Proceedings of the Royal Society of Edinburgh, page 1 of 25
DOI:10.1017/prm.2024.38

An alternative approach to solenoidal Lipschitz truncation

Stefan Schiffer
Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany(schiffer@mis.mpg.de)

(Received 23 June 2023; accepted 06 March 2024)

In this work, we present an alternative approach to obtain a solenoidal Lipschitz truncation result in the spirit of D. Breit, L. Diening and M. Fuchs [Solenoidal Lipschitz truncation and applications in fluid mechanics. *J. Differ. Equ.* 253 (2012), 1910–1942.]. More precisely, the goal of the truncation is to modify a function $u \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$ that satisfies the additional constraint $\operatorname{div} u = 0$, such that its modification $\tilde{u}$ is Lipschitz continuous and divergence-free. This approach is different to the approaches outlined in the aforementioned work and D. Breit, L. Diening and S. Schwarzacher [Solenoidal Lipschitz truncation for parabolic PDEs. *Math. Models Methods Appl. Sci.* 23 (2013), 2671–2700, Section 4] and is able to obtain the rather strong bound on the difference between u and $\tilde{u}$ from the former article. Finally, we outline how the approach pursued in this work may be generalized to closed differential forms.

Keywords: divergence free truncation; Lipschitz truncation; Whitney extension

2020 *Mathematics Subject Classification:* 26B20; 26B35

1. Introduction

1.1. Lipschitz extensions and truncations

A technique that is important both in functional analytic results and in applications to partial differential equations (PDEs) is *Lipschitz extension*. More precisely, consider a metric space (X, d), a closed subset $Y \subset X$ and a function $u\colon Y \to \mathbb{R}^d$ that is Lipschitz continuous, i.e. there is $L > 0$ such that

$$|u(x) - u(y)| \leqslant Ld(x, y) \quad \forall x, y \in Y.$$

The aim of Lipschitz extension is to find a function $\tilde{u}\colon X \to \mathbb{R}^d$ that coincides with u on Y and still is Lipschitz continuous on X with the same Lipschitz constant (or a constant that is only worse by some additional multiplicative constant). Such an extension result has been achieved by McShane and Kirszbraun [25, 27] and, in a slightly different setting, by Whitney [33, 34].

In contrast, the task for *truncation* is as follows. Given $u \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^d)$, $1 \leqslant p < \infty$ and $L > 0$, find $\tilde{u} \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^d)$, such that

(T1') $\|D\tilde{u}\|_{L^\infty} \leqslant C(d)L$ for a dimensional constant $C(d)$.

(T2') u and $\tilde{u}$ coincide on a set of large measure.

The most common approach to this truncation is to redefine the function u on a rather small set by using aforementioned Lipschitz extension. The basis for this work is the extension due to Whitney [**34**], which has been adapted and refined for truncations in [**1**, **2**, **15**, **35**].

There are different ways to quantify (T2'). The first option is

(T2a') $\mathcal{L}^N(\{u \neq \tilde{u}\}) \leqslant C(d)L^{-p}\|u\|^p_{W^{1,p}}$,

whereas another option is

(T2b') $\mathcal{L}^N(\{u \neq \tilde{u}\}) \leqslant C(d)L^{-p} \int_{\{|u|+|Du|\geqslant L\}} |u|^p + |Du|^p \,\mathrm{d}x.$

We explain the merits of (T2a') and (T2b') later (cf. §§ 1.3), but obviously (T2b') is stronger than (T2a').

From (T2b') we may indeed infer a bound of the $W^{1,p}$-distance between u and $\tilde{u}$ (e.g. [**35**]), namely that

$$\|u - \tilde{u}\|^p_{W^{1,p}} \leqslant C(d) \int_{\{|u|+|Du|\geqslant L\}} |u|^p + |Du|^p \,\mathrm{d}x, \tag{1.1}$$

for (T2a') at least the $W^{1,r}$-distance, $r < p$ can be effectively bounded.

It is worth mentioning, that for sequences of functions u_n there is an even stronger statement (both improving (T1') and (T2b')) that not only bounds the L^p distance, but already the L^∞-distance (cf. [**28**]), i.e.

(T2c') Suppose that $K \subset \mathbb{R}^{N\times d}$ compact and convex and that $u_n \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^d)$ is such that $\mathrm{dist}(Du_n, K) \to 0$ in L^p. Then there is $\tilde{u}_n$, such that $\tilde{u}_n - u_n \to 0$ in $W^{1,p}$ and $\mathrm{dist}(D\tilde{u}_n, K) \to 0$ in L^∞.

1.2. Solenoidal truncation and the main statement

We might be presented with an additional requirement that u is divergence-free, for example if u is the velocity of an incompressible flow. The truncated version $\tilde{u}$ then also shall satisfy this differential constraint. In particular, in this work we give a proof to the following theorem that shows that a truncation satisfying (T1') and (T2b') is possible in a divergence-free setting.

Theorem 1.1. *Let $N \geqslant 2$ and suppose that $1 \leqslant p < \infty$, $u \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$ obeys $\operatorname{div} u = 0$ and that $L > 0$. There is a (dimensional) constant $C > 0$ and a function $\tilde{u} \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, such that*

(T1) $\|\tilde{u}\|_{W^{1,\infty}} \leqslant CL$;

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

(T2) $\mathcal{L}^N(\{u \neq \tilde{u}\}) \leqslant C(d)L^{-p} \int_{\{|u|+|Du| \geqslant L\}} |u| + |Du| \,\mathrm{d}x$;

(T3) $\operatorname{div} \tilde{u} = 0$.

It should be mentioned that (although not explicitly stated in this form), theorem 1.1 was already established by Breit, Diening & Fuchs [7]. We, however, give a different, more geometric construction of the function $\tilde{u}$ in theorem 1.1. We further comment about the applications in the following §§ 1.3. First, we give some variants of theorem 1.1.

Remark 1.2.

(i) If (T2) is replaced by (T2a') and $p > 1$, then [9, Section 4] offers a quite elegant truncation using that $\operatorname{div} u = 0$ is (essentially) equivalent to $u = \operatorname{curl} v$ for a suitable v in space dimension three. The approach of [9] is then further adapted to evolutionary problems, where one, in addition, needs to take care of the time derivative. In the present work, we stick to the simpler stationary framework.

(ii) In principle, the approach pursued here might be applied to solenoidal maps on (orientable) manifolds. As already an unconstrained truncation for maps on manifolds faces some challenges (cf. [31]), we however only consider $\mathbb{R}^N$.

(iii) One may get the result mentioned in theorem 1.1 not only for u satisfying $\operatorname{div} u = 0$, but, in more generality, for differential forms $v\colon \mathbb{R}^N \to \Lambda^r$ that satisfy $dv = 0$, where d denotes the exterior derivative. We refer to §§ 4.1 and proposition 4.1 for further discussion.

(iv) The statement of theorem 1.1 is also relevant for different regularities (e.g. L^∞- instead of $W^{1,\infty}$), cf. [5, 32] for a discussion of (T2b') and [21, 22] for a discussion of (T2c') in that setting.

(v) In principle, it is also imaginable to construct a divergence-free truncation on bounded domains that additionally preserves boundary values (cf. [20] in an unconstrained static setting and [17] for parabolic problems).

We explain the different approaches to proving a statement in the style of theorem 1.1 in § 2. As mentioned in the remark, in [9] the authors use that divergence-free functions u may be written as $u = \operatorname{curl} v$ and then perform a higher-order truncation on v. Our approach is, however, much closer to the treatment by Breit, Diening & Fuchs, [7]. There, the truncation is defined as a modification of the usual Lipschitz truncation. First, one truncates as if the additional constraint $\operatorname{div} u = 0$ is not present and then adds small modifications/corrections to return to solenoidal functions. In [7] this is achieved by application of the *Bogovskiĭ-operator* (cf. [6]). In the present work we propose to use some structure coming from the definition of the truncation and *explicitly* give a quite elementary definition of a local correction. The connection between both approaches might be seen through a careful definition of the Bogovskiĭ-operator for a special class of functions, also cf. [12].

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

1.3. Solenoidal truncation and its applications

While seemingly being only a technical lemma, adaptations of theorem 1.1 have a variety of applications in the calculus of variations and partial differential equations. First and foremost, solenoidal Lipschitz truncation and its generalizations to parabolic (i.e. time-dependent) problems are used to get existence results for non-linear problems, in particular for viscous fluid mechanics. e.g. [7, 9, 15, 16, 19] and [4, 9, 11, 14] for further refinements. The strategy roughly is as follows. The non-linear problem is approximated by a sequence of problems solvable with classical methods (e.g. monotonicity method) that have solutions u_n. The issue is the incompatibility of the nonlinearity (let us call it f) with weak convergence, i.e. if $u_n \rightharpoonup u$, then in general $f(u_n)$ does not converge weakly to $f(u)$. Recall that there are usually effects of *concentrations* and *oscillations* present in weakly converging sequences. While oscillations do not fare well with nonlinearities, concentrations do. As a consequence, Lipschitz truncation *cuts* away concentrations and allows us to focus on oscillation effects (and show that these do not exist). Therefore, one concludes $f(u_n) \rightharpoonup f(u)$.

Consequently, especially for the (non-Newtonian) Navier–Stokes equations (cf. [7, 9, 16]), where one expects concentration effects for a sequence u_n (also see the survey [10]), it is clear that truncation property (T2a') is enough. In particular, when considering such a sequence u_n, for fixed $L > 0$, we *do not* have

$$\int_{\{|u_n|+|Du_n|\geqslant L\}} |u_n| + |Du_n| \,\mathrm{d}x \longrightarrow 0 \quad \text{uniformly in } n \text{ as } L \to \infty,$$

so, in such a context, (T2a') and (T2b') do *not* offer a qualitative difference, and so (T2a') is sufficient (eg. [9]). That theorem 1.1 is proven in [7] with (T2b') instead of (T2a') is, in that context, only an interesting byproduct.

It should also be mentioned that the time-dependent case (cf. [9]) features an additional challenge compared to the present stationary setting (cf. [7]): The time derivative $\partial_t u$ usually is an element of a space like $L^r((0,T);(W^1_r)')$; i.e. it features a negative Sobolev space. This also motivates the *potential truncation* (cf. § 2.3.1), as writing $u = \operatorname{curl} v$ means that $\partial_t v \in L^r((0,T);L^r)$. While this method is very effective in L^p-spaces (cf. [9, 11, 17]), it falls short in spaces, in which classical solution theory to the equation $\operatorname{curl} v = u$ fails, for instance in L^1. It remains to be seen, whether the present 'direct' approach of constructing the truncation can be translated to a time-dependent setting.

In contrast, in problems arising in the broad context of solid mechanics, bound (T2b') is explicitly required, cf. [1, 2, 20, 35] and [29] for an overview. In particular, the goal of ruling out concentrations is different: *Oscillations* do occur and we can ignore concentrations effects for the sequences (e.g. as they are minimizing sequences to certain functionals). So when truncating, we need to further quantify the difference between the function and its truncation, i.e. (T2b'). We refer to [5, 29, 32, 35] for more discussions on that matter.

The even stronger bound (T2c') is for instance relevant for convex integration (e.g. for the (in)compressible Euler equation), cf. [21, 22] for more discussions.

Finally, we shall also mentioned, that apart from $W^{1,p}$-$W^{1,\infty}$ and L^p-L^∞ truncation also other regularities of (solenoidal) truncation may be studied, for example in BV (cf. [**18**, Ch. 6],[**8**]).

1.4. Structure of the article

The remainder of this article is organized as follows. In § 2 we revisit the classical Whitney's extension/truncation theorem and its modern variant with the additional constraint of solenoidality. In more detail, we recall the construction of a Whitney cover in § 2.1 and recall its application in form of Whitney's extension and truncation result in § 2.2. Furthermore, we shortly outline two previous methods to obtain solenoidal Lipschitz truncations and thus also theorem 1.1 in particular: The potential truncation, following [**9**] and a truncation via local corrections, following [**7**].

In § 3 we present a proof to theorem 1.1. First of all, without the usage of differential forms, we give a formula for the truncation in dimension $N = 3$. Higher dimensions require writing divergence-free functions as closed $(N-1)$-forms, hence we take a short detour on differential forms and Stokes' theorem in § 3.2. Then we are ready to define the truncation in § 3.3. Sections 3.4, 3.5 & 3.6 are then devoted to the proof of theorem 1.1.

Finally, in § 4 we discuss some possible extensions, in particular a parallel result to theorem 1.1 for closed differential forms, and open questions.

2. Whitney's truncation theorem and its application to (solenoidal) Lipschitz truncation

In this section, we revisit approaches to obtain a (solenoidal) Lipschitz truncation based on Whitney's extension. First, we recall Whitney's original approach and its consequences for truncation theorems, cf. [**1**, **35**]. Then we shortly discuss the modifications to this approach to obtain a *solenoidal* Lipschitz truncation from [**7**] and [**9**] (in the stationary case) and compare it to the ansatz of the present work.

2.1. Whitney cubes

Our goal is to extend a function from a closed set $X \subset \mathbb{R}^N$ to $\mathbb{R}^N$. In the context of truncation, we leave the function $u\colon \mathbb{R}^N \to \mathbb{R}^N$ untouched on some set X and modify it on X^C. We often refer to X as the 'good set' and to its complement as the 'bad set'. Further, we assume that X is closed and, therefore, X^C is open.

Following Stein's book [**33**, Chapter VI], we can cover X^C by open, dyadic cubes $(Q_i^*)_{i\in\mathbb{N}}$ with the following properties:

(i*) $X^C = \bigcup_{i\in\mathbb{N}} \bar{Q}_i^*$;

(ii*) $Q_i^* \cap Q_j^* = \emptyset$ if $i \neq j$;

(iii*) $1/4 \operatorname{dist}(Q_i^*, X) \leqslant l(Q_i^*) \leqslant 4 \operatorname{dist}(Q_i^*, X)$ where $l(Q_i^*)$ denotes the sidelength of Q_i^*;

(iv*) if $\bar{Q}_i^* \cap \bar{Q}_j^* \neq \emptyset$, then $1/4 l(Q_i^*) \leqslant l(Q_j^*) \leqslant 4 l(Q_i^*)$;

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

(v*) for each $i \in \mathbb{N}$, the number of cubes Q_j^* such that $\bar{Q}_i^* \cap \bar{Q}_j^* \neq \emptyset$ is bounded by a dimensional constant $C(N)$.

Furthermore, for each cube Q_i^*, we denote by c_i the *centre* of the cube Q_i^*. We may also find a *projection point* $z_i \in X$, such that $\operatorname{dist}(Q_i^*, z_i) \leqslant 4 \operatorname{dist}(Q_i^*, X)$. Moreover, we define the measure μ_i as

$$\mu_i = \mathcal{L}^N \left(\frac{1}{2} Q_i^*\right)^{-1} \left(\mathcal{L}^N \llcorner \left(\frac{1}{2} Q_i^*\right)\right),$$

where $1/2 Q_i^*$ is the open cube with axis parallel faces, centre c_i and sidelength $1/2 l(Q_i^*)$.

In addition, we also consider slightly blown-up open cubes $Q_i = (1+\epsilon) Q_i^*$ with the same centre c_i but sidelength $(1+\varepsilon) l(Q_i^*)$. If ϵ is sufficiently small (e.g. $\varepsilon < 1/32$), then these cubes have the following properties:

(i*) $X^C = \bigcup_{i \in \mathbb{N}} Q_i$;

(ii*) for all $i \in \mathbb{N}$, the number of cubes Q_j with $Q_j \cap Q_i \neq \emptyset$ is bounded by $C(N)$;

(iii*) $1/5 \operatorname{dist}(Q_i, X) \leqslant l(Q_i) \leqslant 5 \operatorname{dist}(Q_i, X)$, where $l(Q_i)$ is the sidelength of Q_i ($l(Q_i) = (1+\epsilon) l(Q_i^*)$);

(iv*) if $Q_i \cap Q_j \neq \emptyset$, then $1/4 l(Q_i) \leqslant l(Q_j) \leqslant 4 l(Q_i)$.

We now take $\varphi \in C_c^\infty((-\varepsilon/2, 1+\varepsilon/2)^N)$ with $\varphi \equiv 1$ on $[0, 1]^N$. By translation and scaling we get $\varphi_j^* \in C_c^\infty(Q_j)$ with $\varphi_j^* \equiv 1$ on Q_j^*.

Using the properties of the cubes Q_j we can show (again, cf. [**33**]), that

$$\varphi_j = \frac{\varphi_j^*}{\sum_{i \in \mathbb{N}} \varphi_i^*} \tag{2.1}$$

defines a partition of unity on X^C, i.e.

$$\sum_{j \in \mathbb{N}} \varphi_j(y) = \chi_{X^C}(y) := \begin{cases} 1 & \text{on } X^C, \\ 0 & \text{on } X. \end{cases}$$

Moreover, $\varphi_j \in C_c^\infty(Q_j)$ and they satisfy the bound

$$\|D^k \varphi_j\|_{L^\infty} \leqslant C(N,k) l(Q_j)^{-k}. \tag{2.2}$$

Before we continue with Whitney's extension theorem, we again point out that all the dimensional constants $C(N)$ and $C(N, k)$ in above construction *do not* depend on the regularity of the set X; all we need is the set X to be closed.

2.2. Whitney extension and the related truncation theorem

Given a closed set $X \subset \mathbb{R}^N$ and a Lipschitz continuous function $u\colon X \to \mathbb{R}$ with Lipschitz constant $L > 0$, we define

$$\tilde{u}(y) = \begin{cases} \sum_{i \in \mathbb{N}} \varphi_i(y) u(z_i) & y \in X^C, \\ u(y) & y \in X. \end{cases} \tag{2.3}$$

Lemma 2.1 Whitney truncation. *If $u\colon X \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant L, then $\tilde{u}\colon \mathbb{R}^N \to \mathbb{R}$ is Lipschitz continuous with Lipschitz constant $C_N L$ for a purely dimensional constant C_N.*

A detailed proof can be found in [**33**, Chapter VI]. We give a brief heuristic idea, why lemma 2.1 works, as a similar argument is used in § 3.

Observe that $\tilde{u} \in C^\infty(X^C)$ and that locally, only finitely many terms are nonzero. Therefore, we can compute its derivative for $y \in X^C$:

$$D\tilde{u}(y) = \sum_{i\in\mathbb{N}} D\varphi_i(y)u(x_i) = \sum_{i,j\in\mathbb{N}} \varphi_j D\varphi_i(y)(u(x_i) - u(x_j)).$$

Note that we explicitly used in the second step that $(\varphi_i)_{i\in\mathbb{N}}$ and $(\varphi_j)_{j\in\mathbb{N}}$ are partitions of unity. Applying the bound for the derivative of φ_i and the Lipschitz bound for $u(x_i) - u(x_j)$ yields an L^∞-bound for the derivative of $\tilde{u}$.

Together with the observation, that $\tilde{u}$ is continuous, this leads to Lipschitz continuity of $\tilde{u}$.

This approach can then be applied to Lipschitz *truncation* as follows: First, we find a 'good set' X, on which u is Lipschitz continuous. For the truncation we then take $\tilde{u} \equiv u$ on the good set and redefine is on the bad set as in (2.3). An important part in showing Lipschitz continuity for $\tilde{u}$ is therefore the following lemma 2.3 that connects Lipschitz continuity to the (centred) Hardy–Littlewood maximal function $\mathscr{M}u$ that is defined via

$$\mathscr{M}u(x) := \sup_{r>0} \fint_{B_r(x)} |u(z)| \,\mathrm{d}z,$$

where $\fint$ denotes the average integral. The associated operator then has the following properties.

Lemma 2.2 The maximal function. *The maximal operator is sublinear, i.e.*

$$\mathscr{M}(u+v)(y) \leqslant \mathscr{M}u(y) + \mathscr{M}v(y)$$

for almost every $y \in \mathbb{R}^N$. Moreover, it is bounded as a map from $L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N)$, whenever $1 < p \leqslant \infty$. For $p = 1$, $\mathscr{M}$ is not bounded from $L^1(\mathbb{R}^N)$ to $L^1(\mathbb{R}^N)$, but from $L^1(\mathbb{R}^N)$ to $L^{1,\infty}(\mathbb{R}^N)$, i.e.

$$\mathcal{L}^N(\{|\mathscr{M}u| \geqslant \lambda\}) \leqslant C\lambda^{-1}\|u\|_{L^1}$$

for every $\lambda > 0$ and $u \in L^1(\mathbb{R}^N)$.

Key to the truncation statement then are the following two observations. The first one, proven in [**1**, **26**] proves that a function u is Lipschitz continuous on sublevel sets of its maximal function.

Lemma 2.3. *Let $u \in W^{1,p}(\mathbb{R}^N; \mathbb{R}^d)$ be continuous. There exists a dimensional constant C, such that for all $\lambda > 0$ and all $x, y \in X_\lambda = \{\mathscr{M}(Du) \leqslant \lambda\}$ we have*

$$|u(x) - u(y)| \leqslant C\lambda|x - y|. \tag{2.4}$$

The second ingredient is an estimate on the measure of the 'bad set', the complement of X_λ, cf. [**35**].

Lemma 2.4. *Let $v \in L^p(\mathbb{R}^N;\mathbb{R}^d)$, $1 \leqslant p < \infty$. There is a dimensional constant C, such that for all $\lambda > 0$ we have*

$$\mathcal{L}^N(\{\mathscr{M}v > \lambda\}) \leqslant C\lambda^{-p} \int_{\{|v| \geqslant \lambda/2\}} |v|^p \,\mathrm{d}x. \tag{2.5}$$

These results are combined to get a *Lipschitz truncation* in the fashion we already mentioned: We take $u\colon \mathbb{R}^N \to \mathbb{R}^d$, set $\tilde{u} = u$ on the 'good set' X_λ and then extend this restriction via Lipschitz extension (2.3) to the full space.

Defining the extension as in (2.3) only bears the problem that $u(z_i)$ might not be well-defined for an arbitrarily chosen z_i. There are several workarounds for this problem, e.g. only taking Lebesgue points, considering $u \in C^1(\mathbb{R}^N)$ first and then using a density argument or using an averaged value of the function instead.

In the following, we pursue the third approach. Therefore, we define:

$$T_{\mathrm{Lip}}u(y) = \begin{cases} \sum_{i\in\mathbb{N}} \varphi_i(y) \int u(x_i)\,\mathrm{d}\mu_i(x_i) & y \in X_\lambda^C, \\ u(y) & y \in X_\lambda. \end{cases} \tag{2.6}$$

A suitable adaptation of lemma 2.3 and the estimate (2.5) then yield the following:

Lemma 2.5 Lipschitz truncation. *Let $u \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^d)$. Define the set $X_\lambda = \{\mathscr{M}u \leqslant \lambda\} \cup \{\mathscr{M}(Du) \leqslant \lambda\}$. Then, for a dimensional constant $C > 0$, the truncation $T_{\mathrm{Lip}}u$ as in (2.6) has the following properties*

(i) *the function $T_{\mathrm{Lip}}u$ is Lipschitz continuous and $\|T_{\mathrm{Lip}}u\|_{W^{1,\infty}} \leqslant C\lambda$;*

(ii) *the set $X_\lambda^C \supset \{u \neq T_{\mathrm{Lip}}u\}$ has $\mathcal{L}^N$-measure bounded by*

$$\mathcal{L}^N(X_\lambda^C) \leqslant C\lambda^{-p} \int_{\{|u|\geqslant\lambda\}\cup\{|Du|\geqslant\lambda\}} |u|^p + |Du|^p \,\mathrm{d}x;$$

(iii) *as a consequence of (ii) we have*

$$\|u - T_{\mathrm{Lip}}u\|_{W^{1,p}}^p \leqslant C \int_{\{|u|\geqslant\lambda\}\cup\{|Du|\geqslant\lambda\}} |u|^p + |Du|^p \,\mathrm{d}x.$$

2.3. Solenoidal truncation

In this section we shortly describe previous approaches to solenoidal Lipschitz truncation and compare it to the approach in the present work.

2.3.1. Potential truncation. First of all, in [**9**, Section 4] (see also [**5**, **21**] for related discussions), the truncation is obtained by writing a divergence-free function $u \in W^{1,p}(\mathbb{R}^3;\mathbb{R}^3)$ as

$$u = \operatorname{curl} v$$

for a function $v \in \dot{W}^{2,p}(\mathbb{R}^3;\mathbb{R}^3)$, truncate v to obtain some $\tilde{v} \in W^{2,\infty}(\mathbb{R}^3;\mathbb{R}^3)$ and set $\tilde{u} = \operatorname{curl}\tilde{v}$.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

This approach is particularly well-suited to evolutionary problems (i.e. $u\colon [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$) and can (together with *parabolic Lipschitz truncation*) also tackle the usually quite low regularity of the time derivative $\partial_t u$.

Seeking for a sharper result in *space* and not in *space-time* (which severely complicates the matter), this form of truncation can, however, only achieve property (T2a') and *not* (T2b'). Even if $u \in W^{1,\infty}(\mathbb{R}^3;\mathbb{R}^3)$, its potential v is not in $W^{2,\infty}(\mathbb{R}^3;\mathbb{R}^3)$ in general (Ornstein's non-inequality [**13**, **30**]) and $\tilde{v} \neq v$ regardless of the truncation parameter L.

Let us further note that in dimension $N = 2$, (T2b') *is satisfied*; the reason being that the map $u \mapsto v$ is bounded from $W^{1,\infty}(\mathbb{R}^2;\mathbb{R}^2)$ to $W^{2,\infty}(\mathbb{R}^2)$. Hence, the truncation of [**9**] (together with the correct spaces), actually yields property (T2b') both in space and space-time. Moreover, the truncation constructed in [**9**] in the stationary setting coincides with the truncation constructed in this work in space dimension two.

2.3.2. Truncation via local corrections. In contrast, the work [**7**] is directly suited towards stationary problems in more involved spaces, e.g. Orlicz spaces. In particular, the truncation outlined in [**7**] is able to obtain (T2b'). Instead of writing $u = \operatorname{curl} v$, we work with u directly, find $\tilde{u}$ as in (2.3) and then add corrector terms to restore solenoidality. As the 'good set' one takes the set where the maximal function of u and of Du is small[1] .

In more detail, we first add a global corrector term Πu, such $\tilde{u} + \Pi u =: \tilde{T} u$ is still a $W^{1,\infty}$-truncation of u. In the second step, one adds local corrector terms $\mathrm{Cor}_i \in W_0^{1,\infty}(Q_i;\mathbb{R}^3)$, that satisfy the divergence-equation

$$\operatorname{div} \mathrm{Cor}_i = \operatorname{div}(\varphi_i(\tilde{u} + \Pi u)). \tag{2.7}$$

Finally, one obtains the divergence-free truncation via

$$T_{\mathrm{div}} u = \tilde{T} u - \sum_{i \in \mathbb{N}} \mathrm{Cor}_i.$$

As one only modifies the function on the bad set $\{\mathscr{M} u > \lambda\} \cup \{\mathscr{M}(Du) > \lambda\}$, one is able to obtain (T2b'). One crucial observation (cf. [**7**, Lemma 2.11]) to get $T_{\mathrm{div}} u \in W^{1,\infty}$ is the following:

First note that the observation that Cor_i belongs to $W^{1,\infty}$ is *not* trivial; the divergence equation (2.7) and its solution operator (the Bogovskiĭ operator [**6**]) is *not* bounded from L^∞ to $W^{1,\infty}$. Instead, one uses that $\operatorname{div}(\varphi_i(\tilde{u} + \Pi u))$ has a very specific form. It is a much smoother function, hence the solution to the divergence equation is in $W^{1,\infty}$. A *uniform* $W^{1,\infty}$-bound for the solution is achieved via the following argument: The partition of unity is obtained by formula (2.1) for a cover consisting of Q_i^* and associated functions φ_i^*. But actually, as all cubes Q_i^* are dyadic, there are only *finitely* many configurations how the cover can locally look

[1] In the original paper [**7**] the authors only considered the symmetric part Eu of the gradient, but the approach stays the same.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

like. In particular, up to scaling and translation, we only need to solve

$$\operatorname{div} \mathrm{Cor}_i = \operatorname{div}\big(\varphi_i \cdot (\tilde{u} + \Pi u)\big) \quad \text{in } Q_i^* \text{ with zero boundary values}$$

a *finite* amount of times as, up to scaling and translation, there are only finitely many functions the right-hand-side can realize. As we then only argue about *finitely* many different corrector terms, the uniform $W^{1,\infty}$-bound is an easy consequence.

2.3.3. Truncations via local corrections II. In this work, we study a modification of the second approach by explicitly constructing the corrector terms using that our truncation has a very particular form. This approach only relies on

(a) some purely geometric estimates;

(b) the existence of a cover with sets Q_i (that are not necessarily cubes);

(c) a partition of unity φ_i that satisfies estimate (2.2).

In particular, the assumptions on the cover over cubes and the partition of unity are less restrictive than in [**7**], which, for instance, might be useful for geometries on manifolds.

Comparing the present ansatz to [**7**], we also add correctors such that $\tilde{u}$ first given by (2.3) is modified to be divergence-free. Instead of solving the divergence-equation via the Bogovskiĭ operator, we give explicit formulas for the corrector terms that rely on certain cancellations and Stokes' theorem. In space dimension three, those corrections are defined on pairs and tuples of cubes, i.e.

$$T_{\mathrm{div}} u = \tilde{u} + \sum_{i,j \in \mathbb{N}} \mathrm{Cor}_{i,j} + \sum_{i,j,k \in \mathbb{N}} \mathrm{Cor}_{i,j,k}.$$

In general dimension, we need more corrector terms, which is connected to the nice algebraic structure provided for the divergence operator.

2.4. Truncation for differential forms

The construction done for the divergence (in three space dimensions) fits into a more general framework for closed differential forms - we introduce the exact notation in § 3. Recall that a divergence-free function $u \colon \mathbb{R}^N \to \mathbb{R}^N$ can be identified with a closed differential form (meaning the exterior derivative vanishes) $\hat{u} \colon \mathbb{R}^N \to \Lambda^{N-1}$; in particular the divergence operator might be identified with the operator of exterior differentiation.

The approach of constructing the corrector terms heavily builds on this algebraic structure and, correspondingly, on Stokes' theorem. This connection becomes easily visible when dealing with curl-free functions, these are $v \colon \mathbb{R}^N \to \mathbb{R}^N$ obeys

$$(\operatorname{curl} v)_{ij} := \partial_i v_j - \partial_j v_i = 0.$$

These can also be put into the framework of closed forms, i.e. $\hat{v} \colon \mathbb{R}^N \to \Lambda^1$. The results then reads as follows.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

LEMMA 2.6. *Let $N \geqslant 2$ and suppose that $1 \leqslant p < \infty$, $v \in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$ obeys* $\operatorname{curl} v = 0$ *and that $L > 0$. There is a (dimensional) constant $C > 0$ and a function $\tilde{v} \in W^{1,\infty}(\mathbb{R}^N;\mathbb{R}^N)$, such that*

(T1) $\|\tilde{v}\|_{W^{1,\infty}} \leqslant CL$;

(T2) $\mathcal{L}^N(\{v \neq \tilde{v}\}) \leqslant C(d)L^{-p} \int_{\{|v|+|Dv|\geqslant L\}} |v| + |Dv| \,\mathrm{d}x$;

(T3) $\operatorname{curl}\tilde{v} = 0$.

The proof of this result can be achieved quite directly by writing $v = \nabla V$ for some $V \in \dot{W}^{2,p}(\mathbb{R}^N, \mathbb{R}^N)$, truncating V to obtain some $\bar{V}$ and then considering the derivative of $\bar{V}$. The existence of such a V is ensured by the differential constraint $\operatorname{curl} v = 0$.

In particular, on the 'bad set' X^C, we have (recall that z_i was a projection point of Q_i onto X)

$$\bar{V}(y) = \sum_{i\in\mathbb{N}} \varphi_i(y)\big(v(z_i) + Dv(z_i)\cdot(y - z_i)\big).$$

Now, one can calculate the derivative of $\bar{V}$ and then replace Du by v to obtain

$$\bar{v}(y) = \sum_{i\in\mathbb{N}} \varphi_i(y)v(z_i) + \sum_{i,j\in\mathbb{N}} \varphi_j(y)D\varphi_i(y)\int_{[z_i,z_j]} (y-\xi)^T \cdot Dv(z)\cdot \frac{z_i - z_j}{|z_i - z_j|}\,\mathrm{d}\xi; \tag{2.8}$$

if $z_i = z_j$, we define the integral in (2.8) to be zero.

Observe that (2.8) features the previously discussed structure. That is, the first sum in (2.8) coincides with the unconstrained extension and the second is a correction to satisfy the constraint $\operatorname{curl}\bar{v} = 0$.

The general case of differential forms (curl-free functions are closed 1-forms) requires multiple steps of corrections. This is carried out in § 3. On the other hand, such a construction involving corrections does not seem to be limited to differential forms, but probably can be achieved for a wider class of differential operators (cf. § 4.2).

3. Solenoidal Lipschitz truncation

This section is concerned with the proof of theorem 1.1 and its immediate consequences. The proof in general dimension requires some basic methods of differential geometry; to demonstrate the strategy we shortly outline the strategy in 3D (which corresponds to closed 2-forms in higher dimensions), as there a coordinate-wise computation is still feasible.

3.1. Definition of the truncation in three space dimensions

We shortly outline the three-dimensional case. To this end, let $X \subset \mathbb{R}^3$ be a closed set and $(Q_i)_{i\in\mathbb{N}}$ be a Whitney cover of X^C featuring a partition of unity φ_i and measures μ_i (cf. § 2.1). For simplicity, we write

$$\mathrm{d}\mu_{i,j} := \mathrm{d}\mu_i(x_i)\,\mathrm{d}\mu_j(x_j), \quad \mathrm{d}\mu_{i,j,k} = \mathrm{d}\mu_i(x_i)\,\mathrm{d}\mu_j(x_j)\,\mathrm{d}\mu_k(x_k).$$

For points $x_{i_1}, \ldots, x_{i_k} \in \mathbb{R}^3$ denote by

$$\mathrm{Sim}(i_1, \ldots, i_k) = \mathrm{Sim}(x_{i_1}, \ldots, x_{i_k)} := \mathrm{conv}(x_{i_1}, ..., x_{i_k})$$

the convex hull of those points. If $k = 2$, we denote this by the direct line $[x_{i_1}, x_{i_2}] := \mathrm{Sim}(i_1, i_2)$ between these points. If $k = 2, 3, 4$, we call this object a non-degenerate simplex if its dimension is $(k-1)$, i.e. $\mathcal{H}^{k-1}(\mathrm{Sim}(i_1, \ldots, i_k)) > 0$.

Definition 3.1. *Given $u \in W^{1,p}(\mathbb{R}^3;\mathbb{R}^3)$, tuples $(a, b, c) \in \{(1, 2, 3), (2, 3, 1), (3, 1, 2)\}$ and $X \subset \mathbb{R}^3$, we define the truncation operator T as follows:*

$$(Tu(y))_a = \begin{cases} \sum_{i\in\mathbb{N}} \varphi_i(y) \int u_a(x_i)\,\mathrm{d}\mu_i(x_i) + (Su(y))_a + (Ru(y))_a & y \in X^C, \\ u(y) & y \in X, \end{cases} \tag{3.1}$$

where the correction term S is defined as

$$(Su)_a := -\frac{1}{2} \sum_{i,j\in\mathbb{N}} \Big(\varphi_j \partial_b \varphi_i (A_{ab}(i,j) - A_{ba}(i,j))\Big) + \Big(\varphi_j \partial_c \varphi_i (A_{ac}(i,j) - A_{ca}(i,j))\Big), \tag{3.2}$$

$$A_{\alpha\beta}(i,j) := \int \fint_{[x_i,x_j]} Du_\beta(z) \cdot (x_i - x_j)(y-z)_\alpha \,\mathrm{d}\mathcal{H}^1(z)\,\mathrm{d}\mu_{i,j}, \tag{3.3}$$

if $i \neq j$ and $A_{\alpha\beta}(i, i) = 0$ for all $i \in \mathbb{N}$.

The corrector R is defined via

$$(Ru)_a := -\sum_{i,j,k\in\mathbb{N}} \varphi_k \partial_b \varphi_j \partial_c \varphi_i B(i,j,k), \tag{3.4}$$

$$B(i,j,k) := \int \fint_{\mathrm{Sim}(i,j,k)} \frac{1}{2}(x_i - x_j) \times (x_j - x_k) \cdot \Big(\partial_1 u(y-z)_1 + \partial_2 u(y-z)_2 \tag{3.5}$$

$$+ \partial_3 u(y-z)_3\Big)\,\mathrm{d}\mathcal{H}^2(z)\,\mathrm{d}\mu_{i,j,k}, \tag{3.6}$$

if i, j and k are pairwise disjoint and $B(i, i, k) = B(i, j, i) = B(i, j, j) = 0$.

We may prove the following statement, which is part of the higher dimensional treatment in § 3.3 ff.

Theorem 3.2. *If $u \in W^{1,p}(\mathbb{R}^3;\mathbb{R}^3)$ with $\operatorname{div} u = 0$ and $\lambda > 0$, set*

$$X = X_\lambda = \{\mathscr{M}u \leqslant \lambda\} \cup \{\mathscr{M}(Du) \leqslant \lambda\}.$$

Then the truncated Tu defined in definition 3.1 satisfies all the assertions of theorem 1.1 for a.e. $L > 0$ with $L = \lambda/2$.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

This is of course just a special case of the statement for general dimension below, cf. theorem 3.5. In 3D, however, every computation can also be done in the local coordinates. The proof then roughly proceeds as follows:

St.1: Argue that $T_0u \in W^{1,p}(\mathbb{R}^3;\mathbb{R}^3)$ and that Su, $Ru \in W_0^{1,p}(X^C;\mathbb{R}^3)$;

St.2: Compute that in X^C we have $\operatorname{div} Tu = 0$ (which we can do via pointwise computation).

St.3: Show L^∞-bounds on $D(Tu)$.

Furthermore, we mention that the treatment of curl-free functions, i.e. closed 1-forms (lemma 2.8), is also done in a coordinate-wise fashion, cf. formula (2.8). Extending above to arbitrary space dimension, one may get a truncation result for closed *two-forms*. The goal of the following subsections is to generalize it to arbitrary dimension.

3.2. Intermezzo: differential forms and Stokes' theorem

We shortly recall some basic notation and properties of differential forms. Taking $\mathbb{R}^N$ as the physical space, $(\mathbb{R}^N)^*$ denotes its dual and

$$\Lambda^r = \underbrace{(\mathbb{R}^N)^* \wedge \ldots \wedge (\mathbb{R}^N)^*}_{r \text{ copies}}$$

denotes the r-fold wedge product of the dual. Likewise, V^r denotes the r-fold wedge product of $\mathbb{R}^N$ with itself. Observe that $\Lambda^0 = \mathbb{R}$, $\Lambda^1 = (\mathbb{R}^N)^*$ and that Λ^N is isomorphic to $\mathbb{R}$. Moreover, Λ^r is the dual of V^r and we write $\omega[\nu]$ to indicate the dual pairing of $\omega \in \Lambda^r$ with $\nu \in V^r$.

We call a (smooth) map $u\colon \mathbb{R}^N \to \Lambda^r$ an r-form. Moreover, there is the exterior derivative d, that maps r-forms into $(r+1)$-forms with the following properties:

(i) $d \circ d = 0$;

(ii) If $\alpha \in C^\infty(\mathbb{R}^N;\Lambda^r)$ and $\beta \in C^\infty(\mathbb{R}^N;\Lambda^s)$, then

$$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^r \alpha \wedge d\beta$$

(iii) $d\colon C^\infty(\mathbb{R}^N;\Lambda^0) \to C^\infty(\mathbb{R}^N;\Lambda^1)$ is the gradient.

Usually, one may identify a divergence-free function $u\colon \mathbb{R}^N \to \mathbb{R}^N$ with a function $\bar{u} \in \mathbb{R}^N \to \Lambda^1$ obeying $d^*u = 0$ through seeing the divergence as the adjoint to the gradient. Instead of this, we use a different identification with $(N-1)$-forms. In particular, for the standard basis e_i of $\mathbb{R}^N$ denote by dx_i the map in $(\mathbb{R}^N)^*$ given by

$$\sum_{i=1}^N u_i e_i \mapsto u_i.$$

Then the vectors

$$\omega_i := (-1)^i dx_1 \wedge \ldots dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_N$$

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

form a basis of Λ^{N-1} and we may define an isomorphism $\Psi\colon \mathbb{R}^N \to \Lambda^{N-1}$ that is defined on the standard basis as $e_i \mapsto \omega_i$. Then we have

$$\operatorname{div} u = 0 \quad \Longleftrightarrow \quad d(\Psi(u)) = 0$$

and therefore we routinely identify u with the $(N-1)$-form $\Psi(u)$.

One then can formulate Stokes' theorem that, for a $(r+1)$-dimensional manifold M links the integral of du on M to the boundary integral of u. Of special interest in the present setting is the case where M is a simplex; to this end recall that for an index $I = (i_1, \ldots, i_k)$ and $x_{i_1}, \ldots, x_{i_k} \in \mathbb{R}^N$ we denote by

$$\operatorname{Sim}(x_I) = \operatorname{Sim}(x_{i_1}, \ldots, x_{i_k}) := \operatorname{conv}(x_{i_1}, ..., x_{i_k}).$$

If the dimension of $\operatorname{Sim}(x_I)$ is $(k-1)$, we call the simplex non-degenerate. In that case, we define

$$\nu(x_I) = \tfrac{1}{(k-1)!}(x_{i_1} - x_{i_2}) \wedge \ldots \wedge (x_{i_{k-1}} - x_{i_k}) \in V^{k-1}$$

to be the generalized normal. For any $(k-2)$-dimensional face opposite to x_{i_l}, i.e. $\operatorname{Sim}^l(x_I) = \operatorname{Sim}(x_{i_1}, \ldots, x_{i_{l-1}}, x_{i_{l+1}}, \ldots, x_{i_k})$ we can accordingly define a normal vector $\nu^l(x_I)$ on this face. Further observe that $|\nu(x_I)| = \mathcal{H}^{k-1}(\operatorname{Sim}(x_I))$.

Stokes' theorem on simplices now reads as follows:

$$\fint_{\operatorname{Sim}(x_I)} du[\nu(x_I)] \,\mathrm{d}\mathcal{H}^{k-1} = \sum_{l=1}^{k} (-1)^l \fint_{\operatorname{Sim}^l(x_I)} u[\nu^l(x_I)] \,\mathrm{d}\mathcal{H}^{k-2}. \tag{3.7}$$

3.3. Definition of the truncation for divergence-free functions

Similarly to definition 3.1, we are now able to define the truncation through one part, that corresponds to $W^{1,\infty}$-truncation, and another part consisting of a bunch of corrector terms.

From now on, instead of working with $u\colon \mathbb{R}^N \to \mathbb{R}^N$, through the identification discussed in the previous subsection we instead consider a non-renamed $(N-1)$-form $u\colon \mathbb{R}^N \to \Lambda^{N-1}$. The condition of solenoidality then coincides with closedness of the form, i.e. that the exterior derivative vanishes.

To define correct analogues of $A_{\alpha\beta}$ and B as in definition 3.1, for $u\colon \mathbb{R}^N \to \Lambda^{N-1}$ we define the objects

$$Du[\nu] \quad \text{and} \quad Du[\nu] \cdot (y - z)$$

as follows: We first understand $v = Du$ as an element of $\Lambda^1 \otimes \Lambda^{N-1}$ and consider the pairing of the first component of this tensor with some $\nu \in V^r$ as

$$(v_1 \otimes v_2)[\nu] \mapsto (v_1[\nu] \otimes v_2) \in V^{r-1} \otimes \Lambda^{N-1},$$
$$((v_1 \otimes v_2)[\nu])(y - z) \mapsto (v_1[\nu] \wedge (y - z)) \otimes v_2) \in V^r \otimes \Lambda^{N-1}.$$

and then map $V^s \otimes \Lambda^{N-1}$, $s = r-1, r$ to Λ^{N-1-s} through a duality pairing, i.e. finally

$$(v_1 \otimes v_2)[\nu] := v_2\,(v_1(\nu)) \in \Lambda^{N-r},$$
$$(v_1 \otimes v_2)[\nu](y - z) := v_2\,(v_1(\nu) \wedge (y - z)) \in \Lambda^{N-r-1}.$$

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

Corresponding to definition 3.1, let $X \subset \mathbb{R}^N$ be a closed set and cover its complement with Whitney cubes Q_i. Consider a partition of unity φ_i on Q_i, measures μ_i and the product measure μ_I for $I = (i_1, \ldots, i_r)$ and, for simplicity, write

$$\mathrm{d}\mu_I := \mathrm{d}\mu_{i_1}(x_{i_1}) \ldots \mathrm{d}\mu_{i_r}(x_{i_r}).$$

Definition 3.3 Formula for divergence-free extension. *Let* $u \in W^{1,p}(\mathbb{R}^N; \Lambda^{N-1})$, $X \subset \mathbb{R}^N$ *closed. We define*

$$Tu(y) := \begin{cases} T_0 u(y) + \sum_{k=1}^{N-1} S_k u(y) & y \in X^C, \\ u(y) & y \in X, \end{cases} \tag{3.8}$$

where

$$T_0 u(y) = \sum_{i \in \mathbb{N}} \varphi_i(y) \int u(x_i)\, \mathrm{d}\mu_i(x_i) \tag{3.9}$$

and

$$S_k u(y) := \tfrac{(-1)^k (N-k)!}{N!} \sum_{I \in \mathbb{N}^{k+1}} \varphi_{i_{k+1}}(y) \wedge d\varphi_{i_k}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge A_k(I); \tag{3.10}$$

$$A_k(I) := \int ⨏_{\mathrm{Sim}(x_I)} Du[\nu(x_I)](y - z)\, \mathrm{d}\mathcal{H}^k(z)\, \mathrm{d}\mu(x_I). \tag{3.11}$$

We define the integral in (3.11) *to be zero if the simplex* $\mathrm{Sim}(x_I)$ *is degenerate.*

Observe that the definition of $A_k(I)$ is antisymmetric in the index, i.e. if $\sigma \in S(k+1)$ is a permutation and $\tilde{I} = (i_{\sigma(1)}, \ldots, i_{\sigma(k+1)})$, then

$$A_k(I) = \mathrm{sgn}(\sigma) A_k(\tilde{I});$$

and in particular $A_k(I) = 0$ if $i_{j_1} = i_{j_2}$ for some $j_1 \neq j_2$. Further, as a minor note, $S_k u$ maps to Λ^{N-1}: The wedge product of all φ's is an element of Λ^k and, by definition, $A_k(I) \in \Lambda^{N-k-1}$.

Remark 3.4 Definition 3.1 vs definition 3.3. For clarity, we shortly argue why definition 3.3 is the generalization of the 3D-case, definition 3.1. As mentioned before, we identify $(\mathbb{R}^3)^* \wedge (\mathbb{R}^3)^*$ with $\mathbb{R}^3$ in the following way:

$$e_1 \mapsto dx_2 \wedge dx_3, \; e_2 \mapsto dx_3 \wedge dx_1, \; e_3 \mapsto dx_1 \wedge dx_2.$$

The terms for S, R, A and B in definition 3.1 then are nothing else than S_1, S_2, A_1 and A_2 in above definition 3.3 written in a coordinate-wise fashion.

The remainder of this section is devoted to prove the following result.

Theorem 3.5. *Let* $u \in W^{1,p}(\mathbb{R}^N, \Lambda^{N-1})$ *with* $du = 0$ *and* $\lambda > 0$, *set*

$$X = X_\lambda = \{\mathscr{M} u \leqslant \lambda\} \cup \{\mathscr{M}(Du) \leqslant \lambda\}.$$

Then the truncated Tu *defined in definition 3.3 satisfies all the assertions of theorem 1.1 (with the identification of closed* $(N-1)$*-forms to divergence-free functions) for a.e.* $L > 0$ *with* $L = \lambda/2$.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

3.4. Structure of the proof and auxiliary results

In this section, we outline the different steps of the proof. First of all, we remind the reader of the following unconstrained truncation result.

LEMMA 3.6 [**35**]. *Let X be as in theorem 3.5 and $u \in W^{1,p}(\mathbb{R}^N;\Lambda^{N-1})$. Then $\tilde{u}$ defined as*

$$\tilde{u}(y) := \begin{cases} T_0 u(y) & y \in X^C, \\ u(y) & y \in X, \end{cases}$$

satisfies $\tilde{u} \in W^{1,\infty}(\mathbb{R}^N;\Lambda^{N-1})$ and $\|D\tilde{u}\|_{W^{1,\infty}} \leqslant C\lambda$.

Moreover, recall that due to lemma 2.4 we have the following bound on the measure of X_λ.

LEMMA 3.7. *Let $u \in W^{1,p}(\mathbb{R}^N;\Lambda^{N-1})$. Then the set X_λ defined in theorem 3.5 obeys*

$$\mathcal{L}^N(X_\lambda) \leqslant C\lambda^{-p} \int_{\{|u|\geqslant\lambda/2\}\cup\{|Du|\geqslant\lambda/2\}} |u|^p + |Du|^p \,\mathrm{d}x. \tag{3.12}$$

It remains to show that the corrector terms $S_1, \ldots, S_k$ are well-behaved and achieve their goal, i.e. that those terms are bounded in $W^{1,\infty}(\mathbb{R}^N;\Lambda^{N-1})$ and that together with to T_0u they give a solenoidal vector field. We formulate these properties in two separate lemmas.

LEMMA 3.8. *Let $k \in 1, \ldots, N-1$, $u \in W^{1,p}(\mathbb{R}^N;\Lambda^{N-1})$ and X_λ be defined as in theorem 3.5. Then*

(i) $S_k u \in C^\infty(X_\lambda^C;\Lambda^{N-1})$;

(ii) $S_k u \in W_0^{1,1}(X_\lambda^C;\Lambda^{N-1})$;

(iii) $S_k u \in W^{1,\infty}(\mathbb{R}^N;\Lambda^{N-1})$ *and* $\|S_k u\|_{W^{1,\infty}} \leqslant C\lambda$.

The unconstrained truncation T_0u and all S_ku are smooth functions in the open set X_λ^C. Therefore, to prove that $dTu = 0$ globally, it suffices to compute the exterior derivative pointwisely.

LEMMA 3.9. *On the bad set X_λ^C we have $Tu \in C^\infty(X^C;\Lambda^{N-1})$ and, moreover, the strong derivative satisfies*

$$dTu = dT_0u + \sum_{k=1}^{N-1} dS_k u = 0.$$

This lemma is a quite straightforward calculation and depends on certain cancellations and Stokes' theorem. Before proving lemmas 3.8 and 3.9, we shortly realize that these lead directly to the proof of theorem 3.5.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

Proof of theorem 3.5:. Consider Tu as given in (3.8). Lemma 3.6 combined with lemma 3.8 gives that

$$\sum_{k=1}^{N-1} S_k u \in W_0^{1,1}(X_\lambda^C;\Lambda^{N-1})$$

and therefore $Tu \in W^{1,\infty}(\mathbb{R}^N;\Lambda^{N-1})$. Furthermore, we obtain the bound $\|Tu\|_{W^{1,\infty}} \leqslant C\lambda$.

Consequently, to prove that $d(Tu)=0$, it suffices to check that $d(Tu)=0$ pointwise almost everywhere. But we have $d(Tu)(y)=0$ for almost every $y \in X_\lambda^\circ$, as $Tu=u$ in this (open) set. On the other hand, lemma 3.9 shows that $d(Tu)=0$ for every $y \in (X_\lambda^C)$. As for almost every $\lambda>0$ we have $\mathcal{L}^N(\partial X_\lambda)=0$, we are finished in showing that $d(Tu)=0$.

It remains to check the bound for the set where $u \neq Tu$, i.e. (T2). But $\{u \neq Tu\} \subset X_\lambda^C$ and the measure of the latter is bounded in lemma 3.7. $\square$

3.5. L^∞-bounds

This section is devoted to the proof of lemma 3.8. Before proving the lemma, shortly recall that Q_i is a cover of the bad set, every point is only contained in finitely many cubes Q_i and that $\varphi_i \in C^\infty(Q_i;[0,\,1])$ is a partition of unity.

In order to tackle the terms appearing in the definition of A_k in the appropriate fashion, let us formulate the following lemma.

Lemma 3.10. *Let $1 \leqslant k \leqslant N-1$, $Q_1=[-1/2,\,1/2]^N$, $Q_2,\dots Q_{k+1}$ cubes of sidelength $1/4 \leqslant l(Q_r) \leqslant 4$ with $Q_r \subset B_{10}(0)$, $2 \leqslant r \leqslant k+1$. Suppose that $v \in L^1(B_{10}(0))$. Then we can estimate*

$$⨍_{Q_1\times\ldots\times Q_{k+1}} \int_{\mathrm{Sim}(x_1,\ldots,x_{k+1})} |v(z)|\,\mathrm{d}\mathcal{H}^k(z)\,\mathrm{d}x_{k+1}\ldots\mathrm{d}x_1 \leqslant C\int_{B_{10}(0)} |v(z)|\,\mathrm{d}\mathcal{L}^N(z). \tag{3.13}$$

Observe that this inequality scales as follows: If $Q_1=[-1/2\rho,\,1/2\rho]$ and $v \in L^1(B_{10\rho}(0))$, then the left-hand-side scales with a factor of ρ^k and the right-hand-side scales with ρ^N.

It is possible to prove the lemma in various different ways (e.g. by transformation rule or coarea formula), below we use Young's convolution inequality.

Proof of lemma 3.10:. We extend v by zero outside of $B_{10}(0)$. First of all, we rewrite the integral in the simplex by transformation rule

$$\begin{aligned}
&= ⨍_{Q_1\times\ldots\times Q_{k+1}} \int_{\mathrm{Sim}(x_1,\ldots,x_{k+1})} |v(z)|\,\mathrm{d}\mathcal{H}^k(z)\,\mathrm{d}x_{k+1}\ldots\mathrm{d}x_1 \\
&= ⨍_{Q_1\times\ldots\times Q_{k+1}} \int_{\mathbb{D}_k} |v|(t_1x_1+\ldots \\
&\qquad + t_{k+1}x_{k+1})\mathcal{H}^k(\mathrm{Sim}(x_1,\ldots,x_{k+1}))\,\mathrm{d}t\,\mathrm{d}x_{k+1}\ldots\mathrm{d}x_1,
\end{aligned}$$

where $\mathbb{D}_k=\{t \in [0,\,1]^{k+1}\colon t_1+\ldots t_{k+1}=1\}$. As $\mathcal{H}^k(\mathrm{Sim}(x_1,\,\ldots,\,x_{k+1}))$ is uniformly bounded, we can absorb it into a constant. We proceed by estimating the

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

integral over $\mathbb{D}_k^1 = \{t \in \mathbb{D}_k : t_1 \geqslant 1/(k+1)\}$ instead of $\mathbb{D}_k$, the estimate over the full integral follows from symmetry arguments. We now use Fubini's theorem and write

$$
\begin{aligned}
&\fint_{Q_1\times\ldots\times Q_{k+1}} \int_{\mathbb{D}_k^1} |v|(t_1x_1+\ldots+t_{k+1}x_{k+1})\,\mathrm{d}t\,\mathrm{d}x_{k+1}\ldots\mathrm{d}x_1 \\
&\quad = \int_{\mathbb{D}_k^1}\int_{Q_1}\int_{Q_2\times\ldots\times Q_{k+1}} |v|(t_1x_1+t_2x_2+\ldots t_{k+1}x_{k+1})\,\mathrm{d}x_{k+1}\ldots\mathrm{d}x_2\,\mathrm{d}x_1\,\mathrm{d}t \\
&\quad \leqslant \int_{\mathbb{D}_k^1}\int_{\mathbb{R}^N}\int_{(\mathbb{R}^N)^k} |v|(t_1x_1+\tilde{x}_2+\ldots\tilde{x}_{k+1}) \prod_{i=2}^{k+1} t_i^{-N} 1_{t_iQ_i}(\tilde{x}_i)\ \mathrm{d}\tilde{x}_{k+1}\ldots\mathrm{d}\tilde{x}_2\,\mathrm{d}x_1\,\mathrm{d}t,
\end{aligned}
$$

where in this context (and this context only) t_iQ_i is the cube with sidelength $t_il(Q_i)$ and centre t_ic_i. Applying Young's convolution inequality yields that

$$
\begin{aligned}
&\fint_{Q_1\times\ldots\times Q_{k+1}} \int_{\mathbb{D}_k^1} |v|(t_1x_1+\ldots+t_{k+1}x_{k+1})\,\mathrm{d}t\,\mathrm{d}x_{k+1}\ldots\mathrm{d}x_1 \\
&\quad \leqslant C\int_{\mathbb{D}_k^1} \|v(t_1\cdot)\|_{L^1(\mathbb{R}^N)} \leqslant C\|v(\cdot)\|_{L^1(\mathbb{R}^N)},
\end{aligned}
$$

where in the last inequality we use that $t_1 \geqslant 1/k+1$ in $\mathbb{D}_k^1$. □

We are now ready to prove the Lipschitz bounds, i.e. lemma 3.8.

Proof of lemma 3.8:. For (i) observe that all φ_i's are smooth and therefore any summand is smooth. Hence, as locally the sum is finite, we conclude $Tu \in C^\infty(X_\lambda^C;\Lambda^{N-1})$. In particular, only summing over a finite index set (e.g. $I \in \{1,\ldots,M\}^k$), these sums are contained in $C_c^\infty(X_\lambda;\Lambda^{N-1})$.

Due to this observation, for (ii), it suffices to prove that the sum converges absolutely in $W^{1,1}(\mathbb{R}^N;\mathbb{R}^N)$. Fix the first index of I, i.e. fix $i_1 \in \mathbb{N}$. Then only finitely many terms are nonzero. Moreover, $Q_{i_r} \cap Q_{i_1} \neq \emptyset$ and therefore the sidelength of the cubes are comparable. Therefore,

$$
|\varphi_{i_k} \wedge d\varphi_{i_{k-1}} \wedge \ldots \wedge d\varphi_{i_1}| \leqslant Cl(Q_{i_1})^{-(k-1)}. \tag{3.14}
$$

On the other hand, we apply lemma 3.10 to $v = Du[\nu]\cdot(y-z)$. Observe that due to the definition of Q_i, $20Q_i \cap X_\lambda \neq \emptyset$ and therefore, due to the definition of X_λ as a sublevel set of the maximal function of Du and the estimate $|y-z| \leqslant Cl(Q_i)$,

$$
\fint_{20Q_i} |v|\,\mathrm{d}x \leqslant C\lambda l(Q_i).
$$

Therefore,

$$
|A_k(I)| \leqslant C\lambda l(Q_{i_1})^k \tag{3.15}
$$

which leads to

$$
\|\varphi_{i_{k+1}} \wedge d\varphi_{i_k} \wedge \ldots \wedge d\varphi_{i_1} \wedge A_k(I)\|_{L^\infty} \leqslant Cl(Q_{i_1}).
$$

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

The same bounds including the derivative lead to

$$\|D(\varphi_{i_k} \wedge d\varphi_{i_{k-1}} \wedge \ldots \wedge d\varphi_{i_1})\|_{L^\infty} \leqslant Cl(Q_{i_1})^{-k}, \quad \|D_y A_k(I)\|_{L^\infty} \leqslant Cl(Q_{i_1})^{k-1}, \tag{3.16}$$

and we conclude

$$\left\|\varphi_{i_{k+1}}(y) \wedge d\varphi_{i_k}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge A_k(I)\right\|_{W^{1,\infty}} \leqslant C,$$

and, as this function is supported on Q_{i_1}, we obtain

$$\left\|\varphi_{i_{k+1}}(y) \wedge d\varphi_{i_k}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge A_k(I)\right\|_{W^{1,1}} \leqslant Cl(Q_{i_1})^{-N},$$

Again, for only finitely many indices I with this fixed i_1, the summand is nonzero, hence for any $j \in \mathbb{N}$

$$\sum_{I \in \mathbb{N}^k\,:\, i_1 = j} \|\varphi_{i_{k+1}}(y) \wedge d\varphi_{i_k}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge A_k(I)\|_{W^{1,1}} \leqslant Cl(Q_j)^{-N}.$$

Summing over all j and realizing that the Q_j's cover X_λ^C only a finite number of times, we finally obtain

$$\sum_{I \in \mathbb{N}^k} \|\varphi_{i_{k+1}}(y) \wedge d\varphi_{i_k}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge A_k(I)\|_{W^{1,1}} \leqslant \sum_{j \in \mathbb{N}} Cl(Q_j)^{-N} \leqslant C\mathcal{L}^N(X_\lambda^C).$$

Therefore, the sum converges absolutely in $W_0^{1,1}(X_\lambda^C)$, i.e. each $S_k u \in W_0^{1,1}(X_\lambda^C)$.

The $W^{1,\infty}$ bound (iii) now follows from the previously made estimates. □

3.6. Calculations of the pointwise divergence/exterior derivative

In this section we show that for each $y \in X_\lambda^C$ we have $dTu(y) = 0$. This is pure calculation combined with Stokes' theorem. We formulate it in the following lemma. For further reference denote by

$$R_k u := \frac{(-1)^k (N-k)!}{N!} \sum_{I \in \mathbb{N}^{k+1}} \varphi_{i_{k+1}} \wedge d\varphi_{i_k(y)} \wedge \ldots \wedge d\varphi_{i_1} \wedge B_k(I), \tag{3.17}$$

$$B_k(I) := \int \fint_{\mathrm{Sim}(x_I)} Du[\nu(x_I)]\, \mathrm{d}\mathcal{H}^k(z)\, \mathrm{d}\mu_I(x_I). \tag{3.18}$$

LEMMA 3.11. *Let $y \in X^C$. Then the following identities hold for the truncation terms as in definition 3.3*

(i) *We have*

$$dT_0 u = -R_1 u.$$

(ii) *For $1 \leqslant k \leqslant N-2$ we have*

$$dS_k u = R_k u - R_{k+1} u.$$

(iii) *We have*

$$dS_{N-1} u = R_{N-1} u.$$

Observe that lemma 3.9 instantly follows by summing over parts (i)–(ii).

Proof. We start with the exterior derivative of $T_0 u$, i.e. (i). As the only term depending on the variable y is φ_i, we obtain for the derivative with respect to y

$$dT_0u(y) = \sum_{i\in\mathbb{N}} d\varphi_i(y) \wedge \int u(x_i)\,\mathrm{d}\mu_i(x_i).$$

Using that φ_i is a partition of unity, i.e. $\sum_{j\in\mathbb{N}} \varphi_j = 1$ and $\sum_{i\in\mathbb{N}} d\varphi_i = 0$ in the interior of X^C, we obtain:

$$dT_0u(y) = \sum_{i,j\in\mathbb{N}} \varphi_j \wedge d\varphi_i \wedge \left(\int u(x_i)\,\mathrm{d}\mu_i(x_i) - \int u(x_j)\,\mathrm{d}\mu_j(x_j) \right).$$

Application of the fundamental theorem of calculus (i.e. one-dimensional Stokes' theorem) component-wise yields

$$dT_0u(y) = -R_1 u.$$

A similar trick leads to the second statement, (ii). Observe that, in contrast to statement (i), $A_k(I)$ *does* depend on y, hence

$$\begin{aligned} dS_k u = \frac{(-1)^k (N-k)!}{N!} &\sum_{I\in\mathbb{N}^{k+1}} d\varphi_{i_{k+1}}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge A_k(I) \\ + \frac{(-1)^k (N-k)!}{N!} &\sum_{I\in\mathbb{N}^{k+1}} \varphi_{i_{k+1}}(y) \wedge \varphi_{i_k}(y) \ldots \wedge d\varphi_{i_1}(y) \wedge dA_k(I) \end{aligned} \tag{3.19}$$

Now a computation (it, for instance, suffices to check this coordinate-wise and while assuming that $Du[\nu(x_I)] = dx_1 \wedge \ldots \wedge dx_{N-k}$) reveals that

$$dA_k(I) = (N-k)B_k(I).$$

Hence, the second sum in (3.19) equals $R_k u$. For the first term, we observe as φ_{i_r} is a partition of unity on X^C

$$\begin{aligned} &\frac{(-1)^k (N-k)!}{N!} \sum_{I\in\mathbb{N}^{k+1}} d\varphi_{i_{k+1}}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge A_k(I) \\ &\quad = \frac{(-1)^k (N-k)!}{N!} \sum_{\tilde{I}\in\mathbb{N}^{k+2}} \varphi_{i_{k+2}}(y) d\varphi_{i_{k+1}}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge \big(A_k(\tilde{I}_{k+2}) \\ &\quad - \sum_{r=1}^{k+1} A_k(\tilde{I}_r) \big), \end{aligned} \tag{3.20}$$

where $\tilde{I} = (i_1, \ldots, i_{k+2}) \in \mathbb{N}^{k+2}$ and $\tilde{I}_r = (i_1, \ldots, i_{r-1}, i_{r+1}, \ldots, i_{k+2})$. Recall that

$$A_k(\tilde{I}_r) = \int ⨍_{\mathrm{Sim}(x_{\tilde{I}_r})} Du[\nu(x_{\tilde{I}_r})](y-z)\,\mathrm{d}\mathcal{H}^k(z)\,\mathrm{d}\mu_{\tilde{I}_r}(x_{\tilde{I}_r}).$$

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

Thus, the terms in (3.20) form a boundary integral of a $(k+1)$-dimensional simplex. After a suitable identification of spaces (i.e. identifying a wedge product space Λ^{N-1} with suitable (skew-symmetric) elements in $\Lambda^k \otimes \Lambda^{N-1-k}$), we can apply Stokes' theorem by writing

$$\begin{aligned}&\int \fint_{\mathrm{Sim}(x_{\tilde{I}_r})} Du[\nu(x_{\tilde{I}_r})](y-z)\,\mathrm{d}\mathcal{H}^k(z)\,\mathrm{d}\mu_{\tilde{I}_r}(x_{\tilde{I}_r})\\ &\quad= \int \fint_{\mathrm{Sim}(x_{\tilde{I}_r})} F(y,z)[\nu(x_{\tilde{I}_r}]\,\mathrm{d}\mathcal{H}^k(z)\,\mathrm{d}\mu_{\tilde{I}_r}(x_{\tilde{I}_r})\end{aligned}$$

for suitable $F\colon \mathbb{R}^N \to \Lambda^k \otimes \Lambda^{N-k-1}$. Applying Stokes' theorem componentwise, one obtains

$$\begin{aligned}&\frac{(-1)^k(N-k)!}{(N-1)!}\sum_{I\in\mathbb{N}^{k+1}} d\varphi_{i_{k+1}}(y)\wedge\ldots\wedge d\varphi_{i_1}(y)\wedge A_k(I)\\ &\quad= \frac{(-1)^k(N-k)!}{(N-1)!}\sum_{\tilde{I}\in\mathbb{N}^{k+2}} \varphi_{i_{k+2}}(y)\wedge\ldots\wedge d\varphi_{i_1}(y)\\ &\quad\wedge \int \fint d_z F(y,z)[\nu(x_{\tilde{I}})]\,\mathrm{d}\mathcal{H}^{k+1}(z)\,\mathrm{d}\mu_{\tilde{I}}(x_{\tilde{I}}).\end{aligned}$$

Again, a for instance coordinate-wise computation (using that $du=0$) yields that the latter integral is exactly $-(N-k)B_{k+1}(\tilde{I})$.

For (iii) the calculation is exactly the same. It suffices to see that $R_N u = 0$, i.e. that $B_N(I)=0$. Note that for any index I

$$B_N(I) = \int \fint_{\mathrm{Sim}(x_I)} Du[\nu(x_I)]\,\mathrm{d}\mathcal{H}^N(z)\,\mathrm{d}\mu_I(x_I).$$

Considering the definition of $Du[\nu(x_I)]$ for $\nu(x_I)\in V^N \simeq \mathbb{R}$, we obtain $Du[\nu(x_I)] = du(\nu(x_I))$. As u is solenoidal, $du=0$ and hence $B_N(I)=0$. Therefore $R_N=0$, finishing the proof. □

4. Further remarks

In this closing section we discuss some further applications/variants of the technique that was used in § 3 to obtain the truncation.

4.1. Truncation for closed differential forms

The solenoidal truncation is achieved by identifying functions $u\in W^{1,p}(\mathbb{R}^N;\mathbb{R}^N)$ with a non-renamed closed form $u\in W^{1,p}(\mathbb{R}^N;\Lambda^{N-1})$. Via careful construction of the corrector terms using the structure of differential forms we then obtained a solenoidal trunction. This construction with the same arguments can also be repeated if $u\in W^{1,p}(\mathbb{R}^N;\Lambda^{N-1})$ is replaced by a lower-order form, i.e. $u\in W^{1,p}(\mathbb{R}^N;\Lambda^r)$. In particular, we get the following result.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

PROPOSITION 4.1. *Let $N \geqslant 2$ and suppose that $1 \leqslant p < \infty$, $u \in W^{1,p}(\mathbb{R}^N; \Lambda^r)$ obeys $du = 0$ and that $L > 0$. There is a (dimensional) constant $C > 0$ and a function $\tilde{u} \in W^{1,\infty}(\mathbb{R}^N; \Lambda^r)$, such that*

(T1) $\|\tilde{u}\|_{W^{1,\infty}} \leqslant CL$;

(T2) $\mathcal{L}^N(\{u \neq \tilde{u}\}) \leqslant C(d) L^{-p} \int_{\{|u|+|Du| \geqslant L\}} |u| + |Du| \, \mathrm{d}x$;

[(T3) $d\tilde{u} = 0$.

As the proof is very similar to the proof in § 3 we only give a brief sketch. We take the same setup as in § 3.3, i.e. we take the bad set as a sublevel set of the maximal function, cover it with cubes Q_i, consider a partition of unity φ_i and measures μ_i.

Proof(Sketch):. Parallel to the beginning of § 3.3, for $u\colon \mathbb{R}^N \to \Lambda^r$ and a normal $\nu \in V^s$, $1 \leqslant s \leqslant r$, we may give a meaning to

$$Du[\nu] \quad \text{and} \quad Du[\nu] \cdot (y - z)$$

by understanding Du as an element of $\Lambda^1 \otimes \Lambda^r$. Then,

$$Du[\nu] \in \Lambda^{r+1-s} \quad \text{and} \quad Du[\nu] \cdot (y - z) \in \Lambda^{r-s}.$$

Now as in definition 3.3, consider

$$Tu(y) := \begin{cases} T_0(y) + \sum_{k=1}^{r} S_k^r u(y) & y \in X^C, \\ u(y) & y \in X, \end{cases} \tag{4.1}$$

where again $T_0 u$ is the usual Lipschitz extension

$$T_0 u(y) = \sum_{i \in \mathbb{N}} \varphi_i(y) \int u(x_i) \, \mathrm{d}\mu_i(x_i)$$

and

$$S_k^r u(y) := \frac{(-1)^k (r+1-k)!}{(r+1)!} \sum_{I \in \mathbb{N}^{k+1}} \varphi_{i_{k+1}}(y) \wedge d\varphi_{i_k}(y) \wedge \ldots \wedge d\varphi_{i_1}(y) \wedge A_k^r(I); \tag{4.2}$$

$$A_k^r(I) := \int \fint_{\mathrm{Sim}(x_I)} Du[\nu(x_I)](y - z) \, \mathrm{d}\mathcal{H}^k(z) \, \mathrm{d}\mu(x_I). \tag{4.3}$$

Define the integral in (4.3) to be zero if the simplex $\mathrm{Sim}(x_I)$ is degenerate.

We claim that with $X = X_\lambda = \{Mu \leqslant \lambda\} \cup \{\mathscr{M}(Du) \leqslant \lambda\}$ and $\lambda = L/2$, Tu obeys the properties of the proposition.

To show this claim, might argue exactly as in § 3: First of all, parallel to lemma 3.6, T_0 is a sufficient unconstrained truncation and the measure of the bad set is also already bounded; lemma 3.7 still works as it never really used the structure of the target space Λ^{N-1}.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

Furthermore, by using the same calculation, we can show the natural counterpart to lemma 3.8 (the Lipschitz bounds). Finally, we may show the analogue of lemma 3.11, i.e. if for $1 \leqslant k \leqslant r$

$$R_k^r u := \frac{(-1)^k (r+1-k)!}{(r+1)!} \sum_{I \in \mathbb{N}^{k+1}} \varphi_{i_{k+1}} \wedge d\varphi_{i_k(y)} \wedge \ldots \wedge d\varphi_{i_1} \wedge B_k^r(I),$$

$$B_k(I) := \int \fint_{\mathrm{Sim}(x_I)} Du[\nu(x_I)]\, \mathrm{d}\mathcal{H}^k(z)\, \mathrm{d}\mu_I(x_I).$$

then for any $1 \leqslant k \leqslant r-1$ we have

$$dT_0 = -R_1^r u, \quad dS_k^r u = R_k u - R_{k+1}^r u, \quad dS_r^r u = R_r^r u.$$

As seen in § 3, these statements then complete the proof, as we can show the Lipschitz bound, the bound on the set $\{u \neq Tu\}$ and closedness of the differential form Tu. □

4.2. General differential constraints

Generalizing above question, taking some $u \in W^{1,p}(\mathbb{R}^N; \mathbb{R}^d)$, one may ask to truncate under the first-order differential constraint

$$\mathscr{A} u = 0,$$

where

$$\mathscr{A} u = \sum_{\alpha=1}^{N} A_\alpha \partial_\alpha u, \quad A_\alpha \in \mathrm{Lin}(\mathbb{R}^d; \mathbb{R}^m).$$

Unconstrained truncation corresponds to $\mathscr{A} = 0$ and the case of differential forms is represented by $\mathscr{A} = d$ and, in particular, closed $(N-1)$ forms by $\mathscr{A} = \mathrm{div}$. While this general case is out of reach with the results of this paper, the technique of constructing corrector terms does not seem to be entirely hopeless. Meanwhile, similar to [**5**, Conjecture 6.4], we conjecture that a truncation result á la 1.1 is possible for some $\mathscr{A}$, whenever $\mathscr{A}$ satisfies the complex constant rank condition (i.e. the Fourier symbol has constant rank over all *complex* Fourier modes).

4.3. Lower regularity

Corresponding to the $W^{1,p}$-case one might ask to truncate divergence-free L^p functions to be in L^∞ (cf. [**32**]). While these questions turn out to be quite different, they both fit in a larger common framework. In our setting of divergence-free $W^{1,p}$-truncation, consider $w = Du$. The solenoidal Lipschitz truncation now corresponds to a L^∞-truncation of w under the following constraints

$$\mathrm{tr}(w) = 0, \quad \mathrm{curl}\, w = 0, \quad \mathrm{div}^T w = 0,$$

i.e. curl is taken column-wise and div row-wise. This can be abstractly set as $\mathscr{B} w = 0$. Consequently, Lipschitz truncation of a function u under a constraint $\mathscr{A} u = 0$ is closely connected to the L^∞-truncation of w $(= Du)$ under some constraint $\mathscr{B} w =$

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press

0. As mentioned before, the latter has been examined in [**5**, **32**] in greater detail; in particular (as above) the authors in [**5**] have conjectured that a low-regularity truncation is possible whenever $\mathscr{B}$ obeys the complex constant rank condition.

4.4. Solenoidal extensions for $W^{1,\infty}$

As alluded to in § 2.2, extensions and truncations are connected from a technical perspective. Indeed, in the *unconstrained* setting, truncation is derived by extending the function of a cleverly chosen subset to the full space. As it becomes visible through the formula for the truncation (3.8), the same analogy *is not* true for *constrained* truncation and extension. In particular, all the approaches to constrained truncation (cf. [**7**, **9**] etc. and (3.8)) use to a certain extent that the function is already defined on the bad set. Therefore, constructing an extension is actually more challenging than constructing a truncation, and while certain key ideas stay the same, there are a lot of challenges coming from the geometric properties of the underlying domain. We refer to the forthcoming article [**23**] for a thorough investigation of solenoidal *extension*, especially for the borderline cases $p = 1$ and $p = \infty$ (also cf. [**3**, **24**] for $1 < p < \infty$).

Acknowledgements
The author would like to thank Franz Gmeineder and the anonymous referee for helpful suggestions. The research conducted in this work was achieved while the author was still supported by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) through the graduate school BIGS of the Hausdorff Center for Mathematics (GZ EXC 59 and 2047/1, Projekt-ID 390685813).

References

1 E. Acerbi and N. Fusco. Semicontinuity problems in the calculus of variations. *Arch. Rational Mech. Anal.* **86** (1984), 125–145.

2 E. Acerbi and N. Fusco, An approximation lemma for W^1, p functions. In *Material instabilities in continuum mechanics (Edinburgh, 1985–1986)*, Oxford Sci. Publ. (Oxford Univ. Press, New York, 1988), pp. 1–5.

3 G. Acosta, R. G. Durán and M. A. Muschietti. Solutions of the divergence operator on John domains. *Adv. Math.* **206** (2006), 373–401.

4 E. Bäumle and M. Růžička. Existence of weak solutions for unsteady motions of micropolar electrorheological fluids. *SIAM J. Math. Anal.* **49** (2017), 115–141.

5 L. Behn, F. Gmeineder and S. Schiffer. On symmetric div-quasiconvex hulls and divsym-free L^∞-truncations. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* **40** (2023), 1267–1317.

6 M. E. Bogovskiĭ. Solution of the first boundary value problem for an equation of continuity of an incompressible medium. *Dokl. Akad. Nauk SSSR* **248** (1979), 1037–1040.

7 D. Breit, L. Diening and M. Fuchs. Solenoidal Lipschitz truncation and applications in fluid mechanics. *J. Differ. Equ.* **253** (2012), 1910–1942.

8 D. Breit, L. Diening and F. Gmeineder. The Lipschitz truncation of functions of bounded variation. *Indiana Univ. Math. J.* **70** (2021), 2237–2260.

9 D. Breit, L. Diening and S. Schwarzacher. Solenoidal Lipschitz truncation for parabolic PDEs. *Math. Models Methods Appl. Sci.* **23** (2013), 2671–2700.

10 T. Buckmaster and V. Vicol. Convex integration and phenomenologies in turbulence. *EMS Surv. Math. Sci.* **6** (2019), 173–263.

11 M. Bulíček, J. Burczak and S. Schwarzacher. A unified theory for some non-Newtonian fluids under singular forcing. *SIAM J. Math. Anal.* **48** (2016), 4241–4267.

12 M. Costabel and A. McIntosh. On Bogovskiĭ and regularized Poincaré integral operators for de Rham complexes on Lipschitz domains. *Math. Z.* **265** (2010), 297–320.

13 K. de Leeuw and H. Mirkil. Majorations dans L_∞ des opérateurs différentiels à coefficients constants. *C. R. Acad. Sci. Paris* **254** (1962), 2286–2288.

14 L. Diening, C. Kreuzer and E. Süli. Finite element approximation of steady flows of incompressible fluids with implicit power-law-like rheology. *SIAM J. Numer. Anal.* **51** (2013), 984–1015.

15 L. Diening, J. Málek and M. Steinhauer. On Lipschitz truncations of Sobolev functions (with variable exponent) and their selected applications. *ESAIM Control Optim. Calc. Var.* **14** (2008), 211–232.

16 L. Diening, M. Ružička and J. Wolf. Existence of weak solutions for unsteady motions of generalized Newtonian fluids. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **9** (2010), 1–46.

17 L. Diening, S. Schwarzacher, B. Stroffolini and A. Verde. Parabolic Lipschitz truncation and caloric approximation. *Calc. Var. Partial Differ. Equ.* **56** (2017), 120.

18 L. C. Evans and R. F. Gariepy, *Measure theory and fine properties of functions*. Textbooks in Mathematics, revised ed. (CRC Press, Boca Raton, FL, 2015).

19 J. Frehse, J. Málek and M. Steinhauer. On analysis of steady flows of fluids with shear-dependent viscosity based on the Lipschitz truncation method. *SIAM J. Math. Anal.* **34** (2003), 1064–1083.

20 G. Friesecke, R. D. James and S. Müller. A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.* **55** (2002), 1461–1506.

21 D. Gallenmüller. Müller-Zhang truncation for general linear constraints with first or second order potential. *Calc. Var. Partial Differ. Equ.* **60** (2021), 118.

22 D. Gallenmüller and E. Wiedemann. Which measure-valued solutions of the monoatomic gas equations are generated by weak solutions?. *Arch. Ration. Mech. Anal.* **247** (2023), 61.

23 F. Gmeineder and S. Schiffer, Extensions of divergence-free fields in L^1-based spaces. *in preparation*, 2024.

24 T. Kato, M. Mitrea, G. Ponce and M. Taylor. Extension and representation of divergence-free vector fields on bounded domains. *Math. Res. Lett.* **7** (2000), 643–650.

25 M. D. Kirszbraun. Über die zusammenziehende und Lipschitzsche Transformation. *Fund. Math.* **22** (1934), 77–108.

26 F. C. Liu. A Luzin type property of Sobolev functions. *Indiana Univ. Math. J.* **26** (1977), 645–651.

27 E. J. McShane. Extension of range of functions. *Bull. Amer. Math. Soc.* **40** (1934), 837–842.

28 S. Müller. A sharp version of Zhang's theorem on truncating sequences of gradients. *Trans. Amer. Math. Soc.* **351** (1999), 4585–4597.

29 S. Müller, Variational models for microstructure and phase transitions. In *Calculus of variations and geometric evolution problems (Cetraro, 1996)*, volume 1713 of *Lecture Notes in Math.* (Springer, Berlin, 1999), pp. 85–210.

30 D. Ornstein. A non-equality for differential operators in the L_1 norm. *Arch. Rational Mech. Anal.* **11** (1962), 40–49.

31 D. M. Roberts and A. Schmeding. Extending Whitney's extension theorem: nonlinear function spaces. *Ann. Inst. Fourier (Grenoble)* **71** (2021), 1241–1286.

32 S. Schiffer. L^∞-truncation of closed differential forms. *Calc. Var. Partial Differ. Equ.* **61** (2022), 135.

33 E. M. Stein. *Singular integrals and differentiability properties of functions*, Princeton Mathematical Series, Vol. 30 (Princeton University Press, Princeton, NJ, 1970), pp. xiv+290.

34 H. Whitney. Analytic extensions of differentiable functions defined in closed sets. *Trans. Amer. Math. Soc.* **36** (1934), 63–89.

35 K. Zhang. A construction of quasiconvex functions with linear growth at infinity. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)* **19** (1992), 313–326.

https://doi.org/10.1017/prm.2024.38 Published online by Cambridge University Press