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A GENERALISATION OF THE FROBENIUS RECIPROCITY THEOREM

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Abstract

Let G be a locally compact group and K a closed subgroup of G. Let γ, π be representations of K and G respectively. Moore's version of the Frobenius reciprocity theorem was established under the strong conditions that the underlying homogeneous space G/K possesses a right-invariant measure and the representation space $H(\gamma)$ of the representation γ of K is a Hilbert space. Here, the theorem is proved in a more general setting assuming only the existence of a quasi-invariant measure on G/K and that the representation spaces $\mathfrak{B}(\gamma)$ and $\mathfrak{B}(\pi)$ are Banach spaces with $\mathfrak{B}(\pi)$ being reflexive. This result was originally established by Kleppner but the version of the proof given here is simpler and more transparent.

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1. Introduction

Our aim is to give a proof of the Frobenius Reciprocity theorem for locally compact groups in a setting where the representation spaces are Banach spaces. The version given here is a generalisation of a theorem of Moore [10]. In fact, this generalisation was already proved by Kleppner in [6]. However, the ideas and proof given there are much more demanding than is really required; in particular, relying heavily on Kleppner's idea of intertwining forms. The proof given here follows Moore's original approach and relies in places on some of Moore's arguments. It uses standard machinery from the theory of locally compact groups such as the existence of quasi-invariant measures and regular cross-sections.

Let G be a locally compact group and $K \subset G$ a closed subgroup. We consider representations of these groups by isometries on Banach spaces. The Banach space associated with a representation π is denoted by $\mathfrak{B}(\pi)$. For γ and π representations of K and G respectively, let U^γ denote the induced representation of γ to G (see [7]) and $(\pi)_K$ the restriction of the representation π to K. The Frobenius Reciprocity theorem states that, under suitable conditions, there is an isomorphism

$$\psi : \mathrm{Hom}_K(\gamma, (\pi)_K) \to \mathrm{Hom}_G(U^\gamma, \pi), \tag{1.1}$$

where, for two representations π_1 and π_2 of the same group H, $B \in \mathrm{Hom}_H(\pi_1, \pi_2)$ means that $B : \mathfrak{B}(\pi_1) \to \mathfrak{B}(\pi_2)$ and $B\pi_1(h) = \pi_2(h)B$ for $h \in H$.

This result is classical for the case where G is finite and γ, π are finite dimensional representations (over $\mathbb{C}$) (see [7]). Mackey [8] and Mautner [9] discussed generalisations of the theorem to the case where G is locally compact but with restrictions including, but not exclusively, that the representation spaces are Hilbert spaces. Moore [10] proved the theorem as stated in (1.1) when G is locally compact and separable, the representation space $\mathfrak{B}(\gamma)$ is a Hilbert space, $\mathfrak{B}(U^\gamma)$ is a Banach space and the homogeneous space G/K possesses a right *invariant* measure. An essential feature of Moore's version of the theorem, is that the induced representation space $\mathfrak{B}(U^\gamma)$ consists of L_1 functions rather than the more usual L_2 functions (L_1-*inducing*), together with a standard covariance condition (see [10], page 360, (1)). Kleppner [6] extended the result to Banach spaces and removed the invariant measure condition from Moore's version, replacing it by the weaker requirement of a quasi-invariant measure. This imposes essentially no conditions on the closed subgroup; such measures exist in generality for our situation. Fontenot and Schochetman [3] have proved the theorem for L_1 spaces where the covariance condition is modified to include the ratio of the modular functions of the group and the subgroup. The result is also established for L_p-inducing spaces for $1 < p < \infty$ by Jaming and Moran [5] but with the added constraint that G/K is compact.

1.1. Preliminaries and notation. Let G be a second countable locally compact group and $K \subset G$ a closed subgroup. All representations γ are by isometries of separable Banach spaces $\mathfrak{B}(\gamma)$ (see [4], Chapter IV). For representations π and τ of G on Banach spaces, a bounded linear operator T from $\mathfrak{B}(\pi)$ to $\mathfrak{B}(\tau)$ is an *intertwining operator* for π and τ if $T\pi(x) = \tau(x)T$ for $x \in G$. The Banach space of all intertwining operators from $\mathfrak{B}(\pi)$ to $\mathfrak{B}(\tau)$ is denoted by $\mathrm{Hom}_G(\pi, \tau)$.

For γ a representation of K and μ a quasi-invariant measure with a continuous ρ-function ρ [4, 7] on the homogeneous space $X = G/K$ of right cosets, $\lambda(\cdot, y)$ is the Radon–Nikodym derivative of the right translation of μ by μ. Note that $\lambda(x, y) = \rho(xy)/\rho(x)$ [4]. We denote by $L_1(\gamma, \mu)$ the set of all functions f from G to the Banach Space $\mathfrak{B}(\gamma)$ such that:

(a) $\langle f(x), v\rangle$ is a Borel function of x for all v in the dual $\mathfrak{B}(\gamma)^*$ of $\mathfrak{B}(\gamma)$;
(b) f satisfies the covariance condition $f(kx) = \gamma_k f(x)$ for all $k \in K$ and $x \in G$; and
(c) $\|f\|_1 = \int_X \|f(x)\|\, d\mu(z) < \infty$.

Note that the integrand in the above integral is constant on each right coset Kx because of (b). With the usual identification of functions equal almost everywhere, $L_1(\gamma, \mu)$ is a Banach space under the norm defined by (c). Moreover it is nonempty because if ℓ is a continuous complex-valued function with compact support on G and $u \in \mathfrak{B}(\gamma)$, then $W(\ell, u)(s) = \int_K \ell(ks)\gamma_k^{-1}(u)\, d\nu_K(k)$ defines an element of $L_1(\gamma, \mu)$. Here, ν_K is the right-invariant Haar measure on K.

The representation ${}^{\mu}_{1}U^\gamma : y \to {}^{\mu}_{1}U^\gamma_y$ of G on $L_1(\gamma, \mu)$ *induced by* γ is defined by

$$({}^{\mu}_{1}U^\gamma_y f)(x) := \lambda(x, y) f(xy), \quad (x, y \in G, f \in L_1(\gamma, \mu)).$$

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It is easy to check that ${}^{\mu}_{1}U^{\gamma}$ is a representation of G on $L_1(\gamma,\mu)$ and, for any two quasi-invariant measures μ and μ' on X, the two representations ${}^{\mu}_{1}U^{\gamma}$ and ${}^{\mu'}_{1}U^{\gamma}$ are equivalent (see [1], Theorems 3.3.8 and 3.3.9). We write, more simply, U_1^{γ} for the induced representation of γ.

2. Frobenius reciprocity theorem

Theorem 2.1. *Let G be a locally compact group and $K \subset G$ a closed subgroup. Let γ and π be representations of K and G respectively, where $\mathfrak{B}(\pi)$ is reflexive. If $(\pi)_K$ denotes the restriction of π to K, then*

$$\mathrm{Hom}_K(\gamma,(\pi)_K) \cong \mathrm{Hom}_G(U_1^{\gamma},\pi).$$

Proof. Fix a regular Borel section S of K in G (see [8], Lemma 1.1), so that $G/K \simeq S$ *qua* Borel spaces and μ can be regarded as residing on S. For $B \in \mathrm{Hom}_K(\gamma,(\pi)_K)$, we define a mapping $\psi : \mathrm{Hom}_K(\gamma,(\pi)_K) \mapsto \mathrm{Hom}_G(U_1^{\gamma},\pi)$ by

$$\psi(B)f = \int_S \pi_s^{-1} B f(s)\, d\mu(s) \quad \text{for } f \in L_1(\gamma,\mu).$$

We prove that ψ is an isometry. The proof is in three parts.

Part (1). First we observe that the mapping ψ is well defined. The integrand can be regarded as a function on the coset space and $\psi(B)$ is bounded with

$$\|\psi(B)\| \le \|B\|. \tag{2.1}$$

To show that $\psi(B) \in \mathrm{Hom}_G(U_1^{\gamma},\pi)$, consider

$$\pi_t\psi(B)f = \int_S \pi_{st^{-1}}^{-1} B f(s)\, d\mu(s) \quad (t \in G, f \in L_1(\gamma,\mu)).$$

Changing variables $s \mapsto st$, gives

$$\int_S \pi_s^{-1} B \lambda_K(s,t) f(st)\, d\mu(s) = \psi(B)(U_1^{\gamma}(t)f), \quad (f \in L_1(\gamma,\mu)),$$

proving that $\psi(B) \in \mathrm{Hom}_G(U_1^{\gamma},\pi)$.

Part (2). Next we prove that ψ is surjective; that is, for a bounded linear operator $T : L_1(\gamma,\mu) \to \mathfrak{B}(\pi)$ there exists a $B \in \mathrm{Hom}_K(\gamma,(\pi)_K)$ such that $\psi(B) = T$.

Let $T \in \mathrm{Hom}_G(U_1^{\gamma},\pi)$. We note, as in Moore [10], that $L_1(S,\mathfrak{B}(\gamma),\mu)$ is isomorphic to $L_1(\gamma,\mu)$ by the map $f \mapsto f|S$; this can easily be seen using the regularity of the cross-section for continuous functions with compact support and thence by density for all of L_1. Let Φ be the mapping from the set of continuous functions with compact support in $L_1(S,\mathfrak{B}(\gamma),\mu)$ to that in $L_1(\gamma,\mu)$ and let $f = \Phi g$. Clearly, $g(s) = f(s)$ for $s \in S$. Consider $f \in L_1(\gamma,\mu)$ where $f = \Phi g.u$ with $g \in L_1(S,\mu)$ and $u \in \mathfrak{B}(\gamma)$. Then

$$Tf = T\Phi(g.u).$$

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For $u \in \mathfrak{B}(\gamma)$ define $\tilde{T}_u$ by

$$\tilde{T}_u(g) = T\Phi(g.u).$$

Then $\tilde{T}_u$ is a linear map from $L_1(S, \mu)$ to $\mathfrak{B}(\pi)$ and

$$\|\tilde{T}_u(g)\| = \|T\Phi(g.u)\| \le \|T\Phi\|\,\|g\|\,\|u\|,$$

giving $\|\tilde{T}_u\| \le \|T\Phi\|\,\|u\|$. Therefore $\tilde{T}_u$ is bounded. Since $\mathfrak{B}(\pi)$ is reflexive (hence bounded sets are relatively weakly compact), $\tilde{T}_u$ is a weakly compact operator (as it maps bounded sets into relatively weakly compact sets). Applying [2, Theorem 10, page 507] yields an essentially unique Borel function $\chi_u : S \to \mathfrak{B}(\pi)$ such that

$$T\Phi(g.u) = \tilde{T}_u(g) = \int_S \chi_u(s)g(s)\,d\mu(s)$$

with

$$\operatorname{ess\,sup} \|\chi_u(s)\| = \|\tilde{T}_u\| \le \|T\Phi\|\,\|u\|.$$

Now, following Moore, it is easy to see that, save for a null set $N \subset S$, the map $u \mapsto \chi_u(s)$ is linear and bounded. We call this map $D(s) : \mathfrak{B}(\gamma) \to \mathfrak{B}(\pi)$ and note that its norm is less than or equal to $\|T\Phi\|$ for $s \notin N$. Now define $D(s) = 0$ for $s \in N$, $D(s)u = \chi_u(s)$ a.e. for each u so that

$$\tilde{T}_u(g) = \int_S g(s)D(s)u\,d\mu(s).$$

Finally, if $g \in L_1(S, \mathfrak{B}(\gamma), \mu)$,

$$T\Phi(g) = \int_S D(s)g(s)\,d\mu(s)$$

since both sides represent bounded linear transformations into $\mathfrak{B}(\pi)$ which agree on the dense subspace of $L_1(S, \mathfrak{B}(\gamma), \mu)$ consisting of sums of functions of the form $g.u$ for $u \in \mathfrak{B}(\gamma)$ and $g \in L_1(S, \mu)$. The argument has also shown that $D(s)$ is essentially unique and that

$$\operatorname{ess\,sup} \|D(s)\| \le \|T\phi\| \le \|T\|.$$

Since the set of continuous functions with compact support is dense in $L_1(\gamma, \mu)$ and also noting the fact that $f(s) = g(s)$ for $s \in S$, for $f \in L_1(\gamma, \mu)$,

$$Tf = T\Phi(g) = \int_S D(s)g(s)\,d\mu(s) = \int_S D(s)f(s)\,d\mu(s).$$

Let $s \in S$ and $t \in G$, and consider the action of S on G. Using the Borel isomorphism $G \simeq K \times S$, we can define Borel functions $\omega : G \to K$ and $\alpha : G \to S$ such that

$$s.t = \omega(s.t)\alpha(s.t).$$

This Borel isomorphism maps null sets of G with respect to Haar measure to null sets of the product of Haar measure on K with μ.

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Writing $B(s) = \pi_s D(s)$, for any $y \in G$,

$$\pi_y(Tf) = \int_S \pi^{-1}_{sy^{-1}} B(s) f(s)\, d\mu(s) = \int_S \pi^{-1}_{\alpha(s.y^{-1})} \pi^{-1}_{\omega(s.y^{-1})} B(s) f(s)\, d\mu(s). \tag{2.2}$$

Let $v = \alpha(s.y)$. Since $s.y = \omega(s.y)\alpha(s.y)$, we have $\omega(s.y)^{-1}s = vy^{-1}$. Hence, $\omega(s.y)^{-1} = \omega(v.y^{-1})$, and $s = \alpha(v.y^{-1})$. We make the above substitutions and change variables $s \mapsto \alpha(s.y)$ to obtain

$$\begin{aligned} T(U_1^\gamma(y)f) &= \int_S \pi_s^{-1} B(s) \lambda(s,y) \gamma_{\omega(s.y)} f(\alpha(s.y))\, d\mu(s) \\ &= \int_S \pi^{-1}_{\alpha(v.y^{-1})} B(\alpha(v.y^{-1})) \gamma^{-1}_{\omega(v.y^{-1})} f(v)\, d\mu(v). \end{aligned} \tag{2.3}$$

Using (2.2), (2.3) and Fubini's theorem, together with the fact that $T \in \mathrm{Hom}_G(U_1^\gamma, \pi)$, for almost all $s \in S$ and almost all $y \in G$,

$$B(\alpha(s.y^{-1}))\gamma^{-1}_{\omega(s.y^{-1})} = \pi^{-1}_{\omega(s.y^{-1})} B(s). \tag{2.4}$$

From the Borel isomorphism of G with $K \times S$, for fixed $s \in S$, the map $y \mapsto (\omega(s.y^{-1}), \alpha(s.y^{-1}))$ is also a Borel isomorphism of G with $K \times S$, again preserving null sets. From (2.4), setting $\omega(s.y^{-1})$ to be one of the almost all available values of k, we see that $B(\cdot)$ is constant (say, B) almost everywhere. This also leads to

$$B\gamma_k^{-1} = \pi_k^{-1} B \tag{2.5}$$

for almost all $k \in K$. Now (weak) continuity of γ_k implies that (2.5) holds for all $k \in K$. So $B \in \mathrm{Hom}_K(\gamma, (\pi)_K)$ and

$$Tf = \int_S \pi_t^{-1} B f(t)\, d\mu(t),$$

for some $B \in \mathrm{Hom}_K(\gamma, (\pi)_K)$. The uniqueness of B is implied by the uniqueness of D; hence the above argument implies that ψ is surjective.

Part (3). Finally,

$$\|\psi(B)\| = \|T\| \geq \operatorname{ess\,sup} \|D(s)\| = \operatorname{ess\,sup} \|B(s)\| = \|B\|. \tag{2.6}$$

We also have $\|\psi(B)\| \leq \|B\|$ by (2.1). The inequalities (2.6) and (2.1) then imply that ψ is an isometry, which establishes the theorem. □

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