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FUGLEDE-PUTNAM'S THEOREM FOR p-HYPONORMAL OR log-HYPONORMAL OPERATORS

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Abstract. Let T be p-hyponormal or log-hyponormal on a Hilbert space $\mathcal{H}$. Then we have $XT = T^*X$ whenever $XT^* = TX$ for some $X \in \mathcal{B}(\mathcal{H})$. This is an extension of Patel's result. Also for p-hyponormal or log-hyponormal T^*, dominant S and any $X \in \mathcal{B}(\mathcal{H})$ such that $XT = SX$, we have $XT^* = S^*T$.

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1. Introduction. For complex Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, $\mathcal{B}(\mathcal{H})$, $\mathcal{B}(\mathcal{K})$ and $\mathcal{B}(\mathcal{H},\mathcal{K})$ denote the set of all bounded linear operators on $\mathcal{H}$, the set of all bounded linear operators on $\mathcal{K}$ and the set of all bounded linear transformation from $\mathcal{H}$ to $\mathcal{K}$ respectively. Throughout this paper, $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, and Hilbert spaces mean complex Hilbert spaces. A bounded linear operator T on a complex Hilbert space $\mathcal{H}$ is called *normal* if $T^*T = TT^*$. Also T is called *p-hyponormal* for $p > 0$ if $(T^*T)^p \geq (TT^*)^p$, *log-hyponormal* if T is an invertible operator which satisfies $\log(T^*T) \geq \log(TT^*)$. Throughout this paper, we consider the case where $p \in (0, 1]$. T is called *hyponormal* iff it is 1-hyponormal. We say that T is *M-hyponormal* for $M > 0$ if $(T - \lambda)(T - \lambda)^* \leq M(T - \lambda)^*(T - \lambda)$ for all $\lambda \in \mathbb{C}$, and is *dominant* if $\operatorname{ran}(T - \lambda) \subset \operatorname{ran}(T - \lambda)^*$, for all $\lambda \in \mathbb{C}$. If T satisfies $|T^2| \geq T^*T$, then we say that T belongs to the class$A$ (or simply, T is *class$A$*). We also say that T is *co-hyponormal*, *co-M-hyponormal*, *co-dominant*, *co-p-hyponormal* and *co-*log-*hyponormal* if T^* is hyponormal, M-hyponormal, dominant, p-hyponormal and log-hyponormal respectively. It is well known that M-hyponormal is dominant and also well-known that p-hyponormal and log-hyponormal are class$A$. By definition, the restriction of an M-hyponormal (resp. dominant) operator to an invariant subspace is always M-hyponormal (resp. dominant). The parallel results for p-hyponormal (resp. class$A$) have been obtained by the author ([**18**], [**19**]), i.e., it is true that the restriction of p-hyponormal (resp. class$A$) to an invariant subspace is always p-hyponormal (resp. class$A$).

The following Fuglede-Putnam's theorem is famous.

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THEOREM. (Fuglede-Putnam's theorem [**4**], [**12**]). *Let $A \in \mathcal{B}(\mathcal{H})$ and $B \in \mathcal{B}(\mathcal{K})$ be normal operators on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, respectively. Let $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be an operator which satisfies $CA = BC$. Then $CA^* = B^*C$.*

Many mathematicians have extended this theorem to various classes of operators. The following is one of them.

THEOREM. (Duggal [**3**], Yoshino [**21**]) *Let $A^* \in \mathcal{B}(\mathcal{H})$ be M-hyponormal and $B \in \mathcal{B}(\mathcal{K})$ be dominant. Let $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ be an operator which satisfies $CA = BC$. Then $CA^* = B^*C$.*

We say that a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$, invariant under T, is a *normal part of* T if the restriction $T|_{\mathcal{M}}$ of T to $\mathcal{M}$ is normal. It is a famous result of Stampfli [**15**] that every normal part of a dominant operator B is always a reducing subspace of B.

Recently, Patel [**10**] has proved the following result.

THEOREM. *Let T be an injective p-hyponormal operator on $\mathcal{H}$ with the property that every normal part of T reduces T. Let X be a bounded linear operator on $\mathcal{H}$ such that $TX = XT^*$. Then $T^*X = XT$.*

In this paper, we shall show that if T is p-hyponormal or log-hyponormal then every normal part of T is a reducing subspace of T. Consequently the conclusion of the theorem of Patel [**10**] above remains true without the assumption of injectivity or reduceness of the normal parts. Further, the conclusion of the theorem remains true if the hypothesis of p-hyponormality of the operator is replaced by that of log-hyponormality. Finally we shall prove the following partial generalization of the theorem of Duggal [**3**] and Yoshino [**21**] stated above.

THEOREM. *Let $A^* \in \mathcal{B}(\mathcal{H})$* be *p-hyponormal or log-hyponormal and $B \in \mathcal{B}(\mathcal{K})$ be dominant. If $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $CA = BC$, then $CA^* = B^*C$.*

2. Preliminaries The following lemmas are well known except Lemma 3. For the sake of convenience, we state them without proof.

LEMMA 1. ([**13**]). *If N is a normal operator on $\mathcal{H}$, then we have*

$$\bigcap_{\lambda \in \mathbb{C}} (N - \lambda)\mathcal{H} = \{0\}.$$

LEMMA 2. ([**1**], [**17**]). *If T is p-hyponormal for $0 < p < 1$ (resp. log-hyponormal) and $T = U|T|$ is the polar decomposition of T, then the Aluthge transform $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ of T is hyponormal if $p \geq \frac{1}{2}$ and $(p + \frac{1}{2})$-hyponormal if $0 < p \leq \frac{1}{2}$ (resp. $\frac{1}{2}$-hyponormal).*

In [**11**], Patel showed that a p-hyponormal operator is normal whenever its Aluthge transform is normal. The following is an extension of Patel's result.

LEMMA 3. *Let T be a p-hyponormal (respectively log-hyponormal) operator on $\mathcal{H}$ and let $U|T|$ be the polar decomposition of T. Let $\mathcal{M}$ is a closed subspace of $\mathcal{H}$ such* *that the Aluthge transform* $\widetilde{T}$ *is of the form* $\widetilde{T} = N \oplus T'$ *on* $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$, *where* N *is a normal operator on* $\mathcal{M}$. *Then* T *and* U *are of the form* $T = N \oplus T_1$ *and* $U = U_{11} \oplus U_{22}$, *where* T_1 *is* p*-hyponormal (resp. log-hyponormal) and* $N = U_{11}|N|$ *is the polar decomposition of* N.

In particular, if the Aluthge transform $\widetilde{T}$ *of* T *is normal, then* T *is normal.*

Proof. For p-hyponormal or log-hyponormal T, it was shown by Aluthge [**1**] and Tanahashi [**17**] that

$$|\widetilde{T}| \geq |T| \geq |\widetilde{T}^*|.$$

Hence, we have

$$|N| \oplus |T'| \geq |T| \geq |N| \oplus |T'^*|$$

by assumption. This implies that $|T|$ is of the form $|N| \oplus L$, for some positive operator L. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ be the 2×2 matrix representation of U with respect to the decomposition $\mathcal{H} = M \oplus M^{\perp}$. Then the definition $\widetilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ means that

$$\begin{pmatrix} N & 0 \\ 0 & T' \end{pmatrix} = \begin{pmatrix} |N|^{\frac{1}{2}} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix}\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}\begin{pmatrix} |N|^{\frac{1}{2}} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix}.$$

Hence, we have

$$N = |N|^{\frac{1}{2}}U_{11}|N|^{\frac{1}{2}}, \tag{1}$$

$$|N|^{\frac{1}{2}}U_{12}L^{\frac{1}{2}} = 0, \tag{2}$$

$$L^{\frac{1}{2}}U_{21}|N|^{\frac{1}{2}} = 0. \tag{3}$$

If T is p-hyponormal, then $\mathrm{ran}\, U = \overline{\mathrm{ran}\, T} \subset \overline{\mathrm{ran}\, |T|}$. Since $\mathrm{Ker}\, U = \mathrm{Ker}\, T = \mathrm{Ker}\, |T|$ we also have

$$\mathrm{Ker}\, N \subset \mathrm{Ker}\, U_{11}, \mathrm{Ker}\, U_{21} \tag{4}$$

$$\mathrm{ran}\, U_{11}, \mathrm{ran}\, U_{12} \subset \overline{\mathrm{ran}}\, |N| = \overline{\mathrm{ran}}\, N \tag{5}$$

$$\mathrm{Ker}\, L \subset \mathrm{Ker}\, U_{12}, \mathrm{Ker}\, U_{22} \tag{6}$$

$$\mathrm{ran}\, U_{21}, \mathrm{ran}\, U_{22} \subset \overline{\mathrm{ran}}\, L. \tag{7}$$

(1), (4) and (5) imply that $N = U_{11}|N|$.
(2), (5) and (6) imply that $U_{12} = 0$.
(3), (4) and (7) imply that $U_{21} = 0$.

Hence U is of the form $U = U_{11} \oplus U_{22}$, and so we obtain

$$T = U|T| = U_{11}|N| \oplus U_{22}L = N \oplus T_1,$$

where $T_1 = U_{22}L$. The p-hyponormality of T_1 is immediate from that of T. Hence the assertion holds for p-hyponormal operators.

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If T is log-hyponormal, then N and L are invertible, since T is invertible. Hence (1) implies $N = U_{11}|N|$ and (2), (3) imply that $U_{12} = 0$ and $U_{21} = 0$. By the same argument as above, we have the conclusion. □

Lemma 4. (Putnam [**14**]). *Let* $T \in \mathcal{B}(\mathcal{H})$, $D \in \mathcal{B}(\mathcal{H})$ *with* $0 \le D \le M(T-\lambda)\,(T-\lambda)^*$ *for all* λ *in* $\mathbb{C}$, *where* M *is a positive real number. Then, for every* $x \in D^{\frac{1}{2}}\mathcal{H}$ *there exists a bounded function* $f : \mathbb{C} \to \mathcal{H}$ *such that* $(T-\lambda)f(\lambda) \equiv x$.

Lemma 5. ([**5**], [**6**]). *Every p-hyponormal and every log-hyponormal operator is classA.*

Lemma 6. (Löwner-Heinz's inequality [**9**], [**8**]). *Let* $A \in \mathcal{B}(\mathcal{H})$, $B \in \mathcal{B}(\mathcal{H})$. *If* $0 \le A \le B$ *and* $\delta \in (0, 1]$, *then* $0 \le A^{\delta} \le B^{\delta}$.

Lemma 7. (Hansen's inequality [**7**]) *If* $A, B \in \mathcal{B}(\mathcal{H})$ *satisfy* $A \ge 0$ *and* $\|B\| \le 1$, *then* $(B^*AB)^{\delta} \ge B^*A^{\delta}B$, *for all* $\delta \in (0, 1]$.

Lemma 8. (Douglas's theorem [**2**]). *For* $A, B \in \mathcal{B}(\mathcal{H})$*, the following are equivalent.*
(1) $AA^* \le \lambda BB^*$.
(2) $\mathrm{ran}A \subset \mathrm{ran}B$.
(3) $A = BC$ *for some* $C \in \mathcal{B}(\mathcal{H})$.

The following result is well known but we have been unable to find an explicit reference. A proof is included for completeness.

Lemma 9. *Let* $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ *be a positive operator. Then* $\mathrm{ran}B \subset rmA^{\frac{1}{2}}$. *In fact,* $B = A^{\frac{1}{2}}DC^{\frac{1}{2}}$, *for some contraction* $D \in \mathcal{B}(\mathcal{K}, \mathcal{H})$.

Proof. Let $\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$ be a positive operator on $\mathcal{H} \oplus \mathcal{K}$. Then for every $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathcal{H} \oplus \mathcal{K}$, we have

$$0 \le \left\langle \begin{pmatrix} A & B \\ B^* & C \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}, \begin{pmatrix} x \\ y \end{pmatrix} \right\rangle = \left\| A^{\frac{1}{2}}x \right\|^2 + 2Re\langle x, By\rangle + \left\| C^{\frac{1}{2}}y \right\|^2.$$

This implies that

$$\left\| A^{\frac{1}{2}}x \right\|^2 - 2|\langle x, By\rangle| + \left\| C^{\frac{1}{2}}y \right\|^2 \ge 0, \text{ for every } x \in \mathcal{H} \text{ and } y \in \mathcal{K}.$$

If we replace y by ty for $t > 0$, then we have

$$t^2\left\| C^{\frac{1}{2}}y \right\|^2 - 2t|\langle x, By\rangle| + \left\| A^{\frac{1}{2}}x \right\|^2 \ge 0, \text{ for all } t > 0,$$

and this is equivalent to

$$|\langle x, By\rangle| \le \left\| A^{\frac{1}{2}}x \right\| \left\| C^{\frac{1}{2}}y \right\|, \text{ for all } x \in \mathcal{H} \text{ and } y \in \mathcal{K}.$$

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By the inequality above, we see that

$$\mathrm{ran}A^{\frac{1}{2}} \times \mathrm{ran}C^{\frac{1}{2}} \ni \left(A^{\frac{1}{2}}x, C^{\frac{1}{2}}y\right) \longmapsto \langle x, By \rangle \in \mathbb{C}$$

is a continuous sesqui-linear form (with its norm less than or equal to 1) and so it can be extended uniquely to a continuous sesqui-linear form on $\overline{\mathrm{ranA}^{\frac{1}{2}}} \times \overline{\mathrm{ranC}^{\frac{1}{2}}} = \overline{\mathrm{ranA}} \times \overline{\mathrm{ranC}}$. Hence, there exists a contraction $D' \in \mathcal{B}\left(\overline{\mathrm{ranC}}, \overline{\mathrm{ranA}}\right)$ such that

$$\langle x, By \rangle = \left\langle A^{\frac{1}{2}}x, D'C^{\frac{1}{2}}y \right\rangle \quad \text{for all } x \in \mathcal{H} \text{ and } y \in \mathcal{K},$$

by Riesz's representation theorem. Let $P \in \mathcal{B}(\mathcal{K})$ be the orthagonal projection onto $\overline{\mathrm{ranC}}$ and let $D = D'P$. Then $D \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ and

$$\langle x, By \rangle = \left\langle A^{\frac{1}{2}}x, DC^{\frac{1}{2}}y \right\rangle, \text{ for all } x \in \mathcal{H} \text{ and } y \in \mathcal{K}.$$

Thus we have $B = A^{\frac{1}{2}}DC^{\frac{1}{2}}$. This completes the proof. □

It is well known, by [**16**], that a hyponormal operator which is quasi-similar to a normal operator is always normal. The following is an extension of this result to the case of p-hyponormal or log-hyponormal operators.

THEOREM 1. *Let T be p-hyponormal or* log-*hyponormal, N be normal on $\mathcal{H}$ and $\mathcal{K}$ respectively. Let $X \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ be injective with dense range which satisfies $TX = XN$. Then $T^*X = XN^*$.*

Proof. First, we prove the case in which T is p-hyponormal and $p \geq \frac{1}{2}$. The p-hyponormality of T implies that Ker$T$ reduces T. Also Ker$N$ reduces N, since N is normal. Using the orthogonal decompositions $\mathcal{H} = [|T|\mathcal{H}] \oplus \ker T$ and $\mathcal{K} = [N\mathcal{K}] \oplus N$, we can represent T and N as follows.

$$T = \begin{pmatrix} T_1 & 0 \\ 0 & 0 \end{pmatrix} \tag{8}$$

$$N = \begin{pmatrix} N_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{9}$$

where T_1 is injective and p-hyponormal on $[|T|\mathcal{H}]$ and N_1 is injective and normal on $[N\mathcal{H}]$. The assumption $TX = XN$ implies that X maps N to ran$T \subset [|T|\mathcal{H}]$ and Ker$N$ to Ker$T$. Hence X is of the form

$$X = \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix}, \tag{10}$$

where $X_1 \in \mathcal{B}([N\mathcal{K}], [|T|\mathcal{H}])$, $X_2 \in \mathcal{B}(\mathrm{Ker}N, \mathrm{Ker}T)$. Since $TX = XN$, we have that

$$T_1X_1 = X_1N_1. \tag{11}$$

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Since X is injective with dense range, X_1 is also injective with dense range. Put $W = X_1{}^*|T_1|^{\frac{1}{2}}$. Then $W : [|T|\mathcal{H}] \to [N\mathcal{K}]$ is a one-to-one mapping which has dense range and satisfies $W\widetilde{T_1}{}^* = N_1{}^*W$. Here $\widetilde{T_1}$ is the Aluthge transform of T_1. Since $\widetilde{T_1}$ is hyponormal, for every $x \in (\widetilde{T_1}{}^*\widetilde{T_1} - \widetilde{T_1}\widetilde{T_1}{}^*)^{\frac{1}{2}}\mathcal{H}$, there exists a bounded function $f : \mathbb{C} \to \mathcal{H}$ such that $(\widetilde{T_1}{}^* - \lambda)f(\lambda) \equiv x$, for all $\lambda \in \mathbb{C}$, by Lemma 4. Hence

$$\begin{aligned} Wx &= W\Big(\widetilde{T_1}{}^* - \lambda\Big)f(\lambda) \\ &= \big(N_1^* - \lambda\big)Wf(\lambda) \\ &\in \mathrm{ran}\big(N_1^* - \lambda\big), \text{ for all } \lambda \in \mathbb{C}. \end{aligned}$$

By Lemma 1, we have $Wx = 0$, and hence $x = 0$ because W is one-to-one. This implies that $\widetilde{T_1}$ is normal. By Lemma 3, T_1 is normal and therefore $T = T_1 \oplus 0$ is also normal. The assertion is immediate from Fuglede-Putnam's theorem.

Next, we prove the cases in which T is p-hyponormal for $p \leq \frac{1}{2}$ or log-hyponormal. Let T_1, N_1, X_1 and W be as above. Then $\widetilde{T_1}$ is $\frac{1}{2}$-hyponormal and $W^* : [N\mathcal{K}] \to [|T|\mathcal{H}]$ is a one-to-one mapping with dense range that satisfies

$$\widetilde{T_1}W^* = W^*N_1.$$

By using a previous argument we see that $\widetilde{T_1}$ is normal. Hence T_1 is normal by Lemma 3. This implies that T is normal. The assertion follows by Fuglede-Putnam's theorem. □

3. Main theorems In order to obtain our generalization of Patel's result [**10**] discussed earlier we require some preliminary lemmas.

LEMMA 10. (Stampfli-Wadhwa [**16**]) *Let $T \in \mathcal{B}(\mathcal{K})$ be dominant and $S \in \mathcal{B}(\mathcal{K})$ be co-hyponormal. If $W \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is a one-to-one mapping with dense range and $WS = TW$, then T and S are normal.*

LEMMA 11. *Let $T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}$ be a class A operator on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $\mathcal{M}$ is a T-invariant subspace such that the restriction $T_1 = T|_M$ is normal. Then the range of S is included in* $\mathrm{Ker}T_1$. *In particular, if T is injective, every normal part of T reduces T.*

Proof. Let P be the orthogonal projection onto $\mathcal{M}$. Then we have

$$\begin{aligned} \begin{pmatrix} T_1^*T_1 & 0 \\ 0 & 0 \end{pmatrix} &= PT^*TP \leq P\big|T^2\big|P \quad (\text{since } T \text{ is class} A) \\ &\leq \begin{pmatrix} (T_1^{*2}T_1^2)^{\frac{1}{2}} & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{by Hansen's inequality}) \\ &= \begin{pmatrix} T_1^*T_1 & 0 \\ 0 & 0 \end{pmatrix} \quad (\text{since } T_1 \text{ is normal}). \end{aligned}$$

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Let $|T^2| = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$ be the 2×2 matrix representation of $|T^2|$ or $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$. Then we have $X = T_1{}^*T_1$ by the inequality above. Since $|T^2|^2 = T^{*2}T^2$, we have

$$\begin{pmatrix} X^2 + YY^* & XY + YZ \\ ZY^* + Y^*X & Y^*Y + Z^2 \end{pmatrix} = \begin{pmatrix} T_1{}^{*2}T_1{}^2 & T_1{}^{*2}T_1S \\ S^*T_1{}^*T_1{}^2 & S^*S + T_2{}^{*2}T_2{}^2 \end{pmatrix},$$

and hence $X^2 + YY^* = T_1{}^{*2}T_1{}^2 = (T_1^*T_1)^2 = X^2$. This implies that $Y = 0$. Thus we have

$$|T^2| = \begin{pmatrix} T_1{}^*T_1 & 0 \\ 0 & Z \end{pmatrix} \geq T^*T = \begin{pmatrix} T_1{}^*T_1 & T_1{}^*S \\ S^*T_1 & S^*S + T_2{}^*T_2 \end{pmatrix}$$

and hence $T_1{}^*S = 0$. Thus the range of S is included in $\mathrm{Ker}T_1{}^* = \mathrm{Ker}T_1$. If T is one-to-one, then T_1 is also one-to-one. Hence the second statement of Lemma 11 follows trivially. □

LEMMA 12. *If T is p-hyponormal or log-hyponormal, then every normal part of T reduces T.*

Proof. If T is log-hyponormal, then T is invertible. Hence the assertion holds for log-hyponormal operators by Lemma 11.

Now, we assume that T is p-hyponormal. Let $\mathcal{M}$ be a normal part of T. By Lemmas 5 and 11, T is of the form $\begin{pmatrix} N & S \\ 0 & T_1 \end{pmatrix}$ on $\mathcal{M} \oplus \mathcal{M}^\perp$, where N is normal and $\mathrm{ran}S \subset \mathrm{Ker}N$. It is easy to see that

$$T^*T = \begin{pmatrix} |N|^2 & 0 \\ 0 & S^*S + T_1^*T_1 \end{pmatrix},$$

$$TT^* = \begin{pmatrix} |N|^2 + SS^* & ST_1^* \\ T_1S^* & T_1T_1^* \end{pmatrix}.$$

Put $(TT^*)^p = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}$. Then the p-hyponormality of T implies that

$$(T^*T)^p = \begin{pmatrix} |N|^{2p} & 0 \\ 0 & (S^*S + T_1{}^*T_1)^p \end{pmatrix} \geq \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix} = (TT^*)^p.$$

We have $\mathrm{ran}Y \subset \mathrm{ran}X^{\frac{1}{2}}$ by Lemma 9 and $\mathrm{ran}X^{\frac{1}{2}} \subset \mathrm{ran}|N|^p$ by Lemma 8. Hence we have $\mathrm{ran}X, \cup \mathrm{ran}Y \subset \mathrm{ran}X^{\frac{1}{2}} \subset \mathrm{ran}|N|^p$. Put $(TT^*)^{1-p} = \begin{pmatrix} A & B \\ B^* & C \end{pmatrix}$. Hence

$$TT^* = (TT^*)^p(TT^*)^{1-p} = \begin{pmatrix} X & Y \\ Y^* & Z \end{pmatrix}\begin{pmatrix} A & B \\ B^* & C \end{pmatrix}.$$

This implies that $|N|^2 + SS^* = XA + YB^*$. Therefore,

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$$\operatorname{ran}(SS^*) \subset \operatorname{ran}|N|^2 + \operatorname{ran}X + \operatorname{ran}Y \subset \operatorname{ran}|N|^p \subset \overline{\operatorname{ran}N},$$

while, $\operatorname{ran}(SS^*) \subset \operatorname{ran}S \subset \operatorname{Ker}N$. This shows that $\operatorname{ran}(SS^*) = \{0\}$ and therefore $S = 0$. This completes the proof. □

THEOREM 2. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-hyponormal or* log-*hyponormal and $L \in \mathcal{B}(\mathcal{H})$ be a self-adjoint operator which satisfies $TL = LT^*$. Then $T^*L = LT$.*

Proof. We first show that if $TL = LT^* = 0$ then $T^*L = LT = 0$. Since $\operatorname{Ker}T$ reduces T, $TL = 0$ implies that $\operatorname{ran}L \subset \operatorname{Ker}T \subset \operatorname{Ker}T^*$ and (by taking orthogonal complements) $\overline{\operatorname{ran}T} \subset \operatorname{Ker}L$. Hence we have $T^*L = LT = 0$.

Next, we prove the case in which $TL \neq 0$. Assume that T is p-hyponormal. Using the decomposition $\mathcal{H} = \overline{\operatorname{ran}L} \oplus \operatorname{Ker}L$, the operators L and T can be represented as follows:

$$L = \begin{pmatrix} L_1 & 0 \\ 0 & 0 \end{pmatrix}, \tag{12}$$

$$T = \begin{pmatrix} T_1 & S \\ 0 & T_2 \end{pmatrix}, \tag{13}$$

where L_1 is self-adjoint with $\operatorname{Ker}L_1 = \{0\}$ (hence it has dense range) and T_1 is also p-hyponormal by [**18**]. The assumption $TL = LT^*$ implies that $T_1L_1 = L_1T_1{}^*$. Since $\operatorname{Ker}T_1$ reduces T_1 and L_1, they are of the form $T_1 = T_{11} \oplus 0$ and $L_1 = L_{11} \oplus L_{22}$ on $\overline{\operatorname{ran}L} = \overline{\operatorname{ran}|T_1|} \oplus \operatorname{Ker}T_1$. It is easy to see that T_{11} is an injective p-hyponormal operator and L_{11} is an injective self-adjoint operator which satisfies $T_{11}L_{11} = L_{11}T_{11}{}^*$. If $p \geq \frac{1}{2}$, then $\widetilde{T_{11}}W = W\widetilde{T_{11}}^*$, where $W = |T_{11}|^{\frac{1}{2}}L_{11}|T_{11}|^{\frac{1}{2}}$ is injective self-adjoint and $\widetilde{T_{11}}$ is hyponormal. We have $\widetilde{T_{11}}$ is normal by Lemma 10 and T_{11} is also normal by Lemma 3. Hence $T_1 = T_{11} \oplus 0$ is also normal. By Fuglede-Putnam's theorem we see that $T_1{}^*L_1 = L_1T_1$. Since T_1 is normal $S = 0$ by Lemma 12, so we have $T^*L = LT$. Hence the assertion holds for p-hyponormal operators for $p \geq \frac{1}{2}$. If $0 < p < \frac{1}{2}$, $\widetilde{T_{11}}$ is an injective $(p + \frac{1}{2})$-hyponormal. Using the previous argument, we have that $\widetilde{T_{11}}$ is normal and hence T_1 is normal. By the same reasoning as above, the assertion holds for p-hyponormal operators for $0 < p < \frac{1}{2}$.

If T is log-hyponormal, then the Aluthge transform $\tilde{T}$ of T is $\frac{1}{2}$-hyponormal. Moreover it satisfies

$$|\tilde{T}| \geq |T| \geq |\tilde{T}^*|. \tag{14}$$

See [**17**]. Put $W = |T|^{\frac{1}{2}}L|T|^{\frac{1}{2}}$. Then W is self-adjoint and satisfies

$$\tilde{T}W = W\tilde{T}^*. \tag{15}$$

By the previous argument, we have that the restriction $\tilde{T}|_{\overline{\operatorname{ran}W}}$ of $\tilde{T}$ to its invariant subspace $\overline{\operatorname{ran}W}$ is normal and

$$\tilde{T}^*W = W\tilde{T}. \tag{16}$$

Hence $\overline{\operatorname{ran}W}$ reduces $\tilde{T}$, by Lemma 12, and so $\tilde{T}$ is of the form $\tilde{T} = N \oplus S$ on $\overline{\operatorname{ran}}W \oplus \operatorname{Ker}W$, where N is normal. By Lemma 3, $T = N \oplus B$, for some log-hyponormal operator B. Let $W = W_1 \oplus 0$ and

$$L = \begin{pmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{pmatrix}$$

on $\overline{\operatorname{ran} W} \oplus \operatorname{Ker} W$. Then $L_{12} = 0, L_{21} = 0$ and $L_{22} = 0$ follows from the equality $W = |T|^{\frac{1}{2}} L |T|^{\frac{1}{2}}$. By assumption, $NL_{11} = L_{11}N^*$, we have $N^*L_{11} = L_{11}N$ by Fuglede-Putnam's theorem and therefore $T^*L = LT$. □

COROLLARY 1. *Let $T \in \mathcal{B}(\mathcal{H})$ be p-hyponormal or log-hyponormal. If $X \in \mathcal{B}(\mathcal{H})$ and $TX = XT^*$, then $T^*X = XT$.*

Proof. Let $X = L + iK$ be the Cartesian decomposition of X. Then we have $TL = LT^*$ and $TJ = JT^*$, by the assumption. By Theorem 2, we have $T^*L = LT$ and $T^*J = JT$. This implies that $T^*X = XT$. □

REMARK 1. If we use Patel's result and Lemma 12, the assertion of Theorem 2 for p-hyponormal is immediate, since T and L are of the form $T = T_1 \oplus 0$ and $L = L_1 \oplus L_2$ on $\overline{\operatorname{ran}|T|} \oplus \operatorname{Ker} T$, where T_1 is an injective p-hyponormal operator.

If we use the 2×2 matrix trick, we easily deduce the following result.

COROLLARY 2. *Let $T^* \in \mathcal{B}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{K})$ be p-hyponormal (resp. log-hyponormal). If $X \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ and $XT = SX$, then $XT^* = S^*X$.*

Proof. Put $A = \begin{pmatrix} T^* & 0 \\ 0 & S \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 0 \\ X & 0 \end{pmatrix}$ on $\mathcal{H} \oplus \mathcal{K}$. Then A is a p-hyponormal (resp. log-hyponormal) operator on $\mathcal{H} \oplus \mathcal{K}$ that satisfies $BA^* = AB$. Hence we have $BA = A^*B$, by Corollary 1, and therefore $XT^* = S^*X$. □

LEMMA 13. *Let $T^* \in \mathcal{B}(\mathcal{H})$ be p-hyponormal(resp. log-hyponormal) and $U|T|$ be the polar decomposition of T. Let $\mathcal{M}$ be a closed subspace of $\mathcal{H}$ such that the Aluthge transform $\tilde{T}$ is of the form $\tilde{T} = N \oplus T'$ on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where N is a normal operator on $\mathcal{M}$. Then T and U are of the form $T = \begin{pmatrix} N & A \\ 0 & T_1 \end{pmatrix}$ and $U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$(resp $N \oplus T_1$ and $U = U_{11} \oplus U_{22}$) on $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^\perp$, where $N = U_{11}|N|$ is the polar decomposition of N* and $\operatorname{ran} U_{12} \subset \operatorname{Ker} N$.

In particular, if N is one-to-one, then $T = N \oplus T_1$ and $U = U_{11} \oplus U_{22}$ on $= M \oplus M^\perp$.

Proof. Since T^* is p-hyponormal or log-hyponormal,

$$|\tilde{T}| \le |T| \le |\tilde{T}^*|,$$

by Aluthge [**1**] and Tanahashi [**17**]. Hence, we have

$$|N| \oplus |T'| \le |T| \le |N| \oplus |T'^*|,$$

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by assumption. This implies that $|T|$ is of the form $|N| \oplus L$, for some positive operator L. Let $U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$ be the 2×2 matrix representation of U with respect to the decomposition $\mathcal{H} = \mathcal{M} \oplus \mathcal{M}^{\perp}$. Then the definition $\tilde{T} = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}$ means that

$$\begin{pmatrix} N & 0 \\ 0 & T' \end{pmatrix} = \begin{pmatrix} |N|^{\frac{1}{2}} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} |N|^{\frac{1}{2}} & 0 \\ 0 & L^{\frac{1}{2}} \end{pmatrix},$$

Hence, we have

$$N = |N|^{\frac{1}{2}} U_{11} |N|^{\frac{1}{2}}, \tag{17}$$

$$|N|^{\frac{1}{2}} U_{12} L^{\frac{1}{2}} = 0, \tag{18}$$

$$L^{\frac{1}{2}} U_{21} |N|^{\frac{1}{2}} = 0. \tag{19}$$

Since $\mathrm{Ker} U = \mathrm{Ker} T = \mathrm{Ker} |T|$, we have

$$\mathrm{Ker} N \subset \mathrm{Ker} U_{11}, \mathrm{Ker} U_{21}, \tag{20}$$

$$\mathrm{Ker} L \subset \mathrm{Ker} U_{12}, \mathrm{Ker} U_{22}. \tag{21}$$

Let $N = V|N|$ be the polar decomposition of N. Then $\mathrm{ran}(U_{11} - V) \subset \mathrm{Ker} N$ by (17) and (20). Hence, for arbitrary $x \in \mathrm{ran} N$, we have

$$\begin{aligned} \|x\|^2 \geq \|U_{11}x\|^2 &= \|Vx\|^2 + \|(U_{11} - V)x\|^2, \text{ by Pythagoras' theorem,} \\ &= \|x\|^2 + \|(U_{11} - V)x\|^2, \text{ since } V \text{ is unitary on } \overline{\mathrm{ran}} N, \\ &\geq \|x\|^2. \end{aligned}$$

Therefore, we obtain $U_{11} = V$. Since

$$\|x\|^2 = \|Ux\|^2 = \|U_{11}x\|^2 + \|U_{21}x\|^2 = \|x\|^2 + \|U_{21}x\|^2 \text{ for } x \in \mathrm{ran} N,$$

we have $U_{21} = 0$ by (20). Also, we see that $\mathrm{ran} U_{12} \subset \mathrm{Ker} N$ by (18) and (21). Hence,

$$T = U|T| = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix} \begin{pmatrix} |N| & 0 \\ 0 & L \end{pmatrix} = \begin{pmatrix} N & U_{12}L \\ 0 & U_{22}L \end{pmatrix}.$$

In particular, if T^* is log-hyponormal, then N and L are invertible. Hence $U_{12} = 0$ and $U_{21} = 0$ immediately from (18) and (19). This completes the proof of the first statement.

The second statement is trivial, since $U_{12} = 0$ is immediate from $\mathrm{ran} U_{12} \subset \mathrm{Ker} N = \{0\}$. □

THEOREM 3. *Let $A \in \mathcal{B}(\mathcal{H})$ be such that A^* is p-hyponormal or* log-*hyponormal. Let $B \in \mathcal{B}(\mathcal{K})$ be dominant. Then $CA^* = B^*C$ whenever $CA = BC$, for some $C \in \mathcal{B}(\mathcal{H}, \mathcal{K})$.*

Proof. Let A^* be a p-hyponormal operator for $p \geq \frac{1}{2}$ and $U|A|$ be the polar decomposition of A. Then the Aluthge transform $\widetilde{A}$ of A is co-hyponormal and satisfies

$$|\tilde{A}|^2 \leq |A|^2 \leq |\tilde{A}^*|^2, \tag{22}$$

$$C'\tilde{A} = BC', \tag{23}$$

where $C' = CU|A|^{\frac{1}{2}}$. Using the decompositions $\mathcal{H} = \mathrm{Ker}C'^{\perp} \oplus \mathrm{Ker}C'$ and $\mathcal{K} = \overline{\mathrm{ran}C'} \oplus \mathrm{ranC}'^{\perp}$, we see that $\widetilde{A}$, B and C' are of the form

$$\tilde{A} = \begin{pmatrix} A_1 & 0 \\ S & A_2 \end{pmatrix}, \; B = \begin{pmatrix} B_1 & T \\ 0 & B_2 \end{pmatrix}, C' = \begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix},$$

where, A_1 is co-hyponormal, B_1 is dominant and C_1 is a one-to-one mapping with dense range. Since $C'\widetilde{A} = BC'$, we have

$$C_1A_1 = B_1C_1. \tag{24}$$

Hence A_1 and B_1 are normal by Lemma 10, so that $S = 0$, by Lemma 12 and $T = 0$ by [**15**]. Thus $|A| = |A_1| \oplus L$, for some positive L, by (22) and $U = \begin{pmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{pmatrix}$ by Lemma 13. Let $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ be a 2×2 matrix representation of C with respect to the decompositions $\mathcal{H} = \mathrm{Ker}C'^{\perp} \oplus \mathrm{Ker}C'$ and $\mathcal{K} = \overline{\mathrm{ran}C'} \oplus \mathrm{ran}C'^{\perp}$. Then, $C' = CU|A|^{\frac{1}{2}}$ implies that $C_1 = C_{11}U_{11}|A_1|^{\frac{1}{2}}$ and hence $\mathrm{Ker}A_1 \subset \mathrm{Ker}C_1 = \{0\}$. This shows that A_1 is one-to-one (hence, it has dense range), so that $U_{12} = 0$ and $A = A_1 \oplus A_3$, for some co-p-hyponormal operator A_3 by Lemma 13. Since,

$$\begin{pmatrix} C_1 & 0 \\ 0 & 0 \end{pmatrix} = C' = CU|A|^{\frac{1}{2}} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \begin{pmatrix} U_{11}|A_1|^{\frac{1}{2}} & 0 \\ 0 & U_{22}|A_3|^{\frac{1}{2}} \end{pmatrix},$$

we deduce the following statements.

$$C_{12}U_{22}|A_3|^{\frac{1}{2}} = 0; \quad \text{hence} \quad C_{12}A_3 = 0 \quad \text{because} \quad A_3 = U_{22}|A_3|. \tag{25}$$

$$C_{21}U_{11}|A_1|^{\frac{1}{2}} = 0; \quad \text{hence} \quad C_{21} = 0 \quad \text{because} \quad U_{11}|A_1|^{\frac{1}{2}} \text{ has dense range.} \tag{26}$$

$$C_{22}U_{22}|A_3|^{\frac{1}{2}} = 0; \quad \text{hence} \quad C_{22}A_3 = 0. \tag{27}$$

The assumption $CA = BC$ tells us that,

$$C_{11}A_1 = B_1C_{11}, \tag{28}$$

$$C_{12}A_3 = B_1C_{12} = 0, \quad \text{by (25)}, \tag{29}$$

$$C_{22}A_3 = B_2C_{22} = 0, \quad \text{by (27)}. \tag{30}$$

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Since A_1 and B_1 are normal we have $C_{11}A_1{}^* = B_1{}^*C_{11}$, by Fuglede-Putnam's theorem. The p-hyponormality of $A_3{}^*$ shows that $\mathrm{ran}A_3{}^* \subset \overline{\mathrm{ran}A_3}$. Also we have $\mathrm{Ker}B_2 \subset \mathrm{Ker}B_2{}^*$ from the fact that B_2 is dominant. Hence, we also have $C_{12}A_3{}^* = B_1{}^*C_{12} = 0$ and $C_{22}A_3{}^* = B_2{}^*C_{22} = 0$. This implies that $CA^* = C_{11}A_1{}^* \oplus 0 = B_1{}^*C_{11} \oplus 0 = B^*C$.

Next, we prove the case where A^* is p-hyponormal for $0 < p \leq \frac{1}{2}$. Let C' be as above. Then $\tilde{A}$ is co-$(p+\frac{1}{2})$-hyponormal and satisfies $C'\tilde{A} = BC'$. Use the same argument as above. We obtain $\tilde{A} = A_1 \oplus A_2$ on $\mathcal{H} = \mathrm{Ker}C'^{\perp} \oplus \mathrm{Ker}C'$ and $B = B_1 \oplus B_2$, where A_1 is an injective normal operator and B_1 is also normal. Hence, we have $A = A_1 \oplus A_3$ for some co-p-hyponormal A_3, by Lemma 13. Again using the same argument as above, we obtain $C_{21} = 0$, $C_{11}A_1{}^* = B_1{}^*C_{11}$, $C_{12}A_3{}^* = B_1{}^*C_{12} = 0$ and $C_{22}A_3{}^* = B_2{}^*C_{22} = 0$, where $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$ is the 2×2 matrix representation of C with respect to the decompositions $\mathcal{H} = \mathrm{Ker}C'^{\perp} \oplus \mathrm{Ker}C'$ and $\mathcal{K} = \overline{\mathrm{ran}C'} \oplus \mathrm{ran}C'^{\perp}$. Hence we have $CA^* = B^*C$.

Finally, we assume that A^* is log-hyponormal. Let $\tilde{A}$ and C' be as above. Then $C'\tilde{A} = BC'$ and $\tilde{A}^*$ is $\frac{1}{2}$-hyponormal and satisfies

$$|\tilde{A}| \leq |A| \leq |\tilde{A}^*|. \tag{31}$$

By the same argument as above, we have $\tilde{A} = A_1 \oplus A_2$ on $\mathcal{H} = \mathrm{Ker}C'^{\perp} \oplus \ker C'$ and $B = B_1 \oplus B_2$ on $\mathcal{K} = \overline{\mathrm{ran}C'} \oplus \mathrm{ran}C'^{\perp}$, where A_1 is an invertible normal operator, B_1 is normal, A_2 is invertible, co-$\frac{1}{2}$-hyponormal and B_2 is dominant. By Lemma 13, we have that A is of the form $A = A_1 \oplus A_3$, for some log-hyponormal $A_3{}^*$. Let $C = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$. Then $C' = CU|A|^{\frac{1}{2}}$ implies that $C_{12} = 0$, $C_{21} = 0$ and $C_{22} = 0$. The assumption $CA = BC$ implies that $C_{11}A_1 = B_1C_{11}$; hence $C_{11}A_1{}^* = B_1{}^*C_{11}$ by Fuglede-Putnam's theorem. Thus we have $CA^* = C_{11}A_1{}^* \oplus 0 = B_1{}^*C_{11} \oplus 0 = B^*C$. This completes the proof. □

Remark 2. Let T be an operator such that $\mathrm{Ker}T$ does not reduce T and let P be the orthogonal projection onto $\mathrm{Ker}T$. Then P does not commute with T; otherwise $\mathrm{ran}P = \mathrm{Ker}T$ reduces T. Hence $PT \neq 0 = TP$. It is easy to see that $TP = PT^* = 0$ but $T^*P \neq PT (\neq 0)$ because $\mathrm{ran}T^*P \subset \mathrm{ran}T^* \subset \mathrm{Ker}T^{\perp} = (1-P)$. Hence the assertion of Theorem 2 does not hold for such T. Also, if we put $A = T^*$, $B = 1 - P$ and $C = P$, then

$$CA = PT^* = 0 = (1-P)P = BC.$$

However

$$CA^* = PT \neq 0 = (1-P)P = B^*C.$$

Hence the assertion of Theorem 3 does not hold for such T.

This is an example of a class$A$ operator T such that T does not reduce T.

Example 1. Let $\{e_n\}_{n=-\infty}^{\infty}$ be a complete orthonormal system for $\mathcal{H}$. We denote the orthogonal projection onto $\mathbb{C}e_n$ by P_n. Let W be a weighted shift on $\mathcal{H}$ defined by

$$We_n = \begin{cases} \sqrt{2}e_{n+1} & (n \geq 0), \\ e_{n+1} & (n < 0). \end{cases}$$

Then $W^*W - WW^* = P_0$. Define an operator T on a Hilbert space $\mathcal{K} = \mathcal{H} \oplus \mathbb{C}e_0$ by

$$T = \begin{pmatrix} W & P_0 \\ 0 & 0 \end{pmatrix}.$$

Then

$$\begin{aligned} T^{*2}T^2 - (T^*T)^2 &= T^*\{T^*T - TT^*\}T \\ &= \begin{pmatrix} W^* & 0 \\ P_0 & 0 \end{pmatrix} \begin{pmatrix} 0 & W^*P_0 \\ P_0W & P_0 \end{pmatrix} \begin{pmatrix} W & P_0 \\ 0 & 0 \end{pmatrix} \\ &= \begin{pmatrix} W^* & 0 \\ P_0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ P_0W^2 & P_0WP_0 \end{pmatrix} = 0. \end{aligned}$$

Hence $T^{*2}T^2 = (T^*T)^2$ and therefore $|T^2| = T^*T$. This shows that T is class$A$. It is easy to see that

$$\operatorname{Ker} T = \mathbb{C}(-e_{-1} \oplus e_0) \text{ and } \operatorname{Ker} T^* = \{0\} \oplus \mathbb{C}e_0.$$

Hence T does not reduce T and therefore the assertions of Theorems 2 and 3 are not necessarily true for class$A$ operators.

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