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WREATH DECOMPOSITIONS OF FINITE PERMUTATION GROUPS

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To Professor B. H. Neumann for his eightieth birthday

There is a familiar construction with two finite, transitive permutation groups as input and a finite, transitive permutation group, called their wreath product, as output. The corresponding 'imprimitive wreath decomposition' concept is the first subject of this paper. A formal definition is adopted and an overview obtained for all such decompositions of any given finite, transitive group. The result may be heuristically expressed as follows, exploiting the associative nature of the construction. Each finite transitive permutation group may be written, essentially uniquely, as the wreath product of a sequence of wreath-indecomposable groups, and the two-factor wreath decompositions of the group are precisely those which one obtains by bracketing this many-factor decomposition.

If both input groups are nontrivial, the output above is always imprimitive. A similar construction gives a primitive output, called the wreath product in product action, provided the first input group is primitive and not regular. The second subject of the paper is the 'product action wreath decomposition' concept dual to this. An analogue of the result stated above is established for primitive groups with nonabelian socle.

Given a primitive subgroup G with non-regular socle in some symmetric group S, how many subgroups W of S which contain G and have the same socle, are wreath products in product action? The third part of the paper outlines an algorithm which reduces this count to questions about permutation groups whose degrees are very much smaller than that of G.

1. INTRODUCTION

All groups considered in this paper will be finite.

Counting the number of mathematical objects of a certain kind is often undertaken as a test problem ("if you can't count them, you don't really know them"): not so much because we want the answer, but because the attempt focuses attention on gaps in our understanding, and the eventual proof may embody insights beyond those which are capable of concise expression in displayed theorems.

The O'Nan-Scott Theorem (see Liebeck, Praeger, Saxl [4] for the most recent and detailed treatment) and related developments have given formal expression to several features of finite primitive permutation groups. In a recent paper [3] the author explored, in terms of 'blow-up' decompositions, primitive subgroups G of wreath products

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W in product action such that the socles of G and of W are the same and this common socle is not regular. A forthcoming paper [**5**] of Praeger investigates the general, qualitative 'inclusion problem': what kind of primitive groups can contain any given primitive group? (Blow-up decompositions reduce this to questions concerning almost simple groups, which are then sorted out on the basis of the classification of finite simple groups.) The subject of the present paper is a particular, quantitative inclusion problem, a kind of converse to the issues explored in [**3**]: given a primitive subgroup G with non-regular socle in some symmetric group S, how many subgroups W of S which contain G and have the same socle, are wreath products in product action? The answer is given as an informal algorithm which reduces this count to questions about permutation groups whose degrees are very much smaller than that of G.

As always, one has to be careful not to count any one object twice: so one needs to be able to recognise when W appears as a wreath product in product action in two different ways. To cope with this, one needs a formal *product action wreath decomposition* concept, and an overview of all such decompositions of a primitive group with non-regular socle. It turns out to be sufficient to assume that the socle is nonabelian. This socle is then a direct product of isomorphic nonabelian simple groups which are permuted by (the conjugation action of) W. If this action is intransitive, then the socle is non-regular. For primitive W which have at least one product action wreath decomposition and whose socle is non-regular, it follows from the results of [**3**] that W has a unique finest blow-up decomposition, and the (not necessarily simple) direct factors in the corresponding direct decomposition of the socle are transitively permuted by W. The product action wreath decompositions of W are closely related to what one might call the *imprimitive wreath decompositions* of the transitive group obtained in one or the other of these ways. Consequently, there is also a need for a formal concept of decompositions of the latter kind, and for an overview of all such decompositions of an arbitrary transitive group. While I have not been able to find explicit references, I expect that some of this must have been at least intuitively known already to Jordan and to many others since him.

To get a flavour of the conceptual development, consider informally the kind of duality which is at play here. Primitivity is defined in terms of certain partitions of the permuted set, partitions which are called systems of imprimitivity. A partition is a coproduct decomposition in the category of sets; the dual concept is product decomposition in that category. The components of a coproduct decomposition are the relevant inclusion maps; the components of a product decomposition are also maps, call them 'coordinate projections'. In the first context, it is often convenient to focus on a subset (a block of imprimitivity) rather than on its inclusion map. In the second context, it is likewise convenient to speak of an *equivalence relation* instead of a coordinate projection: call two points (that is, elements of the permuted set) equivalent if they have a common image under the coordinate projection in question. [Loosely speaking, the projection can be recovered from the equivalence relation as the obvious map to the set of equivalence classes, namely the map which takes each point to the equivalence class containing that point; strictly speaking, the projection factors uniquely through this map.] A subset is a block of imprimitivity for G if its G-translates form a nontrivial coproduct decomposition, and then the 'normaliser' of this decomposition is an imprimitive wreath product (of two symmetric groups) which contains G. Dually, an equivalence relation will be called a *block of product-imprimitivity* for G if its G-translates form a nontrivial product decomposition, and then the 'normaliser' of this decomposition is a wreath product in product action (of two symmetric groups) which contains G. A transitive G is called primitive if there is no block of imprimitivity for G; a primitive G may be called *product-primitive* if there is no block of product-imprimitivity for G. Needless to say, blocks of product-imprimitivity for any primitive G play a critical rôle in the investigation of wreath products in product action which contain that G.

In writing about these matters, one cannot make do without separate notation for imprimitive wreath products and for wreath products in product action. The latter are (called exponentiation groups and) written in exponential form in the book [**2**] of James and Kerber, but that does get inconvenient when iterations are involved. It seems preferable to have a notation which does not force the use of superscripts and which reminds one of the underlying (co)product construction: to break the horizontal line of the (co)product sign and use the stylised fragments as in $\lfloor \mathrm{wr} \rfloor$ and $\lceil \mathrm{wr} \rceil$.

The canonical identification of $(X \times Y) \times Z$ with $X \times (Y \times Z)$ leads to the associative law of imprimitive wreath products which is so familiar that it needs no reference:

$$(1.1) \qquad (A \lfloor \mathrm{wr} \rfloor B) \lfloor \mathrm{wr} \rfloor C = A \lfloor \mathrm{wr} \rfloor (B \lfloor \mathrm{wr} \rfloor C).$$

The canonical identification of $\left(X^Y\right)^Z$ with $X^{(Y \times Z)}$ leads equally directly to

$$(1.2) \qquad (A \lceil \mathrm{wr} \rceil B) \lceil \mathrm{wr} \rceil C = A \lceil \mathrm{wr} \rceil (B \lfloor \mathrm{wr} \rfloor C),$$

so this will also be taken for granted even though no reference seems to exist. While these laws will only be used here in the discussion (as distinct from the proofs), they will have a fundamental influence on the direction of the work.

The organisation of the paper will reverse the order of the motivating discussion above. Imprimitive wreath decompositions will be dealt with first; the statement and discussion of the results in Section 2, the proofs in Section 3. The treatment of product action wreath decompositions will be similarly divided between Section 4 and Section 5, and the counting problem left for the last Section 6.

2. Imprimitive wreath decompositions: definitions and results

The first problem is to establish a convenient language.

Informally speaking, a wreath decomposition of an abstract group G consists of a semidirect decomposition of G and of a suitable direct decomposition of the normal semidirect factor. This semidirect factor is the *base group*, the other is the *top group*, and the given direct factors of the base group are the *coordinate subgroups*.

For $C \leqslant \operatorname{Sym} \Gamma$ and $D \leqslant \operatorname{Sym} \Delta$, the group commonly called the (permutational or non-standard) wreath product of C and D is a certain subgroup of the symmetric group $\operatorname{Sym} (\Gamma \times \Delta)$ on the cartesian product $\Gamma \times \Delta$; this is the group which is here, to avoid ambiguity, called the *imprimitive wreath product* and written as $C \lfloor \mathrm{wr} \rfloor D$. Accordingly, one might say at first that an imprimitive wreath decompositon of a transitive group G on a set Ω should consist of an abstract wreath decomposition of G and of a matching (cartesian) product decomposition of Ω. As the chosen notation suggests, (at least) the present context will be better served by considering $\Gamma \times \Delta$ as the disjoint union $\coprod \{ \Gamma \times \{\delta\} \mid \delta \in \Delta \}$, a coproduct in the category of sets. When C and D are transitive, the $\Gamma \times \{\delta\}$ are precisely the orbits of the base group; as orbits of a normal subgroup in a transitive group, they form a system of imprimitivity. An element of the top group which stabilises a block setwise also stabilises it pointwise, so the action of the top group provides a coherent set of bijections between the blocks of this system, just what one needs for the reconstruction of the product decomposition $\Gamma \times \Delta$ from the coproduct decomposition given by the system. The base group consists precisely of the elements which stabilise each block setwise. Calling the set of points actually moved (that is, not fixed) by at least one permutation in a group the *support* of that group, one can also name the blocks as the supports of the coordinate subgroups. Conversely, each coordinate subgroup is the pointwise stabiliser of the complement of a block. Note that in this way the base group and its coordinate subgroups may be recognised from the system of imprimitivity even without reference to the top group. (The proofs of these simple observations will be left to the reader.) After a slight shift in notation, the definition so motivated may be stated as follows.

Definition 2.1: An *imprimitive wreath decomposition* of a transitive group G on a finite set Ω consists of a system of imprimitivity Δ and a subgroup D such that

(2.1a) $$D \cap \mathrm{C}_G(\Delta) = 1, \quad D\mathrm{C}_G(\Delta) = G,$$

(2.1b) $$\mathrm{C}_G(\Delta) = \prod \{ \mathrm{C}_G(\Omega \setminus \Gamma) \mid \Gamma \in \Delta \},$$

and

(2.1c) $$(\forall\, \Gamma \in \Delta) \quad \mathrm{N}_D(\Gamma) = \mathrm{C}_D(\Gamma).$$

Here $\mathrm{C}_G(\Delta)$ stands for the kernel of the 'restriction' map $\downarrow \Delta : G \to \operatorname{Sym} \Delta$, while $\mathsf{N}(\Gamma)$ and $\mathsf{C}(\Gamma)$ denote the setwise stabiliser and the pointwise stabiliser of Γ, respectively. Thus also $\mathsf{C}_G(\Delta) = \bigcap \{ \mathsf{N}_G(\Gamma) \mid \Gamma \in \Delta \}$, and $\mathsf{C}(\Gamma)$ is the kernel of the restriction $\downarrow \Gamma : \mathsf{N}(\Gamma) \to \operatorname{Sym} \Gamma$. Note that the product on the right hand side of (2.1b) is automatically a direct product (regardless of whether any of the stated conditions hold). It is natural to call D the ***top group***, $\mathsf{C}_G(\Delta)$ the ***base group***, and the $\mathsf{C}_G(\Omega \setminus \Gamma)$ the ***coordinate subgroups*** of the decomposition. At times it will be convenient to refer to imprimitive wreath decompositions simply as $\lfloor\mathrm{wr}\rfloor$-decompositions.

The preceding discussion has shown that each $\lfloor\mathrm{wr}\rfloor$-product one constructs 'comes with' a decomposition of this kind. The definition would not be of much use if the converse failed to hold.

THEOREM 2.2. *Given an imprimitive wreath decomposition Δ, D of a transitive group G on a finite set Ω, to each Γ in Δ there is a bijection $\Omega \to \Gamma \times \Delta$ which conjugates G to the imprimitive wreath product $(\mathsf{C}_G(\Omega \setminus \Gamma) \downarrow \Gamma) \lfloor\mathrm{wr}\rfloor (G \downarrow \Delta)$, matching the given decomposition of G to the decomposition which this wreath product comes with (and consequently matching top group to top group, base group to base group, and coordinate subgroups to coordinate subgroups).*

Note that while the bijection must depend on D (else one could not ensure that D is the subgroup matched to the top group), the wreath product itself does not; of course, $G \downarrow \Delta = D \downarrow \Delta$. The proofs of all displayed statements are left to Section 3.

It is immediate that if Δ, D is an imprimitive wreath decomposition of G then so is Δ, D^g whenever $g \in G$. Note that (2.1c) remains valid if D is replaced by any subgroup of any conjugate of D. This comment has a very useful converse.

LEMMA 2.3. *Let G be a transitive group on a finite set Ω and Δ, D an imprimitive wreath decomposition of G. A subgroup R of G is conjugate to a subgroup of D if and only if*

$$(\forall\, \Gamma \in \Delta)\, \mathsf{N}_R(\Gamma) = \mathsf{C}_R(\Gamma). \tag{2.3a}$$

The key to working with imprimitive wreath decompositions is that one can restrict attention to a single block of imprimitivity.

LEMMA 2.4. *If Δ, D is an imprimitive wreath decomposition of a transitive subgroup G of $\operatorname{Sym} \Omega$, then*

$$\mathsf{N}_G(\Gamma) \downarrow \Gamma = \mathsf{C}_G(\Omega \setminus \Gamma) \downarrow \Gamma \tag{2.4a}$$

for each Γ in Δ. Conversely, let Γ be a block of imprimitivity for G such that (2.4a) *holds, and let Δ denote the system of imprimitivity consisting of the G-translates of*

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Γ: *then there exist* D *such that* Δ, D *is an imprimitive wreath decomposition of* G, *and these* D *form one conjugacy class of subgroups in* G.

The proof of this lemma will actually show how to construct, using Γ and an arbitrary transversal of $\mathsf{N}_G(\Gamma)$ in G, a semidirect complement D to $\mathsf{C}_G(\Delta)$ satisfying (2.1c).

Given two permutation groups, each with an imprimitive wreath decomposition, one naturally will consider the two decompositions isomorphic if they are appropriately intertwined by some bijection between the sets the groups act on. Accordingly, call two imprimitive wreath decompositions of G *isomorphic* if there is a permutation in $\mathsf{N}_{\mathrm{Sym}\,\Omega}(G)$ which translates the system of imprimitivity Δ of the first decomposition to that of the other and conjugates the top group D of the first to that of the other. If such a permutation can be found in G itself, call the two decompositions *conjugate*. In these terms, the last statement of Lemma 2.4 means that two decompositions of G are conjugate if and only if they have the same system of imprimitivity. Consequently, an overview of all conjugacy types of imprimitive wreath decompositions of G may be envisaged in terms of choosing an arbitrary ω in Ω and considering all blocks of imprimitivity which contain ω and satisfy (2.4a). The answer lies in the following.

LEMMA 2.5. *If* G *is a transitive permutation group on a finite set* Ω, *then any two blocks of imprimitivity* Γ *which satisfy* (2.4a) *and contain a given point* ω *are comparable (in the sense that one of them is a subset of the other).*

Another way of putting this is to say that the blocks of imprimitivity which satisfy (2.4a) and contain ω form a (possibly empty) chain (with respect to order by set inclusion). In view of Lemma 2.4, the length of this chain is the number of conjugacy classes of $\lfloor \mathrm{wr} \rfloor$-decompositions of G. As Ω is assumed finite, two comparable but distinct subsets cannot be translates under any permutation in $\mathrm{Sym}\,\Omega$: so it follows in particular that *any two isomorphic imprimitive wreath decompositions of* G *are conjugate*. Another consequence of Lemma 2.5 is that *any two imprimitive wreath decompositions of a transitive group have comparable base groups.*

THEOREM 2.6. *Let* G *be a transitive group on a finite set* Ω, *let* m *be the number of conjugacy classes of imprimitive wreath decompositions of* G, *and suppose that* $m > 0$. *Then there is a sequence of subgroups* $G_0, \dots, G_m$ *such that, whenever* $1 \leqslant k \leqslant m$, *there is an imprimitive wreath decomposition in which* $\langle G_0, \dots, G_{k-1} \rangle$ *is a coordinate subgroup and* $\langle G_k, \dots, G_m \rangle$ *is the top group. Moreover, if* C *is a coordinate subgroup and* D *is the top group in an imprimitive wreath decomposition of* G, *then there is a (unique)* k *and an element* g *in* G *such that* $C = \langle G_0, \dots, G_{k-1} \rangle^g$ *and* $D = \langle G_k, \dots, G_m \rangle^g$. *Another sequence* $H_0, \dots, H_n$ *of subgroups of* G *has this property if and only if* $n = m$ *and there exists an element* g *in* G *such that*

$H_0 = G_0^g, \dots, H_m = G_m^g$.

[As usual, $\langle G_0, \dots, G_{k-1} \rangle$ stands for the subgroup generated by $G_0, \dots, G_{k-1}$, and so on.]

COROLLARY 2.7. *Let G be a transitive group on a finite set Ω; suppose that G is not $\lfloor\mathrm{wr}\rfloor$-indecomposable, and consider the $\lfloor\mathrm{wr}\rfloor$-decompositions Δ, D of G with $\lfloor\mathrm{wr}\rfloor$-indecomposable $\mathsf{N}_G(\Gamma) \downarrow \Gamma$ (for $\Gamma \in \Delta$): there do exist such decompositions, and they form a single conjugacy class.*

Let G, m, $G_0, \dots, G_m$ be as in Theorem 2.6. For $i = 0, \dots, m$, let Ω_i be a non-singleton orbit of G_i. One could go on to show that each $G_i \downarrow \Omega_i$ is $\lfloor\mathrm{wr}\rfloor$-indecomposable and that there is a bijection $\Omega \to \Omega_0 \times \cdots \times \Omega_m$ which conjugates G to

$$(G_0 \downarrow \Omega_0) \lfloor\mathrm{wr}\rfloor \cdots \lfloor\mathrm{wr}\rfloor (G_m \downarrow \Omega_m).$$

(By (1.1), this m-factor $\lfloor\mathrm{wr}\rfloor$-product needs no brackets.) It should be possible to formalise a 'many-factor' $\lfloor\mathrm{wr}\rfloor$-decomposition concept so that Theorem 2.6 could be rephrased as follows: *each finite transitive group which is not $\lfloor\mathrm{wr}\rfloor$-indecomposable has a unique conjugacy class of (many-factor) $\lfloor\mathrm{wr}\rfloor$-decompositions with all factors $\lfloor\mathrm{wr}\rfloor$-indecomposable, and the two-factor $\lfloor\mathrm{wr}\rfloor$-decompositions of the group are precisely those which arise by suitable 'bracketings' of these.*

Theorem 2.6 is just one of many cases where it is expedient to single out one of the coordinate subgroups. Such contexts could be better served by defining a two-factor decomposition to consist of two subgroups: a coordinate subgroup and the top group. In the appropriate many-factor extension of such a convention, the sequence $G_0, \dots, G_m$ itself would be called a decomposition of G. The associative law (1.1) is, of course, paralleled by a similarly basic rule concerning decompositions; that rule could then be given a particularly simple form. However, the present discussion has already gone too far for the immediate applications envisaged here.

3. IMPRIMITIVE WREATH DECOMPOSITIONS: PROOFS

The proofs of the displayed statements of Section 2 are next on the agenda; the reader may wish to skip to Section 4 where the general discussion continues.

The choice cannot be put off any longer: let all permutations be written on the right, and their composites accordingly. To save a lot of writing, put $B = \mathsf{C}_G(\Delta)$, $C = \mathsf{C}_G(\Omega \setminus \Gamma)$, and $N = \mathsf{N}_G(\Gamma)$. In this notation, (2.1a) says that

(3.1a) $$D \cap B = 1 \text{ and } DB = G,$$

(2.1b) yields that

(3.1b) $$B = \mathsf{C}_B(\Gamma) \times C,$$

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while (2.1c) amounts to

$$(\forall\, g \in G) \quad D \cap N^g = \mathrm{C}_D(\Gamma g). \tag{3.1c}$$

Proof of Theorem 2.2: This will only be sketched. The wreath product in question may now be written as $(C \downarrow \Gamma) \lfloor \mathrm{wr} \rfloor (D \downarrow \Delta)$ but it will be better to have a still shorter name: call it W. Note first that the order $|G|$ of G can be calculated in terms of $|C|$, $|D|$, and the cardinality $|\Delta|$ of Δ. The relevant restrictions being one-to-one on C and on D, one finds that $|G| = |W|$.

Since G is transitive, so is $G \downarrow \Delta$, that is, $D \downarrow \Delta$: thus the maps

$$D \to \Delta, \quad d \mapsto \Gamma d \qquad \text{and} \qquad \Gamma \times D \to \Omega, \quad (\gamma, d) \mapsto \gamma d$$

are surjective. Of course

$$\Gamma d' = \Gamma d \qquad \Longleftrightarrow \qquad N d' = N d,$$

and it is easy to see that

$$\gamma' d' = \gamma d \qquad \Longleftrightarrow \qquad \gamma' = \gamma \ \ \& \ \ N d' = N d;$$

hence $(\gamma, \Gamma d) \mapsto \gamma d$ defines a bijection $\Gamma \times \Delta \to \Omega$. The inverse $\beta : \Omega \to \Gamma \times \Delta$ is the (only) map for which

$$\beta : \omega \mapsto \left(\omega d^{-1}, \Gamma d\right) \quad \text{whenever} \quad d \in D \quad \text{and} \quad \omega \in \Gamma d.$$

Thus for an arbitrary element $(\gamma, \Gamma d)$ of $\Gamma \times \Delta$ and for each d' in D one has

$$(\gamma, \Gamma d)\left(\beta^{-1} d' \beta\right) = (\gamma d d')\beta = (\gamma, \Gamma d d') = \left(\gamma, (\Gamma d)(d' \downarrow \Delta)\right),$$

which shows that $\beta^{-1} D \beta$ is contained in the top group of W. Consider next an element c of C. If $\Gamma d \neq \Gamma$ then $\gamma d \in \Omega \setminus \Gamma$ and so

$$(\gamma, \Gamma d)\left(\beta^{-1} c \beta\right) = ((\gamma d)c)\beta = (\gamma d)\beta = (\gamma, \Gamma d),$$

while if $\Gamma d = \Gamma$ then $\gamma' c \in \Gamma d$ and so

$$(\gamma, \Gamma d)\left(\beta^{-1} c \beta\right) = ((\gamma d)c)\beta = (\gamma d c)\beta = (\gamma c, \Gamma d).$$

This shows that $\beta^{-1} C \beta$ is contained in a coordinate subgroup of W. (The coordinate subgroups of W are usually indexed by Δ: in that sense, the relevant coordinate

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subgroup is that labelled by the element Γ of Δ.) Consequently $\beta^{-1}G\beta \leqslant W$; as $|G| = |W|$, one must in fact have $\beta^{-1}G\beta = W$.

The rest of the proof of Theorem 2.2 is left to the reader. ∎

Proof of Lemma 2.3: The 'if' claim will be established by an application of the first sentence of part (3) of Theorem 5.5 in Gross and Kovács [1]. In this application, the present

$$RB, \quad B, \quad \Delta, \quad \mathsf{C}_B(\Gamma), \quad D \cap RB, \quad \text{and} \quad R$$

take on the rôles there played by

$$G, \quad M, \quad I, \quad K_i, \quad H, \quad \text{and} \quad R,$$

respectively; there will be no need to specify J. It is easy to see that now the rôle of N_i belongs to $\mathsf{N}_{RB}(\Gamma)$, and that the hypothesis $(***)$ is satisfied. The assumption $G = HM$ amounts to $RB = (D \cap RB)B$ and so is simply a case of Dedekind's Law; the assumption concerning $H \cap M$ holds trivially because now $H \cap M = 1$. To complete the application, it will be more than sufficient to verify that $R \cap N_i$ is actually a subgroup of $(H \cap N_i)K_i$ for all i in I, that is, $(\forall\, \Gamma \in \Delta)\, \mathsf{N}_R(\Gamma) \leqslant \mathsf{N}_{D\cap RB}(\Gamma)\mathsf{C}_B(\Gamma)$.

To see this, note that $N \geqslant B$, and argue that

$$\begin{aligned}
\mathsf{C}_D(\Gamma)\mathsf{C}_B(\Gamma) &\subseteq \mathsf{C}_G(\Gamma) \\
&\leqslant N \\
&= (D \cap N)B && \text{by (3.1a) and Dedekind's Law} \\
&= \mathsf{C}_D(\Gamma)B && \text{by (3.1c)} \\
&= \mathsf{C}_D(\Gamma)\mathsf{C}_B(\Gamma)C && \text{by (3.1b),}
\end{aligned}$$

therefore by Dedekind's Law $\mathsf{C}_G(\Gamma) = \mathsf{C}_D(\Gamma)\mathsf{C}_B(\Gamma)\big(\mathsf{C}_G(\Gamma) \cap C\big)$. The last intersection is trivial (because its components have disjoint supports), so the conclusion is that $\mathsf{C}_G(\Gamma) = \mathsf{C}_D(\Gamma)\mathsf{C}_B(\Gamma)$. Consider an arbitrary element r of $\mathsf{N}_R(\Gamma)$; by (2.3a) this lies in $\mathsf{C}_G(\Gamma)$, and hence by the last conclusion $r = db$ for some d and b in $C_D(\Gamma)$ and $\mathsf{C}_B(\Gamma)$, respectively. Now $d = rb^{-1}$ shows that in fact $d \in \mathsf{N}_{D\cap RB}(\Gamma)$, and this completes the proof of the 'if' claim in Lemma 2.3.

As noted in Section 2, the 'only if' claim is obvious. ∎

Proof of Lemma 2.4: Let G be a transitive subgroup of $\operatorname{Sym}\Omega$, and Δ, D an imprimitive wreath decomposition of G. In the second paragraph of the proof of Lemma 2.3, it was already seen that $N = \mathsf{C}_D(\Gamma)\mathsf{C}_B(\Gamma)C$, whence (2.4a) follows.

For the rest of the proof, suppose that Γ is a block of imprimitivity satisfying (2.4a), and let Δ be defined as the system of imprimitivity consisting of the G-translates of Γ. Choose a (right) transversal, T say, for N in G; then Ω is the disjoint union $\coprod\{\,\Gamma t \mid t\in T\,\}$, while $T\to\Delta$, $t\mapsto\Gamma t$ is a bijection. For each g in G, let $\overline{g}$ denote the element of $\operatorname{Sym} T$ such that

$$(\Gamma t)g=\Gamma(t\overline{g}). \tag{3.2a}$$

Define a subset D in G by

$$D=\bigl\{\,g\in G \bigm| (\forall\,\gamma\in\Gamma)\,(\forall\,t\in T)\quad(\gamma t)g=\gamma(t\overline{g})\,\bigr\}. \tag{3.2b}$$

It is easy to see that the D so defined is in fact a subgroup, and that both (2.1c) and the first part of (2.1a) hold. The missing point is the second half of (2.1a), namely $G=DB$. Let g be an arbitrary element of G. By (3.2a) we have that $\Gamma t=\Gamma tt^{-1}(t\overline{g})g^{-1}$ so $t^{-1}(t\overline{g})g^{-1}\in\mathsf{N}_G(\Gamma t)$; of course, (2.4a) implies that also $\mathsf{N}_G(\Gamma t)\downarrow(\Gamma t)=\mathsf{C}_G(\Omega\setminus\Gamma t)\downarrow(\Gamma t)$, so one can choose an element b_t in $\mathsf{C}_G(\Omega\setminus\Gamma t)$ which acts on Γt exactly as $t^{-1}(t\overline{g})g^{-1}$ does. Let b be the product of these b_t (as the factors come from distinct direct factors of B, their order of listing is immaterial). On a γt, the $b_{t'}$ with $t'\neq t$ act trivially [because $\gamma t\in\Omega\setminus\Gamma t'$ and $b_{t'}\in\mathsf{C}_G(\Omega\setminus\Gamma t')$], so b acts as b_t, that is, as $t^{-1}(t\overline{g})g^{-1}$. Therefore one can argue that

$$(\gamma t)(bg)=((\gamma t)b)g=\bigl((\gamma t)\bigl(t^{-1}(t\overline{g})g^{-1}\bigr)\bigr)g=\gamma(t\overline{g})=\gamma\bigl(t\overline{bg}\bigr).$$

This holds for all γ and for all t, hence first $bg\in D$, then $g\in DB$, and finally $DB=G$ follows.

This has shown that the set of the imprimitive wreath decompositions involving the given Δ is nonempty. The claim that they form a single conjugacy class then follows from Lemma 2.3. ∎

PROOF OF LEMMA 2.5: For an argument by contradiction, let $\omega\in\Omega$, and let Γ, Γ^* be incomparable blocks of imprimitivity containing ω and such that (2.4a) holds for Γ and also with Γ^* in place of Γ. Let Δ and Δ^* denote the systems of imprimitivity consisting of the G-translates of Γ and of Γ^*, respectively, and let B, B^* be defined accordingly. By Lemma 2.4, there exist D and D^* such that Δ, D satisfy (2.1) and so do Δ^*, D^* in place of Δ, D. Of course $\Gamma\cap\Gamma^*$ is either a singleton or a block of imprimitivity. In the latter case one can switch attention to the restriction of G to the set of the G-translates of $\Gamma\cap\Gamma^*$: since by assumption Γ consists of more than one such translate, it yields a block of imprimitivity in this set of blocks; indeed, one for which the analogue of (2.4a) holds; and the same can be said for Γ^*. Therefore it is sufficient to derive a contradiction under the additional assumption that $\Gamma\cap\Gamma^*$ is

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a singleton. Now $B \cap B^*$ fixes the singleton $\Gamma \cap \Gamma^*$ and is therefore trivial (because a nontrivial normal subgroup of a transitive group can have no fixed points): in particular, B centralises B^*. On the other hand, the base group of a wreath product contains its own centraliser: so $B \leqslant B^*$, whence $B \leqslant B \cap B^* = 1$ and $G = D$. This is impossible because the transitive G must be able to map one point of Γ to another while (3.1c) shows that D cannot do that. ∎

PROOF OF THEOREM 2.6: Choose a point ω; then the blocks of imprimitivity for G which satisfy (2.4a) and contain ω form a chain of length m: say, $\Gamma_1 \subset \cdots \subset \Gamma_m$. For $i = 1, \ldots, m$, define Δ_i, B_i, C_i, N_i, and choose a D_i, accordingly. Set $G_0 = C_1$ and $G_m = D_m$. If $m = 1$ then the claims all follow from Lemma 2.4; hence assume that $m \geqslant 2$.

Let $1 \leqslant i \leqslant m-1$. Obviously, $C_i \leqslant C_{i+1}$ and $B_i \leqslant B_{i+1}$. Further, $N_i \leqslant N_{i+1}$ because if $g \in N_i$ then $\Gamma_i = \Gamma_i g \subset \Gamma_{i+1} g$ shows that $\Gamma_{i+1} g$ is not disjoint from and hence must be the same as Γ_{i+1}. Let g be any element of G; then

$$\mathsf{N}_{D_{i+1}}(\Gamma_i g) = D_{i+1} \cap N_i^g \leqslant D_{i+1} \cap N_{i+1}^g = \mathsf{C}_{D_{i+1}}(\Gamma_{i+1} g) \leqslant \mathsf{C}_{D_{i+1}}(\Gamma_i g).$$

By Lemma 2.3 one may therefore conclude that D_{i+1} is conjugate to a subgroup of D_i. It follows that, replacing each D_i by a conjugate if necessary, one can arrange that $D_i \geqslant D_{i+1}$ for all i; let this be done. Finally, set $G_i = C_{i+1} \cap D_i$ for $i = 1, \ldots, m-1$.

It will be useful to have short names for $\langle G_0, \ldots, G_{i-1} \rangle$ and for $\langle G_i, \ldots, G_m \rangle$: call them C_i^* and D_i^*, respectively. The next aim is to show, by induction on i, that $C_i^* = C_i$ whenever $1 \leqslant i \leqslant m$. The initial step is a tautology. The inductive step amounts to showing that if $i < m$ then $\langle C_i, G_i \rangle = C_{i+1}$. Observe that $C_{i+1} \downarrow \Gamma_{i+1}$ is transitive, and that Γ_i is a block of imprimitivity also for this group, satisfying the relevant variant of (2.4a): so by Lemma 2.4 there exist imprimitive wreath decompositions of $C_{i+1} \downarrow \Gamma_{i+1}$ in which the system of imprimitivity consists of the C_{i+1}-translates of Γ_i and the top group can be taken in the form $R \downarrow \Gamma_{i+1}$ with $R \leqslant C_{i+1}$. The corresponding base group is simply $B_i \downarrow \Gamma_{i+1}$, and of course $C_i \downarrow \Gamma_{i+1}$ is one of the coordinate subgroups. Now one can use Lemma 2.3 to deduce that R is conjugate to some subgroup of D_i: say, $R^{bd} \leqslant D_i$ with $b \in B_i$ and $d \in D_i$. Of course then also $R^b \leqslant D_i$ so $R^b \leqslant G_i$, and $(R^b) \downarrow \Gamma_{i+1}$, being a conjugate of a top group, is a top group in some other imprimitive wreath decomposition of $C_{i+1} \downarrow \Gamma_{i+1}$ with $C_i \downarrow \Gamma_{i+1}$ as a coordinate subgroup: thus $\langle C_i, R^b \rangle = C_{i+1}$ and *a fortiori* $\langle C_i, G_i \rangle = C_{i+1}$ follows. This completes the inductive step.

It follows that $\langle C_i, D_i^* \rangle = \langle C_i^*, D_i^* \rangle = \langle C_m^*, D_m^* \rangle = \langle C_m, D_m \rangle = G$. As $D_i^* \leqslant D_i$ by definition, and as a coordinate subgroup and a proper subgroup of the top group never generate the whole wreath product, one concludes that $D_i^* = D_i$. This proves the first claim of the theorem.

The proof of the second claim is straightforward: if Δ, D is an $\lfloor\mathrm{wr}\rfloor$-decomposition of G with B the base group and C a coordinate subgroup, then there is a (unique) k such that $\Delta = \Delta_k$ and so $B = B_k$; moreover, in this case $D = D_k^b$ for some b in B while $C = C_k^d$ for a d in D, and then $C = C_k^{db}$ and $D = D_k^{db}$, too.

The 'if' part of the last claim is obvious, so it remains to prove the 'only if' part. This will also be done by induction on m. The initial step concerns the case $m = 1$ and claims nothing more than what was established in the previous paragraph. For the inductive step, let $m > 1$, choose first a b in B_m such that $H_m = G_m^b$, then a d in G_m^b such that $\langle H_0, \dots, H_{m-1}\rangle = C_m^d$, note that now $G_m^d = G_m^{bd}$ and $C_m^d = C_m^{bd}$, and apply the inductive hypothesis to

$$C_m^{bd}\downarrow\Gamma_m d,\quad \Gamma_m d,\quad m-1,\quad G_0^{bd}\downarrow\Gamma_m d,\dots,G_{m-1}^{bd}\downarrow\Gamma_m d,\quad H_0\downarrow\Gamma_m d,\dots,H_{m-1}\downarrow\Gamma_m d$$

in place of

$$G,\qquad \Omega,\qquad m,\qquad G_0,\dots,G_m,\qquad\qquad H_0,\dots,H_m.$$

The conclusion is that $H_0 = G_0^{bdc}, \dots, H_{m-1} = G_{m-1}^{bdc}$ for some element c in C_m^{bd}. It is a general property of wreath products that the normaliser of the top group in the base group (is the corresponding diagonal and therefore) projects fully onto each coordinate subgroup: thus such a c can be chosen in $\mathrm{N}_{B_m}(G_m^{bd})$ instead of in C_m^{bd}. With that choice, $g = bdc$ has the required properties.

This completes the proof of Theorem 2.6. ∎

PROOF OF COROLLARY 2.7: If $C_1\downarrow\Gamma_1$ had a $\lfloor\mathrm{wr}\rfloor$-decomposition, Δ_0, D_0 say, then the blocks of Δ_0 would also satisfy (2.4a) and one of them would contain ω, so it should have been counted among the Γ_i. Thus $C_1\downarrow\Gamma_1$ is $\lfloor\mathrm{wr}\rfloor$-indecomposable; in view of (2.4a), the conjugates of Δ_1, D_1 are therefore decompositions of the kind considered in the corollary. In the third paragraph of the proof of Theorem 2.6 it was noted that each $C_{i+1}\downarrow\Gamma_{i+1}$ is $\lfloor\mathrm{wr}\rfloor$-decomposable: hence by (2.4a) only the conjugates of Δ_1, D_1 can be decompositions of the kind considered there, and the corollary is proved. ∎

4. PRODUCT ACTION WREATH DECOMPOSITIONS: DEFINITIONS AND RESULTS

Just as the principal use of $\lfloor\mathrm{wr}\rfloor$-products is in dealing with transitive groups, so the main applications of $\lceil\mathrm{wr}\rceil$-products are in the context of primitive groups: we shall restrict attention to that case. For a transitive subgroup G of $\mathrm{Sym}\,\Omega$, a system of imprimitivity is a non-singleton G-orbit of non-singleton subsets which gives a coproduct decomposition of Ω. The dual concept for a primitive G will be called a *system* *of product-imprimitivity:* a non-singleton G-orbit of equivalence relations on Ω which gives a product decomposition of Ω. Here one thinks of an equivalence relation σ as a (certain kind of) subset of the cartesian square Ω^2, and forms its G-translates accordingly: so that, with $[\omega]\sigma$ denoting the σ-class containing the point ω, the (σg)-class $[\omega](\sigma g)$ containing ω is $([\omega g^{-1}]\sigma)g$. Given a set Σ of equivalence relations, for each σ in Σ consider the map $\omega \mapsto [\omega]\sigma$ of Ω onto the set Ω/σ of the σ-classes, then combine these maps into a single map from Ω to the product of the Ω/σ: if this combined map is a bijection, write $\Omega = \prod \{ \Omega/\sigma \mid \sigma \in \Sigma \}$ and say that Σ gives a *product decomposition* of Ω. (Note that if G is primitive and Σ is a non-singleton G-orbit, then only the surjectivity of the combined map is in question, because $\bigcap \{ \sigma \mid \sigma \in \Sigma \}$ is a non-universal G-invariant equivalence relation and therefore must be trivial.) Another way of recognising (or of exploiting) the fact that Σ gives a product decomposition of Ω is the following: for each σ in Σ, choose a σ-class, and then form the intersection of these equivalence classes; for all choices, the intersection so obtained must be a singleton. An equivalence relation on Ω will be called a *block of product imprimitivity* for G if its G-translates form a system of product imprimitivity.

Some further points of notation will be needed. The setwise stabiliser of the subset σ of Ω^2 will be written as $\mathsf{N}_G(\sigma)$ (if this is transitive on Ω, it is the largest subgroup of G for which the σ-classes form a system of imprimitivity). The obvious homomorphism $\mathsf{N}_G(\sigma) \to \mathrm{Sym}\,(\Omega/\sigma)$, whose kernel is the intersection $\mathsf{C}_G(\Omega/\sigma)$ of the setwise stabilisers of the σ-classes, will be denoted by $\downarrow(\Omega/\sigma)$. If Σ is setwise stabilised by G, one writes $\mathsf{C}_G(\Sigma)$ for $\bigcap \{\mathsf{N}_G(\sigma) \mid \sigma \in \Sigma\}$.

When $1 < C \leqslant \mathrm{Sym}\,\Gamma$ and $1 < D \leqslant \mathrm{Sym}\,\Delta$, the *wreath product in product action* $C \lceil\mathrm{wr}\rceil D$ is a certain subgroup of the symmetric group $\mathrm{Sym}\,\Gamma^\Delta$ on the set Γ^Δ of all maps $\Delta \to \Gamma$. Just as it was convenient above to think of $\Gamma \times \Delta$ as a set with a distinguished coproduct decomposition, so it will be important here to think of Γ^Δ as having a distinguished product decomposition. Namely, for each δ in Δ, call two elements of Γ^Δ equivalent if they agree at δ (as functions $\Delta \to \Gamma$), and write σ_δ for the equivalence relation so defined. The set $\Sigma = \{ \sigma_\delta \mid \delta \in \Delta \}$ of these equivalence relations is the product decomposition in question.

Put $W = C \lceil\mathrm{wr}\rceil D$ and $\Omega = \Gamma^\Delta$. As is well-known and very easy to see, W is primitive if and only if C is primitive but not regular and D is transitive. Under these conditions, the distinguished product decomposition Σ of Γ^Δ is a system of product imprimitivity for W. The base group B of W may be recognised as $\mathsf{C}_G(\Sigma)$. Let K_δ denote the product of all but one of the coordinate subgroups of W, missing out the coordinate subgroup labelled by δ; the relation σ_δ could now be defined equivalently by declaring that the orbits of K_δ be the σ_δ-classes: $\Omega/\sigma_\delta = \Omega/K_\delta$. Conversely, the 'coordinate kernel' K_δ can be identified as $\mathsf{C}_B(\Omega/\sigma_\delta)$, so the coordinate subgroups of W can also be recognised from Σ, namely as the intersections of all but one of the coordinate kernels. The proofs of these simple observations will be left to the reader. The general defintion so motivated may be stated as follows.

DEFINITION 4.1: A *product action wreath decomposition* of a primitive group G on a finite set Ω consists of a system of product imprimitivity Σ and a subgroup D such that, for $B = \mathrm{C}_G(\Sigma)$,

$$(4.1a) \qquad D \cap B = 1, \quad DB = G,$$

$$(4.1b) \qquad B = \prod \{ B/\mathrm{C}_B(\Omega/\sigma) \mid \sigma \in \Sigma \},$$

and

$$(4.1c) \qquad (\forall\, \sigma \in \Sigma) \quad \mathrm{N}_D(\sigma) = \mathrm{C}_D(\Omega/\sigma).$$

The natural projections $B \to B/\mathrm{C}_B(\Omega/\sigma)$ combine into a single homomorphism from B to the relevant (external) direct product; (4.1b) is just the preferred way here for saying that this homomorphism is bijective. The preceding discussion has shown that each $\lceil \mathrm{wr} \rceil$-product one constructs comes with a decomposition of this kind. It will be convenient to call D the *top group*, B the *base group*, and the $\mathrm{C}_B(\Omega/\sigma)$ the *coordinate kernels* of the decomposition. Set

$$C_\sigma = \bigcap \{ \mathrm{C}_G(\Omega/\rho) \mid \rho \in \Sigma,\ \rho \neq \sigma \}.$$

Since Σ is a G-orbit, an element of G which fixes all ρ has no choice but to fix σ as well: that is, $\bigcap \mathrm{N}_G(\rho) \subseteq \mathrm{N}_G(\sigma)$. Thus in fact

$$\bigcap \{ \mathrm{N}_G(\rho) \mid \rho \in \Sigma,\ \rho \neq \sigma \} = B;$$

and so C_σ is also the intersection of all but one of the coordinate kernels. It follows that the (internal) direct decomposition of B corresponding to (4.1b) may be written as

$$B = \dot{\prod} \{ C_\sigma \mid \sigma \in \Sigma \},$$

and one also has that

$$(4.1bb) \qquad (\forall\, \sigma \in \Sigma) \quad B = C_\sigma \times \mathrm{C}_B(\Omega/\sigma).$$

Accordingly, the C_σ will be called the coordinate subgroups of the decomposition.

The first few results of Section 2 have the following analogues (proofs are deferred to Section 5).

THEOREM 4.2. *Given a product action wreath decomposition* Σ, D *of a primitive group* G *on a finite set* Ω, *to each* σ *in* Σ *and to each right transversal* T *of* $\mathsf{N}_D(\sigma)$ *in* D *such that* $1 \in T$, *there is a bijection* $\tau : \Omega \to (\Omega/\sigma)^\Sigma$ *which conjugates* G *to* $\left(\mathsf{N}_G(\sigma) \downarrow (\Omega/\sigma)\right) \lceil\mathrm{wr}\rceil \left(G \downarrow \Sigma\right)$, *matching the given decomposition of* G *to the decomposition with which this wreath product comes (and consequently matching top group to top group, base group to base group, coordinate kernels to coordinate kernels, and coordinate subgroups to coordinate subgroups).*

Note that the wreath product in question depends only on Σ and σ; in particular, it is independent of D (and of T). Of course, $G \downarrow \Sigma = D \downarrow \Sigma$.

LEMMA 4.3. *Let* G *be a primitive group on a finite set* Ω *and* Σ, D *a product action wreath decomposition of* G. *A subgroup* R *of* G *is conjugate to a subgroup of* D *if and only if*

$$(4.3\text{a}) \qquad (\forall\, \sigma \in \Sigma) \quad \mathsf{N}_R(\sigma) = \mathsf{C}_R(\Omega/\sigma).$$

LEMMA 4.4. *If* Σ, D *is a product action wreath decomposition of a transitive subgroup* G *of* $\mathrm{Sym}\,\Omega$, *then* $\mathsf{N}_G(\sigma) \downarrow (\Omega/\sigma) = C_\sigma \downarrow (\Omega/\sigma)$ *for each* σ *in* Σ.

Conversely, let σ *be a block of product imprimitivity for* G *such that*

$$(4.4\text{a}) \qquad \mathsf{N}_G(\sigma) \downarrow (\Omega/\sigma) = \left(\bigcap \left\{\, \mathsf{C}_G(\Omega/\sigma g) \mid g \in G,\ \sigma g \neq \sigma \,\right\}\right) \downarrow (\Omega/\sigma),$$

and let Σ *denote the system of product imprimitivity consisting of the* G*-translates of* σ: *then there exist* D *such that* Σ, D *is a product action wreath decomposition of* G, *and these* D *form one conjugacy class of subgroups in* G.

The reader will have no problem defining isomorphism and conjugacy of product action wreath decompositions; clearly, conjugate decompositions have equal base groups. In general, there seems to be no parallel for the idea that an overview of all conjugacy types of imprimitive wreath decompositions may be sought in terms of blocks of imprimitivity satisfying (2.4a) and containing a given point ω. Accordingly, there is no *general* analogue for Lemma 2.5, and the analogues of at least some of its consequences have easy counterexamples.

EXAMPLE 4.5: The primitive wreath product in product action $S_3 \lceil\mathrm{wr}\rceil S_2$ (of the two symmetric groups indicated) has only one isomorphism class of product action wreath decompositions, but this breaks up into two conjugacy classes, and nonconjugate decompositions have incomparable base groups.

There is a parallel for that idea, and a valid analogue for Theorem 2.5, under the restriction that only primitive groups with nonabelian socle be considered. These depend on information from [**3**] concerning blow-up decompositions of primitive groups of this restricted kind, and are technical enough to be best left for the next section; suffice it to say that in them the role of (2.4a) is taken on by (4.4a). The consequent analogues of Theorem 2.6 and Corollary 2.7 can be stated here without going into those technicalities.

THEOREM 4.6. *Let G be a primitive group on a finite set Ω such that the socle of G is nonabelian, let m be the number of conjugacy classes of product action wreath decompositions of G, and suppose that $m > 0$. Then there is a sequence of subgroups $G_0, \dots, G_m$ such that, whenever $1 \leqslant k \leqslant m$, there is a product action wreath decomposition in which $\langle G_0, \dots, G_{k-1}\rangle$ is a coordinate subgroup and $\langle G_k, \dots, G_m\rangle$ is the top group. Moreover, if C is a coordinate subgroup and D is the top group in a product action wreath decomposition of G, then there is a (unique) k and an element g in G such that $C = \langle G_0, \dots, G_{k-1}\rangle^g$ and $D = \langle G_k, \dots, G_m\rangle^g$. Another sequence $H_0, \dots, H_n$ of subgroups of G has this property if and only if $n = m$ and there exists an element g in G such that $H_0 = G_0^g, \dots, H_m = G_m^g$.*

COROLLARY 4.7. *Let G be a primitive group on a finite set Ω; suppose that G is not $\lceil \mathrm{wr} \rceil$-indecomposable, and consider the $\lceil \mathrm{wr} \rceil$-decompositions Σ, D of G with $\lceil \mathrm{wr} \rceil$-indecomposable $\mathsf{N}_G(\sigma)\!\downarrow\!(\Omega/\sigma)$ (for $\sigma \in \Sigma$): there do exist such decompositions, and they form a single conjugacy class.*

Let G, m, $G_0, \dots, G_m$ be as in Theorem 4.6, and let Ω_0 be any orbit of G_0. Further, for $i = 1, \dots, m$, let Ω_i be the set of the G_i-conjugates of $\langle G_0, \dots, G_{i-1}\rangle$, and write $G_i\!\downarrow\!\Omega_i$ for the group of permutations induced on Ω_i by the conjugation action of G_i. One could go on to show that $G_0\!\downarrow\!\Omega_0$ is $\lceil \mathrm{wr} \rceil$-indecomposable while the other $G_i\!\downarrow\!\Omega_i$ are $\lfloor \mathrm{wr} \rfloor$-indecomposable, and that there is a bijection $\Omega \to \left(\dots\left(\Omega_0^{\Omega_1}\right)\dots\right)^{\Omega_m}$ which conjugates G to

$$\left(\dots\left((G_0\!\downarrow\!\Omega_0)\ \lceil \mathrm{wr} \rceil\ (G_1\!\downarrow\!\Omega_1)\right)\ \lceil \mathrm{wr} \rceil\ \dots\right)\ \lceil \mathrm{wr} \rceil\ (G_m\!\downarrow\!\Omega_m).$$

Specifying such a 'left-normed' bracketing, it should be possible to formalise a 'many-factor' $\lceil \mathrm{wr} \rceil$-decomposition concept so that Theorem 4.6 could be re-phrased as follows: *each finite primitive group which is not $\lceil \mathrm{wr} \rceil$-indecomposable has a unique conjugacy class of (many-factor) $\lceil \mathrm{wr} \rceil$-decompositions with the first factor $\lceil \mathrm{wr} \rceil$-indecomposable and the other factors $\lfloor \mathrm{wr} \rfloor$-indecomposable, and the two-factor $\lceil \mathrm{wr} \rceil$-decompositions of the group are precisely those which arise by suitable bracketings of these.* In view of (1.2), here a suitable bracketing means

$$\left((G_0\!\downarrow\!\Omega_0)\ \lceil \mathrm{wr} \rceil \cdots \lceil \mathrm{wr} \rceil\ (G_{k-1}\!\downarrow\!\Omega_{k-1})\right)\ \lceil \mathrm{wr} \rceil\ \left((G_k\!\downarrow\!\Omega_k)\ \lfloor \mathrm{wr} \rfloor \cdots \lfloor \mathrm{wr} \rfloor\ (G_m\!\downarrow\!\Omega_m)\right)$$

for some k, with the omitted bracketing of the k-factor $\lceil\mathrm{wr}\rceil$-product understood as left-normed.

The alternative decomposition concept, which in the two-factor case would amount to naming a coordinate subgroup and the top group, has much to recommend it also in this context but again will not be pursued here.

5. Product action wreath decompositions: proofs

This section is devoted to the proofs of the displayed statements of Section 4. The proofs of the first three results follow very closely the pattern set in Section 2; then come a few comments on Example 4.5. Detailed preparation for the proof of Theorem 4.6 follows next, culminating in a restricted analogue of Lemma 2.5 which even the reader intent on skipping to Section 6 may wish to see. Beyond that, the proofs of Theorem 4.6 and of Corollary 4.7 are so predictable that they are left to the reader.

The first need is for extra technicalities concerning wreath products in product action. Let $C \leqslant \operatorname{Sym}\Gamma$, $D \leqslant \operatorname{Sym}\Delta$, and put $\Phi = \Gamma^{\Delta}$ so $C \lceil\mathrm{wr}\rceil D \leqslant \operatorname{Sym}\Phi$. For each d in D, the corresponding element in the top group of $C \lceil\mathrm{wr}\rceil D$ is the permutation d^* of Φ such that

$$(5.1a) \qquad (\forall\, \delta \in \Delta)\,(\forall\, \varphi \in \Phi)\,\delta(\varphi d^*) = \big(\delta d^{-1}\big)\varphi.$$

Similarly, for each c in C, the corresponding element in the coordinate subgroup labelled by δ in $C \lceil\mathrm{wr}\rceil D$ is the permutation c^{δ} of Φ such that

$$(5.1b) \qquad (\forall\, \delta' \in \Delta)\,(\forall\, \varphi \in \Phi)\,\delta'\big(\varphi c^{\delta}\big) = \begin{cases} (\delta'\varphi)c & \text{if } \delta' = \delta, \\ \delta'\varphi & \text{if } \delta' \neq \delta. \end{cases}$$

Naturally, in the context of $\lceil\mathrm{wr}\rceil$-decompositions repeated use will be made of the definition of the action of G on Σ:

$$(5.1c) \qquad [\omega](\sigma g) = \big([\omega g^{-1}]\sigma\big)g.$$

The very easy Proof of the first part of Lemma 4.4 may as well be got out of the way first. Since $B \leqslant \mathsf{N}_G(\sigma) \leqslant G = DB$, Dedekind's Law and (4.1c), (4.1bb) yield that

$$\mathsf{N}_G(\sigma) = \mathsf{C}_D(\Omega/\sigma)B \leqslant \mathsf{C}_B(\Omega/\sigma)\mathsf{C}_B(\Omega/\sigma)C_\sigma,$$

whence $\mathsf{N}_G(\sigma){\downarrow}(\Omega/\sigma) = C_\sigma{\downarrow}(\Omega/\sigma)$ follows directly. ∎

Proof of Theorem 4.2: Throughout this argument, σ will be a fixed element of Σ. In view of the first part of Lemma 4.4, the wreath product in question may be thought of as $(C_\sigma \downarrow (\Omega/\sigma))\ \lceil \mathrm{wr} \rceil\ (D \downarrow \Sigma)$; call this simply W. Choose any right transversal T for $\mathsf{N}_D(\sigma)$, that is, for $\mathsf{C}_D(\Omega/\sigma)$, in D, such that $1 \in T$. Note that $t \mapsto \sigma t$ defines a bijection $T \to \Sigma$, so each element of Σ may be written uniquely in the form σt with $t \in T$.

Define a map $\tau : \Omega \to (\Omega/\sigma)^\Sigma$, $\omega \mapsto \omega\tau$ by setting

$$(\sigma t)(\omega\tau) = [\omega t^{-1}]\sigma \qquad \text{for each element } \sigma t \text{ of } \Sigma,$$

and a map $\tau' : (\Omega/\sigma)^\Sigma \to \Omega$, $\varphi \mapsto \varphi\tau'$ by

$$\{\varphi\tau'\} = \bigcap \{\, ((\sigma t)\varphi)t \mid t \in T \,\}.$$

(The latter makes sense because we are dealing with a product decomposition of Ω and $(\sigma t)\varphi$ is a σ-class so $((\sigma t)\varphi)t$ is, by (5.1c), a (σt)-class: thus the intersection on the right hand side is always a singleton.) Now

$$\begin{aligned} (\sigma t)(\varphi\tau'\tau) &= \big[(\varphi\tau')t^{-1}\big]\sigma && \text{by the definition of } \tau \\ &= (\sigma t)\varphi && \text{since } (\varphi\tau')t^{-1} \in (\sigma t)\varphi \in \Omega/\sigma \quad \text{by the definition of } \tau' \end{aligned}$$

for all σt in Σ; hence $\tau'\tau = 1$. Conversely, the definition of τ' says that $\omega\tau\tau'$ is the unique element of $\bigcap \big((\sigma t)(\omega\tau)\big)t$, while $\omega t^{-1} \in \big((\sigma t)(\omega\tau)\big)$ by the definition of τ, so $\omega \in \big((\sigma t)(\omega\tau)\big)t$ for all t: thus also $\tau\tau' = 1$. This proves that τ is a bijection and $\tau^{-1} = \tau'$.

Next, define a map $G \to \operatorname{Sym} T$, $g \mapsto \overline{g}$ by

(5.2a) $$(\forall\, t \in T) \quad (\sigma t)g = \sigma(t\overline{g});$$

note that this homomorphism and $\downarrow\Sigma$ are intertwined by the bijection $T \to \Sigma$, $t \mapsto \sigma t$. It will be useful to know that

(5.2b) $$(\forall\, \omega \in \Omega)\,(\forall\, t \in T)\,(\forall\, d \in D)\ [\omega(td)^{-1}]\sigma = \big[\omega\big(t\overline{d}\big)^{-1}\big]\sigma.$$

Indeed,

$$\begin{aligned} [\omega(td)^{-1}]\sigma &= \Big([\omega(td)^{-1}]\sigma\Big)td\big(t\overline{d}\big)^{-1} && \text{because } td\big(t\overline{d}\big)^{-1} \in \mathsf{N}_D(\sigma) = \mathsf{C}_D(\Omega/\sigma) \\ &= \big([\omega](\sigma td)\big)\big(t\overline{d}\big)^{-1} && \text{by (5.1c) applied with } g = td \\ &= \Big([\omega]\big(\sigma\big(t\overline{d}\big)\big)\Big)\big(t\overline{d}\big)^{-1} && \text{by (5.2a)} \\ &= \big[\omega\big(t\overline{d}\big)^{-1}\big]\sigma && \text{by (5.1c) applied with } g = t\overline{d}. \end{aligned}$$

The first claim to establish is that $\tau' D\tau$ is the top group of W; more specifically, that $\tau' d^{-1}\tau = \left(d^{-1}\downarrow\Sigma\right)^*$ in the sense of (5.1a) whenever $d \in D$. According to that rule, this claim amounts to

$$(\forall\, \omega \in \Omega)(\forall\, t \in T)(\forall\, d \in D)\,(\sigma t)\left(\omega d^{-1}\tau\right) = (\sigma t d)(\omega\tau),$$

which holds because

$$\begin{aligned}
(\sigma t)\left(\omega d^{-1}\tau\right) &= \left[\omega d^{-1}t^{-1}\right]\sigma && \text{by the definition of } \tau\\
&= \left[\omega\left(t\overline{d}\right)^{-1}\right]\sigma && \text{by (5.2b)}\\
&= \left(\sigma\left(t\overline{d}\right)\right)(\omega\tau) && \text{by the definition of } \tau\\
&= (\sigma t d)(\omega\tau) && \text{by (5.2a).}
\end{aligned}$$

The proof will be completed by showing that $\tau' C\tau$ is a coordinate subgroup of W, namely the coordinate subgroup labelled by σ. Specifically, it will be shown that $\tau' c\tau = \left(c\downarrow(\Omega/\sigma)\right)^\sigma$ in the sense of (5.1b), whenever $c \in C_\sigma$. By that rule, this claim amounts to

$$(\forall\, t \in T)(\forall\, \omega \in \Omega)\,(\sigma t)\left((\omega\tau)(\tau' c\tau)\right) = \begin{cases} \left((\sigma t)(\omega\tau)\right)c & \text{if } \sigma t = \sigma,\\ \left((\sigma t)(\omega\tau)\right) & \text{if } \sigma t \neq \sigma.\end{cases}$$

To see that this holds, calculate

$$\begin{aligned}
(\sigma t)\left((\omega\tau)(\tau' c\tau)\right) &= (\sigma t)(\omega c\tau)\\
&= [\omega c t^{-1}]\sigma && \text{by the definition of } \tau\\
&= \left([\omega]\left(\sigma t c^{-1}\right)\right)(t c_{-1}) && \text{by (5.1c) with } g = tc^{-1}\\
&= \left([\omega](\sigma t)\right)t^{-1}tct^{-1} && \text{since } c^{-1} \in B = \mathrm{C}_G(\Sigma)\\
&= \left([\omega t^{-1}]\sigma\right)tct^{-1} && \text{by (5.1c) with } g = t\\
&= \left((\sigma t)(\omega\tau)\right)tct^{-1} && \text{by the definition of } \tau.
\end{aligned}$$

Of course, $(\sigma t)(\omega\tau) \in \Omega/\sigma$. If $\sigma t \neq \sigma$ then $c \in \mathrm{C}_G(\Omega/\sigma t)$ whence $tct^{-1} \in \mathrm{C}_G(\Omega/\sigma)$, while if $\sigma t = \sigma$ then $t = 1$, so the above calculations confirm the claim.

This completes the proof of Theorem 4.2. ▯

PROOF OF LEMMA 4.3: The 'only if' claim is obvious. The 'if' claim will be established by an application of the first sentence of part (3) of Theorem 5.5 in Gross and Kovács [1]. In this application, the present

$$RB, \qquad B, \qquad \Sigma, \qquad \mathrm{C}_B(\Omega/\sigma), \qquad D\cap RB, \qquad \text{and} \qquad R$$

take on the rôles there played by

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$$G, \qquad M, \quad I, \quad K_i, \qquad H, \qquad \text{and} \quad R,$$

respectively; there will be no need to specify J. It is easy to see that now the rôle of N_i belongs to $\mathsf{N}_{RB}(\sigma)$. The pattern of the proof of Lemma 2.3 may be followed to the end without difficulty. $\square$

PROOF OF THE SECOND HALF OF LEMMA 4.4: Let σ be a block of product imprimitivity satisfying (4.4a), define Σ as the system of product imprimitivity consisting of the G-translates of σ, and define B with reference to this Σ. Choose a right transversal, T say, for $\mathsf{N}_G(\sigma)$ in G; then each element of Σ may be written uniquely in the form σt with $t \in T$. Define $G \to \operatorname{Sym} T$, $g \mapsto \overline{g}$ by (5.2a), and put

$$\text{(5.2bb)} \qquad D = \left\{ d \in G \;\middle|\; (\forall\, \omega \in \Omega)\ (\forall\, t \in T) \quad ([\omega]\sigma)td = ([\omega]\sigma)\big(t\overline{d}\big) \right\}.$$

It is easy to see that the subset D so defined is a subgroup satisfying (4.1c) and the first part of (4.1a): the point to establish is that each element g of G lies in BD. By (5.2a) one has that $t^{-1}(t\overline{g})g^{-1} \in \mathsf{N}_G(\sigma t)$, so by (4.4a) with σt in place of σ one can conclude that there is an element b_t which acts on $\Omega/\sigma t$ as $t^{-1}(t\overline{g})g^{-1}$ does and acts trivially on all $\Omega/\sigma t'$ with $t' \neq t$. Of course such a b_t must lie in B. By (4.1b), there is then a b in B which, for each t, acts on $\Omega/\sigma t$ as $t^{-1}(t\overline{g})g^{-1}$. Now

$$\begin{aligned}
([\omega]\sigma)tbg &= \big([\omega t](\sigma t)\big)bg && \text{by (5.1c) with } \omega t \text{ and } t \text{ in place of } \omega \text{ and } g\\
&= \big([\omega t](\sigma t)\big)t^{-1}(t\overline{g})g^{-1}g && \text{because } [\omega t](\sigma t) \in \Omega/\sigma t\\
&= (\omega\sigma)(t\overline{g}) && \text{by the last application of (5.1c)}\\
&= ([\omega]\sigma)\big(t\overline{bg}\big) && \text{because } \overline{b} = 1
\end{aligned}$$

shows that $bg \in D$, whence $g \in DB$.

This has proved that the set of the product action wreath decompositions involving the given Σ is nonempty. The claim that they form a single conjugacy class then follows from Lemma 4.3. $\square$

COMMENTS ON EXAMPLE 4.5: Let $G = S_3 \,\lceil \mathrm{wr} \rceil\, S_2 \leqslant S_9$, and let M denote the socle of G. Now M is a regular normal subgroup in G, so the set G acts on may be identified with M and then G becomes a subgroup of the holomorph of M. The stabiliser G_1 of the point 1 (that is, of the identity element of M) is then the intersection of G with the automorphism group $GL(2,3)$ of M. Of course this G_1 is a dihedral group of order 8. The base group B (which G has by construction) meets G_1 in a non-cyclic subgroup B_1 of order 4. The normaliser of G_1 in $GL(2,3)$ is a Sylow subgroup of order 16 in which B_1 is conjugate to the other non-cyclic maximal subgroup of G_1. If h is an element of $GL(2,3)$ which normalises G_1 but not B_1,

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then h normalises G but not B: so conjugation by h will change the given $\lceil \mathrm{wr} \rceil$-decomposition of G to one with a different base group. This new decomposition is, by construction, isomorphic to the original, but of course cannot be conjugate to it by any element of G. The verification of the remaining claims made about this group is left to the reader. $\square$

The rest of this section will be concerned with product action wreath decompositions of *primitive groups with nonabelian socle.*

To exploit the additional assumption one has to focus on the socle; this calls for a slight extension and variation of the notation used so far. The socle of an arbitrary group X will be denoted $\operatorname{soc} X$; for brevity, the convention will be that $\operatorname{soc} G = M$. Consider $G = C \lceil \mathrm{wr} \rceil D$ with $1 < C \leqslant \operatorname{Sym} \Gamma$ and $1 < D \leqslant \operatorname{Sym} \Delta$, and suppose that G is primitive: then $M = (\operatorname{soc} C)^{\Delta}$ (see for example (2.1) in [**3**]). For each δ in Δ, let K_δ now denote the kernel $\{ f : \Delta \to \operatorname{soc} C \mid \delta f = 1 \}$ of the relevant coordinate projection in this direct decompositon of M (rather than of the corresponding decomposition of the base group, as before). Since $\operatorname{soc} C$ is transitive on Γ (because C is primitive), two elements of Γ^{Δ} are in the same K_δ-orbit if and only if (as functions on Δ) they agree at δ: thus $\Omega/\sigma_\delta = \Omega/K_\delta$ remains true in spite of this change. Of course, now $K_\delta = \mathrm{C}_M(\Omega/\sigma_\delta)$. It will be useful to see one point stabiliser in M: take any γ in Γ, and consider the constant function $\varphi : \Delta \to \Gamma$ which maps all elements of Δ to this one γ; then one has $\mathrm{C}_M(\varphi) = [\mathrm{C}_{\operatorname{soc} C}(\gamma)]^{\Delta}$.

Conversely, let Σ, D be a product action wreath decomposition of a primitive subgroup G of $\operatorname{Sym} \Omega$. It is easy to see that then M has a direct decomposition

$$M = \prod \{ M/\mathrm{C}_M(\Omega/\sigma) \mid \sigma \in \Sigma \} \tag{5.3}$$

with the following properties. First, the direct factors form a (single, complete) conjugacy class of subgroups in G. Second, a point stabiliser in M is the product of its intersections with the direct factors. Third, Σ can be reconstructed from this direct decomposition (a σ-class being an orbit under the product of all but one of the direct factors). In the case when M is not regular, direct decompositions (of M) having the first two of these properties were called *blow-up decompositions* (of G) in [**3**], and it was proved there (see the Remark after Theorem 2^+) that each such group—if it has a blow-up decomposition at all—has a unique finest blow-up decomposition. In that case, write $M = P \times R$ with R one of the factors, and P the product of all the other factors, in the finest blow-up decomposition of G. The P so defined cannot be transitive (on Ω). [To see this directly, note that $\mathrm{C}_M(\omega) = \mathrm{C}_P(\omega) \times \mathrm{C}_R(\omega)$ by the second property above, so if P were transitive then $|M : \mathrm{C}_M(\omega)| = |P : \mathrm{C}_P(\omega)|$ would yield that $\mathrm{C}_R(\omega) = R$; as in any case M is the product of the $\mathrm{C}_G(\omega)$-conjugates of R, one would get $\mathrm{C}_G(\omega) \geqslant M$, a contradiction.] In the case when M is regular but nonabelian, the unique direct decomposition of M with simple direct factors also has the first two properties above, so one can just write $M = R \times P$ with R simple, and note that P, now a proper subgroup in a regular group, cannot be transitive in this case either.

Let π denote the equivalence relation on Ω whose classes are the orbits of the P so chosen. Since P is not transitive, π is not the universal relation. Note that P and π were defined without reference to the particular product action wreath decomposition Σ, D; nevertheless, all but one of the factors of the direct decompositon (5.3) of M must lie in P, that is, *there is a unique* σ *in* Σ *such that* $\sigma \subseteq \pi$. This is one of the conclusions which has to be carried forward.

Let $\mathcal{M}$ denote the set of G-conjugates of R. Each direct factor in the decompositon of M corresponding to Σ is a product of some members of $\mathcal{M}$, so this decomposition of M yields (and is recoverable from) a system of imprimitivity for the conjugation action of G on $\mathcal{M}$. Call this system Σ^*. Note that here the term 'system of imprimitivity' is being used loosely, in that one cannot exclude the possibility that all blocks of Σ^* are singletons.

For each subset $\mathcal{S}$ of $\mathcal{M}$, let S be the product of the members of $\mathcal{S}$, and $\mathcal{S}\lambda$ the equivalence relation on Ω whose equivalence classes are the S-orbits. The $\mathcal{S}\lambda$ form a sublattice Λ in the lattice of all equivalence relations on Ω, and λ is a lattice isomorphism onto Λ from the Boolean lattice of all subsets of $\mathcal{M}$. For $\sigma = \mathcal{S}\lambda$, let let σ' denote the unique complement $(\mathcal{M} \setminus \mathcal{S})\lambda$ of σ in Λ. As $\bigcap \{\rho \in \Sigma \mid \rho \neq \sigma\}$ is a complement of σ in Λ, it must be this σ'. Further,

$$\Omega/\sigma' = \prod \{\Omega/\rho \mid \rho \in \Sigma,\ \rho \neq \sigma\}$$

shows that

$$\mathrm{C}_G(\Omega/\sigma') = \bigcap \{\mathrm{C}_G(\Omega/\rho) \mid \rho \in \Sigma,\ \rho \neq \sigma\},$$

hence the relevant case of (4.4a) may be written as

$$\mathrm{N}_G(\sigma)\downarrow(\Omega/\sigma) = \mathrm{C}_G(\Omega/\sigma')\downarrow(\Omega/\sigma). \tag{5.4}$$

Consider two blocks θ, σ of product-imprimitivity for G, such that $\pi \supseteq \theta \supseteq \sigma$ and (5.4) holds [for this σ: do *not* assume that it holds also for θ in place of σ]. Let Θ and Σ be the G-orbits of θ and σ. Then Θ^* and Σ^* are sets of subsets of $\mathcal{M}$. Let $\mathcal{T}$ and $\mathcal{S}$ be the members of Θ^* and Σ^*, respectively, which contain the element R of $\mathcal{M}$: then $\mathcal{T}\lambda = \theta'$ and $\mathcal{S}\lambda = \sigma'$, so $\mathcal{T} \subseteq \mathcal{S}$. It is easy to see that $\{\mathcal{T}^g \mid g \in G,\ \mathcal{T}^g \subseteq \mathcal{S}\}$ is a block of imprimitivity, call it $\mathcal{S}^*$, for the action of G on Θ^*. Use (5.4) to prove that (2.4a) holds for Θ^*, $\mathcal{S}^*$ in place of Ω, Γ: this exercise is left for the reader.

It is easy to see that if σ_1, σ_2 are blocks of product-imprimitivity for G satisfying (5.4) in place of σ and such that $\pi \supseteq \sigma_1$ and $\pi \supseteq \sigma_2$, then $\sigma_1 \vee \sigma_2$ (the join formed in Λ) is also a block of product-imprimitivity. Apply the previous paragraph twice, both times with $\theta = \sigma_1 \vee \sigma_2$, to obtain two blocks of imprimitivity $\mathcal{S}_1^*$, $\mathcal{S}_2^*$ for the action of G on Θ^*, with both blocks containing $\mathcal{T}$ and satisfying (2.4a). By Lemma 2.5, one of $\mathcal{S}_1^*$ and $\mathcal{S}_2^*$ must then contain the other: equivalently, *one of σ_1 and σ_2 must be contained in the other*. This is the desired analogue of Lemma 2.5.

Everything is together now for proving Theorem 4.6 and Corollary 4.7 by imitating the proofs of Theorem 2.6 and Corollary 2.7; the details are left to the reader.

An alternative line would be to exploit the relevant version (1.2) of the associative law, as follows. The blocks of product-imprimitivity for G lying in π and satisfying (5.4) must form a chain: say, $\sigma_1 \supset \cdots \supset \sigma_m$. If $m \leqslant 1$ there is nothing more to prove, so suppose $m \geqslant 2$. Let Σ_1, D_1 be a $\lceil \mathrm{wr} \rceil$-decomposition with $\sigma_1 \in \Sigma_1$, and C_1 the coordinate subgroup corresponding to σ_1 in this decomposition; set $G_0 = C_1$. For $i = 2, \ldots, m$, let Σ_i^* be the relevant system of imprimitivity satisfying (2.4a) for the action of G on $\mathcal{M}$. [Note that these are genuine systems of imprimitivity: the blocks of Σ_1^* could perhaps be singletons, but never those of the Σ_i^* with $i \geqslant 2$.] Further, $G \downarrow \mathcal{M} = D_1 \downarrow \mathcal{M}$, and $\downarrow \mathcal{M}$ is one-to-one on D_1. Thus Theorem 2.6 can be invoked, with the transitive $D_1 \downarrow \mathcal{M}$ and $m-1$ in place of G and m, to provide suitable subgroups $G_1, \ldots, G_m$ in D_1. It is then possible to prove along the lines of (1.2) that the $G_0, \ldots, G_m$ so chosen have all the required properties.

6. The counting problem

Consider a primitive subgroup G of $\operatorname{Sym}\Omega$ with non-regular socle M. How can one account for all the subgroups W of $\operatorname{Sym}\Omega$ which contain G and which are wreath products in product action? I can deal with this problem only under the additional restriction that the socle of W is M. For such W, the counting may be done as follows.

Let H be a point stabiliser in G, and K a maximal normal subgroup of M. Let $K_1, \ldots, K_k$ be the maximal normal subgroups of M such that $H \cap K_i = H \cap K$; set $P = \bigcap K_i$, $Q = M\mathsf{N}_H(H \cap K)$, and $R = \mathsf{C}_M(P)$. According to Theorem 2^+ of [**3**], the set of all blow-up decompositions of G is bijective with the set of all subgroups X of G such that $Q \leqslant X < G$. For each such X, consider the direct decomposition of M which is the blow-up decomposition of G corresponding to X, and then the product decomposition Σ_X of Ω corresponding to that. Explicitly, form the normal core of P in X, let σ_X denote the eqivalence relation on Ω whose equivalence classes are the orbits of that normal core, and let Σ_X be the G-orbit of σ_X. Next, form $\mathsf{N}_G(\sigma_X) \downarrow (\Omega/\sigma_X)$, form the normaliser N_X of $\mathsf{N}_M(\sigma_X) \downarrow (\Omega/\sigma_X)$ in $\operatorname{Sym}(\Omega/\sigma_X)$, and count the number

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a_X of the $\lceil \mathrm{wr} \rceil$-indecomposable groups A_X which are such that

$$\mathsf{N}_G(\sigma_X) \downarrow (\Omega/\sigma_X) \leqslant A_X \leqslant N_X .$$

Further, count the number d_X of those subgroups D_X of $\mathrm{Sym}\, \Sigma_X$ which contain $G \downarrow \Sigma_X$.

The result of this section is that *the number of the relevant W is $\sum a_X d_X$, with summation over the indicated range of X.*

For a sketch of the proof, let W be a subgroup of $\mathrm{Sym}\, \Omega$ which is a wreath product in product action, contains G, and has socle M. By Corollary 4.7, all $\lceil \mathrm{wr} \rceil$-decompositions of W with $\lceil \mathrm{wr} \rceil$-indecomposable first factor involve the same product decomposition, Σ say, of Ω. This Σ determines a direct decomposition

$$\begin{aligned} M &= \prod \{ M/\mathsf{C}_M(\Omega/\sigma) \mid \sigma \in \Sigma \} \\ &= \prod \{ \mathsf{C}_M(\mathsf{C}_M(\Omega/\sigma)) \mid \sigma \in \Sigma \} \end{aligned}$$

of M which is such that each point stabiliser $\mathsf{C}_M(\omega)$ is the product of its intersections with the direct factors. Since Σ is a single G-orbit, the direct factors in this direct decomposition of M form a conjugacy class of subgroups of G. Thus what we have is a blow-up decomposition of G, and so Σ is Σ_X for a unique X in the given range. Further, $\mathsf{N}_W(\sigma_X) \downarrow (\Omega/\sigma_X)$ is an A_X and $W \downarrow \Sigma_X$ is a D_X of the kind indicated.

Given any X, A_X, D_X, whether so obtained from a W or not, one can construct a subgroup V in $\mathrm{Sym}\, \Omega$ as follows. Choose a transversal T for $\mathsf{N}_G(\sigma_X)$ in G. Since G is transitive on Σ_X, this T is also a transversal for the normaliser of σ_X in $\mathsf{N}_{\mathrm{Sym}\,\Omega}(\Sigma_X)$. It is not hard to see that (4.4a) is satisfied by $\mathsf{N}_{\mathrm{Sym}\,\Omega}(\Sigma_X)$ and σ_X in the rôles of G and σ. Consequently, $\mathsf{N}_{\mathrm{Sym}\,\Omega}(\Sigma_X)$ has a $\lceil \mathrm{wr} \rceil$-decomposition with base group $\mathsf{C}_{\mathrm{Sym}\,\Omega}(\Sigma_X)$ and a top group D defined by (5.2a) and (5.2bb) [still with $\mathsf{N}_{\mathrm{Sym}\,\Omega}(\Sigma_X)$ and σ_X in the roles of G and σ]. To each element v of $\mathsf{N}_{\mathrm{Sym}\,\Omega}(\Sigma_X)$, there is then a unique d_v in D such that $v \downarrow \Sigma_X = d \downarrow \Sigma_X$. Set

$$V = \left\{ v \in \mathsf{N}_{\mathrm{Sym}\,\Omega}(\Sigma_X) \;\middle|\; v \downarrow \Sigma_X \in D_X \quad \& \quad (\forall\, t \in T) \quad (t d_v^{-1} v t^{-1}) \downarrow (\Omega/\sigma_X) \in A_X \right\}.$$

One can argue, in turn, that this V is a subgroup, that $V \geqslant G$, that Σ_X and $D \cap V$ give a $\lceil \mathrm{wr} \rceil$-decomposition for V showing that $V \cong A_X \lceil \mathrm{wr} \rceil D_X$, and that the socle of V is M. Finally, one notes that if X, A_X, D_X were originally obtained from a W then $W \leqslant V$; as W is also isomorphic to $A_X \lceil \mathrm{wr} \rceil D_X$, it has the same order as V and is therefore equal to V.

This completes the proof of our counting and the details of our accounting. Of course we have done no more than reduce the problem to similar ones in smaller symmetric groups. In paticular, $\mathsf{N}_G(\sigma_X) \downarrow (\Omega/\sigma_X)$ is a primitive group with non-regular socle; the A_X we count are the larger subgroups of the ambient symmetric group which have the same socle, but excluding those which are wreath products in product action. This is reasonable because what we can count we can also discount, and $\mathrm{Sym}\,(\Omega/\sigma_X)$ is much smaller than $\mathrm{Sym}\,\Omega$. Counting the D_X is perhaps a taller order, but then $\mathrm{Sym}\,\Sigma_X$ is very much smaller than $\mathrm{Sym}\,\Omega$.

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