 0}\left(\frac{1}{h}\int_0^\infty e^{-\lambda t}T_{t+h}\psi dt - \frac{1}{h}\int_0^\infty e^{-\lambda t}T_t\psi\, dt\right) \\ &= \lim_{h\to 0}\left(\frac{1}{h}\int_h^\infty e^{-\lambda(t-h)}T_t\psi dt - \frac{1}{h}\int_0^\infty e^{-\lambda t}T_t\psi\, dt\right) \\ &= -\lim_{h\to 0} e^{\lambda h}\frac{1}{h}\int_0^h e^{-\lambda t}T_t\psi dt + \lim_{h\to 0}\frac{1}{h}(e^{\lambda h}-1)\int_0^\infty e^{-\lambda t}T_t\psi dt \\ &= -\psi + \lambda\psi_\lambda. \end{aligned}$$

Hence $\psi_\lambda \in D_A$ and $A\psi_\lambda = -\psi + \lambda S_\lambda(A)\psi$, i.e., for all $\psi \in B$

$$(\lambda I - A)S_\lambda(A)\psi = \psi.$$

So $\lambda I - A$ is surjective for all $\lambda > 0$ and its right inverse is $S_\lambda(A)$.

Our proof is complete if we can show that $\lambda I - A$ is also injective. To establish this, assume that there exists $\psi \in D_A$ with $\psi \neq 0$ such that $(\lambda I - A)\psi = 0$, and define $\psi_t = e^{\lambda t}\psi$ for each $t \geq 0$. Then differentiation yields the initial-value problem

$$\psi_t' = \lambda e^{\lambda t}\psi = A\psi_t$$

with initial condition $\psi_0 = \psi$. But by Theorem 1.3.15, we have $\psi_t = T_t\psi$ for all $t \geq 0$. We then have

$$||T_t\psi|| = ||\psi_t|| = |e^{\lambda t}|\,||\psi||,$$

and so $||T_t|| \geq ||T_t\psi||/||\psi|| = e^{\Re(\lambda)t}$. But this contradicts $||T_t|| \leq Me^{at}$, as is seen by taking $\Re(\lambda) > a + \frac{\log(M)}{t}$. Hence we must have $\psi = 0$, and the proof that $S_\lambda(A) = R_\lambda(A)$ is complete. This establishes (1) and (2), while (3)

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follows from the estimate for $S_\lambda(A)$, which was obtained at the beginning of this proof. □

Note that if $(T_t, t \geq 0)$ is a contraction semigroup, then it follows from Theorem 1.5.25 (1) that

$$\sigma(A) \subseteq (-\infty, 0] \times i\mathbb{R}.$$

If E is a Hilbert space and the semigroup is self-adjoint in that T_t is a self-adjoint operator for all $t \geq 0$, then $\sigma(A) \subseteq (-\infty, 0]$, and we may write the spectral decomposition as

$$T_t = \int_{-\sigma(A)} e^{-\lambda t} E(d\lambda),$$

for each $t \geq 0$, and the generator satisfies the "dissipativity condition"[10] $\langle Af, f \rangle \leq 0$, for all $f \in D_A$. This is why, in the literature, self-adjoint contraction semigroups are often written in the form $T_t = e^{-tB}$, where B is a positive, self-adjoint operator. We will study self-adjoint operators in greater depth in Chapter 4.

Since most semigroups encountered in applications are contraction semigroups, it is a natural question to ask why we bother with the more general C_0-class? One reason, is that the theory (as we are seeing) is very intellectually satisfying. Another reason is that norm-continuous semigroups, which are an important subclass, are not necessarily contractions. Finally if A is the generator of a contraction semigroup $(S_t, t \geq 0)$, then it is natural to want to consider the trivial perturbations $A_c = A + cI$ where $c \in \mathbb{R}$, having domain D_A. It is easy to see that A_c generates the semigroup $(T_t, t \geq 0)$ and that $T_t = e^{ct} S_t$ for all $t \geq 0$, but we can always find sufficiently large c so that T_t is not a contraction. In the next chapter, we will see that the solutions of PDEs may also give rise to C_0-semigroups that are not necessarily contractions.

1.6 Exercises for Chapter 1

1. (a) Let $A \in \mathcal{L}(E)$. Show that the series $\sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ is absolutely convergent (uniformly on finite intervals).
   (b) Deduce that $T_t = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n$ defines a norm-continuous semigroup.
   (c) If E is a Hilbert space and A is self–adjoint, show that the result of (b) agrees with that obtained by defining $T_t = e^{tA}$ using spectral theory.
2. Prove Proposition 1.3.11 and (1.3.6).

[10] We will say more about this, and generalise it to Banach spaces in the next chapter.

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3. Let $(T_t, \geq 0)$ be a family of bounded linear operators in E for which there exists $M \geq 0$ such that $\sup_{0 \leq t \leq 1} ||T_t|| \leq M$. If (S3)$'$ is valid for these operators on a dense subset of E, show that it holds on the whole of E.
4. Prove directly that the translation semigroup of Example 1.2.5 is strongly continuous in both $C_0(\mathbb{R})$ and $L^p(\mathbb{R})$.
[Hint: On L^p, find a suitable dense subspace and use the result of (3)].
5. Suppose that S is a bounded operator on E and that T is a closed operator having domain D_T. Show that $S + T$ and ST are both closed operators, having domain D_T. What can you say when T is only known to be closeable? What can you say about TS?
6. Suppose that $(T_t^{(1)}, t \geq 0)$ and $(T_t^{(2)}, t \geq 0)$ are C_0-semigroups in E for which
$$T_s^{(1)} T_t^{(2)} = T_t^{(2)} T_s^{(1)},$$
for all $s, t \geq 0$.
(a) Show that $(T_t^{(1)} T_t^{(2)}, t \geq 0)$ is a C_0-semigroup on E.
(b) If for $i = 1, 2$, $(T_t^{(i)}, t \geq 0)$ has generator A_i, deduce that $(T_t^{(1)} T_t^{(2)}, t \geq 0)$ has generator $A_1 + A_2$.
7. If $c > 0$ and $(S_t, t \geq 0)$ is a C_0-semigroup with generator A, show that $T_t = e^{-ct} S_t$ also defines a C_0-semigroup. What is the generator of this semigroup? Can you express its domain in terms of that of A?
8. (a) If A is a densely defined linear operator in E such that there exists $K \geq 0$ so that $||A\psi|| \leq K\|| \psi||$ for all $\psi \in D_A$, show that A has a unique bounded extension to a linear operator $\tilde{A}$ defined on the whole of E.
(b) If A is densely defined and D_A is closed, show that $D_A = E$.
(c) If A is the generator of a C_0-semigroup, show that D_A is closed if and only if the semigroup is norm-continuous.
9. Show that if X is a closed linear operator on a Banach space, then its domain is complete under the graph norm.
10. Let $(T_t, t \geq 0)$ be a C_0-semigroup for which $||T_t|| \leq Me^{at}$ for all $t \geq 0$, where $M \geq 1$ and $a \in \mathbb{R}$. Use induction to show that if $\Re(z) > a$ and $n \in \mathbb{N}$ then
$$R_z^n f = \int_0^\infty \frac{t^{n-1}}{(n-1)!} e^{-zt} T_t f dt.$$

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