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RESIDUATED REGULAR SEMIGROUPS

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We prove that in a residuated regular semigroup the elements of the form $x \mathbin{^{\cdot}.} x$ and $x \mathbin{.^{\cdot}} x$ are idempotents, and derive some consequences of this fact. In particular, we show how the maximality of such idempotents is related to the semigroup being naturally ordered, and obtain from this a characterisation of the boot-lace semigroup of [**2**].

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We recall that an ordered semigroup S is said to be *residuated* if for all $x, y \in S$ there exist

$$x \mathbin{.^{\cdot}} y = \max\{z \in S; yz \leqq x\}, \quad x \mathbin{^{\cdot}.} y = \max\{z \in S; zy \leqq x\}.$$

For the basic properties of such semigroups, we refer the reader to [**1**, Chapter 3]. Here we shall be concerned with the case where S is also *regular*. In this case, S is then *principally ordered*, in the sense that for every $x \in S$ there exists

$$x^* = \max\{y \in S; xyx \leqq x\}.$$

In fact, it is clear that $x^* = (x \mathbin{^{\cdot}.} x) \mathbin{.^{\cdot}} x = (x \mathbin{.^{\cdot}} x) \mathbin{^{\cdot}.} x$. For the basic properties of principally ordered regular semigroups, we refer the reader to [**2**, **3**]. For our purposes here, we recall that when S is principally ordered and regular every $x \in S$ has a greatest inverse, namely $x^\circ = x^*xx^*$. Moreover, $x = xx^*x$ and if E is the set of idempotents of S then $xx^\circ = xx^* \in E$ and $x^\circ x = x^*x \in E$.

Theorem 1. *Let S be a residuated regular semigroup. Then*

(1) $(\forall x \in S)\ (x \mathbin{^{\cdot}.} x)x = x = x(x \mathbin{.^{\cdot}} x)$;

(2) $(\forall x \in S)\ x \mathbin{^{\cdot}.} x = (x \mathbin{^{\cdot}.} x) \mathbin{^{\cdot}.} (x \mathbin{^{\cdot}.} x), \quad x \mathbin{.^{\cdot}} x = (x \mathbin{.^{\cdot}} x) \mathbin{.^{\cdot}} (x \mathbin{.^{\cdot}} x)$;

(3) $(\forall x \in S)\ (x \mathbin{^{\cdot}.} x)^* = (x \mathbin{^{\cdot}.} x) \mathbin{.^{\cdot}} (x \mathbin{^{\cdot}.} x), \quad (x \mathbin{.^{\cdot}} x)^* = (x \mathbin{.^{\cdot}} x) \mathbin{^{\cdot}.} (x \mathbin{.^{\cdot}} x)$;

(4) $(\forall x \in S)\ (x \mathbin{^{\cdot}.} x)^\circ = (x \mathbin{^{\cdot}.} x)^*(x \mathbin{^{\cdot}.} x), \quad (x \mathbin{.^{\cdot}} x)^\circ = (x \mathbin{.^{\cdot}} x)(x \mathbin{.^{\cdot}} x)^*$.

Proof. (1) By [**1**, Theorem 22.1] we have

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$$(\forall x, y \in S)\ (yx \,{}^{\bullet}.\, x)x = yx,\ x(xy \,.{}^{\bullet}\, x) = xy.$$

Taking $y = xx^*$ in the first of these, and $y = x^*x$ in the second, we obtain (1).

(2) By (1) and [**1**, Theorem 22.3] we have

$$x \,{}^{\bullet}.\, x = x \,{}^{\bullet}.\, (x \,{}^{\bullet}.\, x)x = (x \,{}^{\bullet}.\, x) \,{}^{\bullet}.\, (x \,{}^{\bullet}.\, x),$$

and similarly $x \,.{}^{\bullet}\, x = (x \,.{}^{\bullet}\, x) \,.{}^{\bullet}\, (x \,.{}^{\bullet}\, x)$.

(3) By (2) we have

$$(x \,{}^{\bullet}.\, x)^* = [(x \,{}^{\bullet}.\, x) \,{}^{\bullet}.\, (x \,{}^{\bullet}.\, x)] \,.{}^{\bullet}\, (x \,{}^{\bullet}.\, x) = (x \,{}^{\bullet}.\, x) \,.{}^{\bullet}\, (x \,{}^{\bullet}.\, x),$$

and similarly $(x \,.{}^{\bullet}\, x)^* = (x \,.{}^{\bullet}\, x) \,{}^{\bullet}.\, (x \,.{}^{\bullet}\, x)$.

(4) This follows by (1) and (3). □

Corollary. $(\forall x \in S)\ x \,{}^{\bullet}.\, x \in E,\ x \,.{}^{\bullet}\, x \in E.$

Proof. To see that $x \,{}^{\bullet}.\, x \in E$, observe that

$$\begin{aligned} x \,{}^{\bullet}.\, x &= (x \,{}^{\bullet}.\, x)[(x \,{}^{\bullet}.\, x) \,.{}^{\bullet}\, (x \,{}^{\bullet}.\, x)] \text{ by Theorem 1(1)} \\ &= (x \,{}^{\bullet}.\, x)(x \,{}^{\bullet}.\, x)^* \text{ by Theorem 1(3).} \end{aligned}$$

Hence $x \,{}^{\bullet}.\, x \in E$, and similarly $x \,.{}^{\bullet}\, x \in E$. □

Theorem 2. *If S is a residuated regular semigroup then associated with every $x \in S$ there are the following chains of idempotents:*

$$xx^* \leqq x \,{}^{\bullet}.\, x \leqq (x \,{}^{\bullet}.\, x)^\circ \leqq (x \,{}^{\bullet}.\, x)^* \leqq x^* \,.{}^{\bullet}\, x^* = x^{**} \,{}^{\bullet}.\, x = (xx^*)^*;$$

$$x^*x \leqq x \,.{}^{\bullet}\, x \leqq (x \,.{}^{\bullet}\, x)^\circ \leqq (x \,.{}^{\bullet}\, x)^* \leqq x^* \,{}^{\bullet}.\, x^* = x^{**} \,.{}^{\bullet}\, x = (x^*x)^*.$$

Proof. We establish the first chain as follows, the second being similar. First, recall that $xx^* \in E$; and that since $xx^*x = x$ we have $xx^* \leqq x \,{}^{\bullet}.\, x$.

Since for every $e \in E$ we have $e \leqq e^\circ \leqq e^*$ (see [**2**]), it follows from the fact that $x \,{}^{\bullet}.\, x \in E$ that

$$x \,{}^{\bullet}.\, x \leqq (x \,{}^{\bullet}.\, x)^\circ \leqq (x \,{}^{\bullet}.\, x)^*.$$

Now by Theorem 1(4) we have $(x \,{}^{\bullet}.\, x)^\circ \in E$ and so, since $x \,{}^{\bullet}.\, x \in E$, it follows by [**3**, Theorem 2.2] that also $(x \,{}^{\bullet}.\, x)^* \in E$.

To see that $(x \,{}^{\bullet}.\, x)^* \leqq x^* \,.{}^{\bullet}\, x^*$, observe that

$$xx^*(x \,{}^{\bullet}.\, x)^*x \leqq (x \,{}^{\bullet}.\, x)(x \,{}^{\bullet}.\, x)^*x$$

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$$=(x\,{}^{\cdot}.\,x)x \quad \text{by Theorem 1(1),(3)}$$

$$=x,$$

and so $x^*(x\,{}^{\cdot}.\,x)^* \leqq x^*$ which gives $(x\,{}^{\cdot}.\,x)^* \leqq x^*\,.{}^{\cdot}\,x^*$.

Next, observe that

$$\begin{aligned}(xx^*)^* &= (xx^*\,.{}^{\cdot}\,xx^*)\,{}^{\cdot}.\,xx^* \\ &= [(xx^*\,.{}^{\cdot}\,x)\,.{}^{\cdot}\,x^*]\,{}^{\cdot}.\,xx^* \\ &= (x^*\,.{}^{\cdot}\,x^*)\,{}^{\cdot}.\,xx^* \quad \text{by [1, Theorem 22.1]} \\ &= [(x^*\,.{}^{\cdot}\,x^*)\,{}^{\cdot}.\,x^*]\,{}^{\cdot}.\,x \\ &= x^{**}\,{}^{\cdot}.\,x.\end{aligned}$$

Now since xx^* is the greatest idempotent in its $\mathscr{R}$-class (see [**2**]), we have $xx^* = xx^*(xx^*)^*$ and therefore $x = xx^*x = xx^*(xx^*)^*x$ which gives $x^*(xx^*)^* \leqq x^*$ and so $(xx^*)^* \leqq x^*\,.{}^{\cdot}\,x^*$. But, by Theorem 1(1),

$$xx^*(x^*\,.{}^{\cdot}\,x^*)xx^* = xx^*xx^* = xx^*$$

and so $x^*\,.{}^{\cdot}\,x^* \leqq (xx^*)^*$. Hence $(xx^*)^* = x^*\,.{}^{\cdot}\,x^* \in E$. □

Corollary. *A residuated regular semigroup is a strong Dubreil–Jacotin semigroup if and only if it has a greatest idempotent.*

Proof. $\Rightarrow$: The bimaximum element ξ of a strong Dubreil–Jacotin regular semigroup is idempotent.

$\Leftarrow$: If S has a greatest idempotent ξ then since $x\,{}^{\cdot}.\,x$ is idempotent, we have

$$x^2 \leqq x \Rightarrow x \leqq x\,{}^{\cdot}.\,x \leqq \xi$$

so $\{x \in S; x^2 \leqq x\}$ has greatest element ξ. That S is strong Dubreil–Jacotin now follows by [**1**, Theorem 25.10]. □

Consider now the subset S° of S given by $S^\circ = \{x^\circ; x \in S\}$. As is shown in [**2**], we have $x^{\circ\circ\circ} = x^\circ$ and so it follows that

$$x \in S^\circ \Leftrightarrow x = x^{\circ\circ}.$$

As is shown in [**4**], for every $x \in S$, we have $(xx^\circ)^\circ = x^{\circ\circ}x^\circ$ and $(x^\circ x)^\circ = x^\circ x^{\circ\circ}$.

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Theorem 3. *If S is a residuated regular semigroup then*

$$x \in S^{\circ} \Leftrightarrow x \mathbin{{}^{\cdot}.} x = (xx^*)^*, \quad x \mathbin{.{}^{\cdot}} x = (x^*x)^*.$$

Proof. $\Rightarrow$: If $x \in S^{\circ}$ then $x = x^{\circ\circ}$ and so, since $x^{\circ}x^{\circ\circ}$ is the biggest idempotent in its $\mathscr{R}$-class, we have

$$x^{\circ}x = x^{\circ}x^{\circ\circ} = x^{\circ}x^{\circ\circ}(x^{\circ}x^{\circ\circ})^{\circ} = x^{\circ}x(x^{\circ}x)^{\circ} = x^*x(x^*x)^*$$

whence $x = x(x^*x)^*$. Consequently $(x^*x)^* \leqq x \mathbin{.{}^{\cdot}} x$, whence we have equality by Theorem 2; and similarly $x \mathbin{{}^{\cdot}.} x = (xx^*)^*$.

$\Leftarrow$: From $x \mathbin{.{}^{\cdot}} x = (x^*x)^*$ we obtain

$$x^*x(x^*x)^* = x^*x(x \mathbin{.{}^{\cdot}} x) = x^*x$$

and so, since x^*x is the biggest idempotent in its $\mathscr{L}$-class,

$$(x^*x)^{\circ} = (x^*x)^*x^*x(x^*x)^* = (x^*x)^*x^*x = x^*x.$$

Thus we see that

$$x^*x = (x^*x)^{\circ} = (x^{\circ}x)^{\circ} = x^{\circ}x^{\circ\circ}$$

and therefore $x = xx^*x = xx^{\circ}x^{\circ\circ}$. Similarly, from $x \mathbin{{}^{\cdot}.} x = (xx^*)^*$ we can deduce that $x = x^{\circ\circ}x^{\circ}x$. It follows that

$$x = xx^{\circ}x^{\circ\circ} = x^{\circ\circ}x^{\circ}xx^{\circ}x^{\circ\circ} = x^{\circ\circ}x^{\circ}x^{\circ\circ} = x^{\circ\circ}$$

and so $x \in S^{\circ}$. □

Our objective now is to show that if a residuated regular semigroup S is orthodox and if $E \subseteq S^{\circ}$ then S is an inverse semigroup. For this purpose, we require the following result.

Theorem 4. *If S is a residated regular semigroup then, for every $e \in E$,*

$$e^{\circ} = (e \mathbin{.{}^{\cdot}} e)e(e \mathbin{{}^{\cdot}.} e).$$

Proof. It is readily seen by Theorem 1(1) that $(e \mathbin{.{}^{\cdot}} e)e(e \mathbin{{}^{\cdot}.} e) \in V(e)$ and therefore

$$(e \mathbin{.{}^{\cdot}} e)e(e \mathbin{{}^{\cdot}.} e) \leqq e^{\circ}.$$

But from $ee^{\circ}e = e$ we have $e^{\circ}e \leqq e \mathbin{.{}^{\cdot}} e$ and $ee^{\circ} \leqq e \mathbin{{}^{\cdot}.} e$, so that

$$e^\circ = e^\circ e e^\circ = e^\circ e \,.\, e \,.\, ee^\circ \leqq (e \,.^{\cdot}\, e)e(e \,{}^{\cdot}.\, e),$$

whence the required equality follows. □

Theorem 5. *If S is a residuated regular semigroup then the following statements are equivalent:*

(1) *S is orthodox and $E \subseteq S^\circ$;*

(2) *S is inverse.*

Proof. (1)⇒(2): Using Theorem 4 we see that

$$\begin{aligned} e^\circ e(e \,{}^{\cdot}.\, e) &= (e \,.^{\cdot}\, e)e(e \,{}^{\cdot}.\, e)e(e \,{}^{\cdot}.\, e) \\ &= (e \,.^{\cdot}\, e)e(e \,{}^{\cdot}.\, e) \\ &= e^\circ. \end{aligned}$$

Since in an orthodox semigroup every inverse of an idempotent is idempotent, it follows that if (1) holds then for every $e \in E$ we have $e^\circ \in E$. We therefore deduce from the above that

$$e^\circ = e^\circ e(e \,{}^{\cdot}.\, e)e^\circ.$$

Consequently, $e(e \,{}^{\cdot}.\, e) \leqq e^{\circ *} = e^{**}$ (see [2]) and so, by Theorems 2 and 3,

$$e \,{}^{\cdot}.\, e \leqq e^{**} \,.^{\cdot}\, e = (e^* e)^* = e \,.^{\cdot}\, e.$$

Similarly, we can show that $e \,.^{\cdot}\, e \leqq e \,{}^{\cdot}.\, e$. Hence $e \,{}^{\cdot}.\, e = e \,.^{\cdot}\, e$ and so

$$e^\circ = (e \,.^{\cdot}\, e)e(e \,{}^{\cdot}.\, e) = (e \,{}^{\cdot}.\, e)e(e \,.^{\cdot}\, e) = e.$$

Thus for every $e \in E$ we have $e = ee^\circ = e^\circ e$, whence every $\mathscr{R}$-class and every $\mathscr{L}$-class contains a single idempotent. Hence S is an inverse semigroup.

(2)⇒(1): This is clear. □

Recall now that the *natural order* on the idempotents of S is given by

$$e \preceq f \Leftrightarrow e = ef = fe,$$

and that S is said to be *naturally ordered* if

$$e \preceq f \Rightarrow e \leqq f.$$

Definition. If S is a residuated regular semigroup then $x \in S$ will be called *concise* if the idempotents $x \mathbin{{}^{\bullet}.} x$ and $x \mathbin{.{}^{\bullet}} x$ are maximal.

We shall denote the set of concise elements of S by $C(S)$. Clearly, $C(S) \neq \emptyset$ if and only if S contains maximal idempotents; moreover, every maximal idempotent belongs to $C(S)$.

Theorem 6. *Let S be a residuated regular semigroup in which $C(S) \neq \emptyset$. Then $C(S)$ is a residuated regular subsemigroup. Moreover, $C(S)$ is naturally ordered and $C(S) = [C(S)]^{\circ}$.*

Proof. Suppose that $x, y \in C(S)$. Then the inequalities

$$xy \mathbin{.{}^{\bullet}} xy = (xy \mathbin{.{}^{\bullet}} x) \mathbin{.{}^{\bullet}} y \geqq y \mathbin{.{}^{\bullet}} y;$$

$$xy \mathbin{{}^{\bullet}.} xy = (xy \mathbin{{}^{\bullet}.} y) \mathbin{{}^{\bullet}.} x \geqq x \mathbin{{}^{\bullet}.} x$$

show that $xy \in C(S)$; and the inequalities

$$(x \mathbin{.{}^{\bullet}} y) \mathbin{.{}^{\bullet}} (x \mathbin{.{}^{\bullet}} y) = x \mathbin{.{}^{\bullet}} y(x \mathbin{.{}^{\bullet}} y) \geqq x \mathbin{.{}^{\bullet}} x;$$

$$(x \mathbin{.{}^{\bullet}} y) \mathbin{{}^{\bullet}.} (x \mathbin{.{}^{\bullet}} y) = [x \mathbin{{}^{\bullet}.} (x \mathbin{.{}^{\bullet}} y)] \mathbin{.{}^{\bullet}} y \geqq y \mathbin{.{}^{\bullet}} y$$

show that $x \mathbin{.{}^{\bullet}} y \in C(S)$. Similarly, $x \mathbin{{}^{\bullet}.} y \in C(S)$ and so $C(S)$ is a residuated subsemigroup. It is also regular, for if $x \in C(S)$ then the inequalities $x \mathbin{{}^{\bullet}.} x \leqq x^* \mathbin{.{}^{\bullet}} x^*$ and $x \mathbin{.{}^{\bullet}} x \leqq x^* \mathbin{{}^{\bullet}.} x^*$ of Theorem 2 show that $x^* \in C(S)$.

To see that $C(S)$ is naturally ordered, suppose that $e, f \in E \cap C(S)$ with $e \preceq f$. Then $e = ef = fe$ gives $f \leqq e \mathbin{.{}^{\bullet}} e$ and so

$$f \mathbin{.{}^{\bullet}} f \leqq (e \mathbin{.{}^{\bullet}} e) \mathbin{.{}^{\bullet}} f = e \mathbin{.{}^{\bullet}} ef = e \mathbin{.{}^{\bullet}} e$$

whence $f \mathbin{.{}^{\bullet}} f = e \mathbin{.{}^{\bullet}} e$. Then $e \leqq e \mathbin{.{}^{\bullet}} e = f \mathbin{.{}^{\bullet}} f$ gives $e = fe \leqq f$.

Finally, if $x \in C(S)$ then by Theorem 2 we have $x \mathbin{{}^{\bullet}.} x = (xx^*)^*$ and $x \mathbin{.{}^{\bullet}} x = (x^*x)^*$ and so, by Theorem 3, $x \in [C(S)]^{\circ}$. □

Definition. A residuated regular semigroup will be called *concise* if every element of S is concise.

Theorem 7. *If S is a residuated regular semigroup then the following statements are equivalent:*

(1) *S is concise;*

(2) *S is naturally ordered and $S = S^{\circ}$.*

Proof. (1)⇒(2): If (1) holds then $S = C(S)$, and (2) follows by Theorem 6.

(2)⇒(1): If (2) holds then by Theorems 2 and 3 we have $x \,.^{\cdot}\, x=(x \,.^{\cdot}\, x)^*$. It follows by [2, 3] that $x \,.^{\cdot}\, x$ is a maximal idempotent; similarly, so is $x \,{}^{\cdot}.\, x$. □

Theorem 8. *If S is a concise residuated regular semigroup then*

$$(\forall x, y \in S)\ x^* \,.^{\cdot}\, y=(xy)^*,\ x^* \,{}^{\cdot}.\, y=(yx)^*.$$

Proof. In general, we have

$$(\forall x, y \in S)\ x \,{}^{\cdot}.\, x \leqq (xy \,{}^{\cdot}.\, y) \,{}^{\cdot}.\, x = xy \,{}^{\cdot}.\, xy$$

and so, if S is concise, we have $x \,{}^{\cdot}.\, x = xy \,{}^{\cdot}.\, xy$. Consequently,

$$(xy)^* = (xy \,{}^{\cdot}.\, xy) \,.^{\cdot}\, xy = (x \,{}^{\cdot}.\, x) \,.^{\cdot}\, xy = x^* \,.^{\cdot}\, y,$$

and similarly $(yx)^* = x^* \,{}^{\cdot}.\, y$. □

We recall now that

(1) if the natural order $\preceq$ on the idempotents of S reduces to equality then S is said to be *completely simple*;

(2) a principally ordered regular semigroup S is said to be *compact* if $e^\circ = e^*$ for every $e \in E$, this being equivalent, by [3, Theorem 3.1], to $x^\circ = x^*$ for every $x \in S$.

The relationships between the above concepts are summarised in the following result, in which $S^* = \{x^*; x \in S\}$. Note that, as observed in [2], we have $x^* = x^{***}$, so that $x \in S^* \Leftrightarrow x = x^{**}$; and $x^* = x^{*\circ\circ}$, so that $S^* \subseteq S^\circ \subseteq S$.

Theorem 9. *If S is a residuated regular semigroup then the following statements are equivalent*:

(1) *S is completely simple and $S = S^\circ$*;

(2) *S is naturally ordered and $S = S^*$*;

(3) *S is concise and compact*;

(4) *$(\forall x \in S)$ xx^* and x^*x are maximal idempotents.*

Proof. (1)⇒(4): On the one hand we have $xx^* = (x \,{}^{\cdot}.\, x)xx^*$; and on the other we have

$$xx^* = xx^*xx^* \leqq xx^*(x \,{}^{\cdot}.\, x) \leqq xx^*(x^* \,.^{\cdot}\, x^*) = xx^*,$$

so that $xx^* = xx^*(x \,{}^{\cdot}.\, x)$. Thus $xx^* \preceq x \,{}^{\cdot}.\, x$. If (1) holds we therefore have $xx^* = x \,{}^{\cdot}.\, x$, which is a maximal idempotent since S is trivially naturally ordered and so is concise. Similarly, x^*x is a maximal idempotent.

(4)⇒(3): If (4) holds then S is concise by Theorem 2. Since $ee^* \leqq e^{**}e^* \in E$ we also have $ee^* = e^{**}e^*$ whence $e^\circ = e^*ee^* = e^*e^{**}e^* = e^*$, so S is compact.

(3)⇒(2): If (3) holds then, by Theorem 7, S is naturally ordered and $S = S^\circ$; and since S is compact we have $S^\circ = S^*$.

(2)⇒(1): If (2) holds then clearly $S = S^\circ$. Suppose now that $e, f \in E$ are such that $e \preceq f$. Since S is naturally ordered, it follows by [3, Theorem 3.3] that $f^* = e^*$. Consequently, since $S = S^*$, we have $e = e^{**} = f^{**} = f$. Hence S is completely simple. □

Corollary. *If S is a residuated regular semigroup that is concise and compact then*

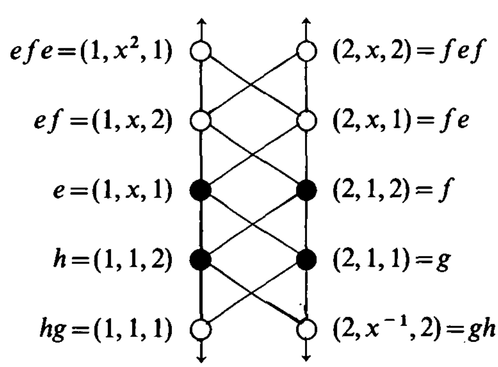
$$(\forall x, y \in S)\ x \,.^{\boldsymbol\cdot}\, y = (x^*y)^*,\ x \,^{\boldsymbol\cdot}.\, y = (yx^*)^*.$$

Proof. Since $S = S^*$, so that $x = x^{**}$ for every $x \in S$, the result follows from Theorem 8 on replacing x by x^*. □

We recall now the *boot-lace semigroup* as constructed in [2]. Let G be an ordered group and let $x \in G$ be such that $1 < x$. If $M = M(G; I, \Lambda; P)$ is the regular Rees matrix semigroup over G with sandwich matrix

$$P = \begin{bmatrix} x^{-1} & 1 \\ 1 & 1 \end{bmatrix}$$

then the *boot-lace semigroup* is the (regular) subsemigroup generated by the four idempotents of M, ordered as in the Hasse diagram



Our objective now is to obtain a characterisation of the boot-lace semigroup as a member of the class of residuated regular semigroups.

Theorem 10. *The boot-lace semigroup is the smallest residuated regular semigroup that is concise and compact, without being strong Dubreil–Jacotin.*

Proof. To see first that the boot-lace is residuated, observe that its elements are of

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the form (i,x^n,λ) where $i,\lambda\in\{1,2\}$ and $n\in\mathbb{Z}$. Routine calculations give the following formulae:

$$(i,x^m,\lambda)\,^{\bullet}.\,(j,x^n,\mu)=\begin{cases}(i,x^{m-n+1-j},j) & \text{if } \lambda=1,\mu=2;\\ (i,x^{m-n+2-j},j) & \text{otherwise,}\end{cases}$$

$$(i,x^m,\lambda)\,.^{\bullet}\,(j,x^n,\mu)=\begin{cases}(\mu,x^{m-n+1-\mu},\lambda) & \text{if } i=1, j=2;\\ (\mu,x^{m-n+2-\mu},\lambda) & \text{otherwise.}\end{cases}$$

By its construction as a Rees matrix semigroup, the boot-lace is completely simple. It is readily seen that for each of the four idempotents i the idempotents $i\,^{\bullet}.\,i$ and $i\,.^{\bullet}\,i$ are maximal. Thus every idempotent is concise and so, by Theorem 6, belongs to S°. It follows that, for every $x\in S$,

$$x=xx^{\circ}x^{\circ\circ}x^{\circ}x=(xx^{\circ})^{\circ\circ}x^{\circ\circ}(x^{\circ}x)^{\circ\circ}=x^{\circ\circ}x^{\circ}x^{\circ\circ}x^{\circ}x^{\circ\circ}=x^{\circ\circ},$$

and that therefore $S=S^{\circ}$. It now follows by Theorem 9 that the boot-lace is a residuated regular semigroup that is concise and compact. It is not strong Dubreil–Jacotin since it has two maximal idempotents (recall the Corollary to Theorem 2). Finally, that it is the smallest such semigroup is a consequence of [**2**, Theorem 6] and [**3**, Theorems 3.3 and 4.2]. □

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