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EFFECTIVE SLIP LENGTH: SOME ANALYTICAL AND NUMERICAL RESULTS

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Abstract

More and more experimental evidence demonstrates that the slip boundary condition plays an important role in the study of nano- or micro-scale fluid. We propose a homogenization approach to study the effective slippage problem. We show that the effective slip length obtained by homogenization agrees with the results obtained by the traditional method in the literature for the simplest Stokes flow; then we use our approach to deal with two examples which seem quite hard by other analytical methods. We also include some numerical results to validate our analytical results.

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1. Introduction

The most commonly used boundary condition in fluid mechanics is the Dirichlet zero-boundary condition, which assumes that the fluid molecules stick to a solid wall and do not slip. This assumption agrees well with the experimental measurements in most cases. However, when the dimension of the experiment goes down to micro- or even nano-scales, more and more evidence shows that the fluid molecules do slip on the solid interface, which corresponds mathematically to the so-called Navier-slip boundary condition: $u = b(\partial u/\partial n)$, where n is the unit normal vector of the boundary pointing into the fluid. Here b, which has unit of length, is called the *slip length*. This length can be visualized if one extrapolates the velocity profile linearly from the solid surface into the solid itself. The depth at which the extrapolated velocity goes to

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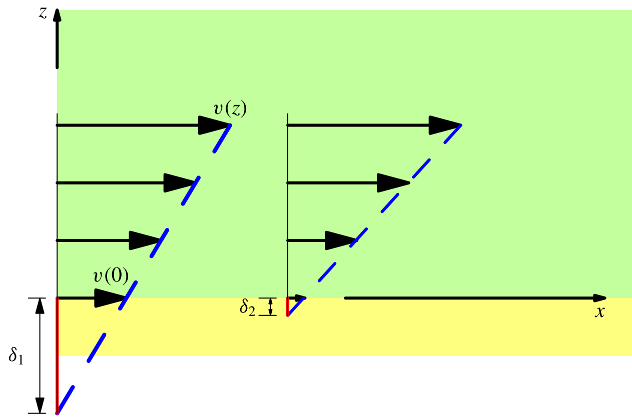

FIGURE 1. Schematic effective slip length.

zero is the slip length b; see Figure 1. We refer the reader to [9] for a review of the experimental results of the Navier-slip boundary condition and to [3, 9] for a review of the theoretical understanding.

Nonetheless, we have to be careful when we talk about the slip length. In fact, we may consider two different notions of slip here. One is *the intrinsic* or *local slip*, which refers to the possibility of fluid molecules slipping against the solid molecules. The other is *the effective slip*, which refers to the macroscopic effect of the local slip to the fluid in the far field.

So, it is interesting to ask if we can derive the effective slip from the local slip, particularly if the local slip length varies across the solid surface. There are two main approaches in the literature. One is the mathematical analysis approach and the other is through numerical methods. In the mathematical analysis category, the first paper is by Philip [12], where he managed to find the explicit solution to the Laplace equation in a rectangular domain and then obtained the effective slip length. Lauga and Stone [10] revisited this problem defined in a circular pipe, and provided some numerical calculation and asymptotic analysis. They have focused on the nonslip (that is, $b = 0$) and perfect slip (that is, $b = \infty$) array of the boundary configuration in order to use the explicit solution approach. Their analysis showed that the effective slip length depends on the flow direction. Hendy and Lund [7] discussed this problem with a general b (not only the 0–∞ array) using an asymptotic expansion. Sbragaglia and Prosperetti [13, 14] used a similar approach to study this problem. The numerical methods in studying the effective slip length include the finite element method, molecular dynamics simulation [4] and some other methods.

In this paper, we will introduce another mathematical analysis method into the study of the effective slip: the homogenization method. This method was introduced in the 1970s [2] and, since then, has become a powerful tool in partial differential equations (PDEs) when studying the global behaviour of the solution from the local one. In Section 2, we will set up the scenario of the effective slip length problem, and

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introduce the effective slip length as used in the literature. We will then introduce the homogenization approach, and demonstrate that, in the simplest case, the effective slip length we obtain from the homogenization method is indeed the effective slip length in the literature. Based on the discussion in Section 2, some results on the effective slip are obtained by the homogenization method in Section 3, including the results for the curved boundary case and the anisotropic slip case, which appear to be very difficult (if not completely untouchable) by other analytical methods. Some finite element numerical results are also included to back up our analysis.

2. Traditional approach and homogenization approach

First, we introduce the concept of effective slip length used in the literature, and then the homogenization approach.

2.1. Effective slip length Suppose that we are given a domain $\Omega = [0, a] \times [0, a] \times [h, H]$ with a constant $a > 0$, a large $H > 0$, the bottom boundary is given by a small-magnitude function $z = h(x, y) = Lh_0(x/L, y/L) \geq 0$ for $0 \leq x \leq a, 0 \leq y \leq a$ and h_0 is a periodic function with a period $[0, 1] \times [0, 1]$. Note that $h(x, y) \to 0$ as $L \to 0$.

Let us start with the simplest partial differential equations describing the fluid dynamics: the Laplace equation $\Delta \mathbf{u} = \mathbf{0}$. In MEMS (microelectromechanical systems), the Reynolds number is often so low that the viscous forces dominate over inertial forces and, by neglecting the advection term and the time dependence, the full incompressible Navier–Stokes equation, which is hard to analyse, can be reduced to a Poisson equation [8, p. 56]

$$\Delta \mathbf{u} = \frac{1}{\mu} \nabla p.$$

If there is no pressure gradient, then this equation becomes the Laplace equation.

If the flow is driven by the motion of the top boundary at $z = H$ and no pressure gradient is present in the domain, then the x-component of the velocity field of the fluid in the flat-bottom case $h(x, y) = 0$ can be described by

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ b(x, y) \dfrac{\partial u}{\partial z} = u & \text{on } z = 0, \\ \dfrac{\partial u}{\partial z} = k & \text{on } z = H, \end{cases} \tag{2.1}$$

where k is a given constant describing the shear rate of the flow at $z = H$ and $b(x, y)$ is a nonnegative periodic function which depends on the position (x, y) and describes the local slip length of the interface between the liquid and the solid. Periodic boundary conditions are imposed on the lateral boundaries, that is,

$$u(0, y, z) = u(a, y, z), \quad u(x, 0, z) = u(x, a, z) \quad \text{for any } x, y, z.$$

For the general nonflat-bottom case $h(x, y) \geq 0$, if there is no pressure gradient in the domain, then the fluid dynamics can be described by [10]

$$\begin{cases} \Delta \mathbf{u} = \mathbf{0}, \quad \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ b(\nabla \mathbf{u} + \nabla \mathbf{u}^T)\mathbf{n} = \mathbf{u} & \text{on } z = h(x, y), \\ \mathbf{u} \text{ is tangent to the surface} & \text{on } z = h(x, y), \\ \dfrac{\partial \mathbf{u}}{\partial z} = k\mathbf{e_1} & \text{on } z = H, \end{cases} \tag{2.2}$$

where $\mathbf{e_1} = (1, 0, 0)$ is the unit vector of the x-axis and periodic boundary conditions are imposed on the lateral boundaries as in the flat-bottom case.

Now we look at the solution u to (2.1) for large H. Because of the boundary condition on the top boundary, the solution is expected to have kz as a part of it. On the other hand, the boundary condition on the bottom should contribute to the solution as well, so the solution u should contain a part of it describing the boundary condition on the bottom boundary. But, as the top boundary gets further and further away from the bottom, it is expected that the solution can only *feel* the overall effects of the slip boundary condition and cannot *feel* the different effect from $b(x, y)$ at different positions on the bottom. Mathematically, this means that as $H \to \infty$, the solution u to (2.1) should asymptotically have the form

$$u = u_s + kz \tag{2.3}$$

for some constant u_s depending on the shear rate k on the top boundary and the local slip $b(x, y)$ on the bottom boundary. One can easily check that u given by (2.3) solves Problem (2.1) if the boundary condition in (2.1) is replaced by

$$b_{\text{eff}} \frac{\partial u}{\partial z} = u \quad \text{on } z = 0 \quad \left(\text{with } b_{\text{eff}} = \frac{u_s}{k}\right). \tag{2.4}$$

This means that, if we are concerned only with the far-field effect of the local slippage, then the local slip boundary condition can be replaced by a constant-coefficient condition (2.4). Therefore, the quantity $b_{\text{eff}} = u_s/k$ is called *the effective slip length* for Problem (2.1) in the literature (see, for example, [4]). We call it the traditional approach to getting the effective slip length.

Similar concepts and explanations apply also to the nonflat-bottom boundary case (2.2).

This definition explicitly expresses the effect of the local slip on the fluid in the far field, considering the limit behaviour of the flow as $H \to \infty$. The homogenization approach, however, takes a different view about the macroscopic effect of the slip.

Remark 2.1. In this paper, we do not include the pressure gradient term, which simplifies the exposition. But the same concept was also proposed for pressure-driven flow in the literature (see, for example, [10]), and our approach in this paper also works for that general case. In physics, the effective slip reflects the interaction between the fluid and the solid wall at the interface between the liquid and the solid, but does not depend on the bulk property of the liquid.

2.2. Homogenization approach Homogenization theory was developed in the 1970s, and it deals with the limit behaviour of solutions to PDEs with rapidly oscillating coefficients (see, for example, [2]). It is a very powerful tool to pass from microscopic-local properties to macroscopic-global ones. Quite often, homogenization theory assumes that the medium has a periodic (or periodic-like) structure, and investigates the limit process as the period length shrinks to zero. One key step in applying homogenization theory is to obtain some a priori bounds of the solutions in a Sobolev space, which are independent of the period length L. With these bounds, one can use some functional analysis tools to prove that the solution sequence as $L \to 0$ has a limit function in some Sobolev space, and one can construct the PDE and the boundary conditions that the limit function satisfies. For example, in (2.1), if we want to exploit homogenization theory, then we need to fix the height H and let the period length L go to zero. If the solution u to (2.1) admits a limit w as $L \to 0$, and supposing that we know the PDE and that the boundary conditions that the limit w satisfies are

$$\begin{cases} \Delta w = 0 & \text{in } \Omega, \\ \tilde{B}\dfrac{\partial w}{\partial z} = w & \text{on } z = 0, \\ \dfrac{\partial w}{\partial z} = k & \text{on } z = H, \end{cases} \tag{2.5}$$

with some constant $\tilde{B}$, then we call $\tilde{B}$ *the effective slip length for Problem* (2.1) *according to the homogenization approach*. The passage from (2.1) to (2.5) is the so-called *homogenization process*.

This definition focuses on the effect of the local slip boundary as the period L shrinks to zero. Intuitively, this approach and the approach exploited in the literature should lead to an identical result. Can we confirm this rigorously?

2.3. Consistency between the two approaches

THEOREM 2.2.

$$\tilde{B} = b_{\text{eff}}.$$

PROOF. If $\tilde{B}$ is given by the homogenization approach, that is, it is given as in (2.5), then the solution to (2.5) is $w = k(z + \tilde{B})$, so the constant u_s in Section 2.1 can be obtained as

$$u_s = k\tilde{B}.$$

Hence, according to the explanation in Section 2.1,

$$\tilde{B} = b_{\text{eff}}.$$ □

REMARK 2.3. In physics, the slip length is a quantity describing the interaction of the fluid and the solid at the interface, and does not depend on the bulk properties of the fluid, so one can expect that the result we prove here can extend to the general case. In other words, it is possible to use the homogenization approach to study the effective slip problem for the general case.

In the next section, we will first exploit the homogenization approach to yield an explicit formula for the effective slip length for Problem (2.2), then demonstrate an application of homogenization theory to a tensorial slip problem in Section 3.2 and present some numerical results obtained by the *traditional* approach (that is, not by the homogenization approach) to back up our homogenization analysis.

3. Some results obtained by the homogenization approach

Once we have established the consistency of our approach with the current one in the literature, we can use the powerful homogenization theory to obtain some interesting results. As we mentioned earlier, homogenization theory [2] of PDEs deals with the asymptotic global behaviour of the solution to some PDEs in some periodically structured domain. In most applications, the coefficients of partial differential operators are periodic functions, so one has to deal with the convergence of the product of two weakly convergent function sequences. But, in our applications here, we assume that only the bottom boundary (not the whole domain) has a periodic structure, which makes our applications not very difficult.

Here we discuss two examples: one deals with a curved boundary and the other with anisotropic slippage.

3.1. Curved boundary We have reported some effective slip results [11] obtained by the homogenization method for a corrugated boundary with some comparison to numerical results, but we did not given a rigorous proof. We will sketch a brief mathematical proof in this section.

We consider the problem (2.2) of x-direction flow (see Figure 2):

$$\begin{cases} \Delta \mathbf{u} = \nabla p, \quad \nabla \cdot \mathbf{u} = 0 & \text{in } \Omega, \\ b(\nabla \mathbf{u} + \nabla \mathbf{u}^T)\mathbf{n} = \mathbf{u} & \text{on } z = h(x,y), \\ \mathbf{u} \text{ is tangent to the surface} & \text{on } \Gamma_{\text{bottom}}: z = h(x,y), \\ \dfrac{\partial \mathbf{u}}{\partial z} = k\mathbf{e_1} & \text{on } \Gamma_{\text{top}}: z = H. \end{cases}$$

Here $\mathbf{u}$ is periodic on the four lateral faces Γ_{lateral}, $h(x,y) = Lh_0(x/L, y/L)$ with periodic functions h_0 and $b = b(x/L, y/L)$. We define the functional space V_0 by

$$V_0 = \{\mathbf{u} \mid \mathbf{u} \in (H^1(\Omega))^3,\ \mathbf{u} \text{ periodic on } \Gamma_{\text{lateral}}\};$$

then the weak formulation of the problem [5] is

$$2\int_\Omega \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) + \int_{\Gamma_{\text{bottom}}} b^{-1}\mathbf{u}\cdot\mathbf{v}\, d\Gamma = \int_{\Gamma_{\text{top}}} k\mathbf{e_1}\cdot\mathbf{v}\, d\Gamma, \tag{3.1}$$

where

$$\mathbf{D}(\mathbf{u}) = \frac{\nabla\mathbf{u} + (\nabla\mathbf{u})^T}{2}$$

and

$$\mathbf{R} : \mathbf{S} \equiv \sum_{i,j=1}^{n} r_{ij}s_{ij}$$

for two matrices $\mathbf{R}$, $\mathbf{S}$.

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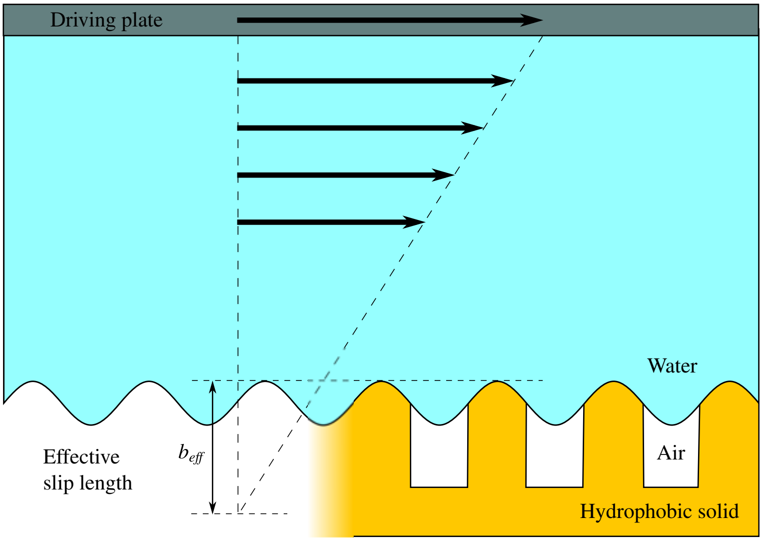

FIGURE 2. Schematic effective slip length for a curved boundary [11].

If we assume that $b(x, y) \geq b_0 > 0$, then it is standard to derive from equation (3.1) that

$$\|\mathbf{u}\|_{W^{1,2}(\Omega)} \leq C$$

for a constant C which is independent of the period L. The bottom boundary is given by a smooth function $z = Lh_0(x/L, y/L)$, so that with appropriate extension of $\mathbf{u}$ to $z = 0$,

$$\|\mathbf{u}\|_{W^{1,2}(\Omega_0)} \leq C,$$

where Ω_0 is the cube $[0, 1]^2 \times [0, H]$. From equation (3.1), we also have

$$2\int_{\Omega} \mathbf{D}(\mathbf{u}) : \mathbf{D}(\mathbf{v}) + \int_{z=0} \frac{\sqrt{1 + h_{0x}'^2 + h_{0y}'^2}}{b} \mathbf{u} \cdot \mathbf{v} \, dx \, dy = \int_{z=H} k\mathbf{e_1} \cdot \mathbf{v} \, dx \, dy.$$

So, as $L \to 0$, the sequence of periodic functions $(\sqrt{1 + h_{0x}'^2 + h_{0y}'^2})/b$ is weakly convergent to its average, while $\mathbf{u}$ is strongly convergent to its limit $\mathbf{u}_0$. This yields the limit equation:

$$2\int_{\Omega_0} \mathbf{D}(\mathbf{u}_0) : \mathbf{D}(\mathbf{v}) + \int_{z=0} \left\langle \frac{\sqrt{1 + h_{0x}'^2 + h_{0y}'^2}}{b} \right\rangle \mathbf{u}_0 \cdot \mathbf{v} \, dx \, dy = \int_{z=H} k\mathbf{e_1} \cdot \mathbf{v} \, dx \, dy,$$

where $\langle f \rangle$ stands for the average of f over its period. This proves the following theorem.

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THEOREM 3.1. *If $b \geq b_0 > 0$ for some constant b_0, then the effective slip length for* (2.2) *is given by*

$$\tilde{b} = \left\langle \frac{\sqrt{1 + h_{0x}'^2 + h_{0y}'^2}}{b} \right\rangle^{-1}.$$

3.2. Anisotropic slip Up to now, the slip b has been assumed to be a function of the location on the surface, but independent of the direction of the flow. We call it the *isotropic slip*. However, even in the very first paper [12] in this field, Philip noticed that the effective slip length depends on the flow direction in some cases. So, it is interesting to ask if we can introduce some concept of local slip depending on the flow direction. Bazant and Vinogradova [1] introduced the concept of tensorial slip along these lines. Bocquet and Barrat [3] also mentioned the possibility of generalizing the concept of slip length to a matrix.

3.2.1. *The analytical results.* Here we will show how to derive the effective slip matrix from the local slip matrix by homogenization. Suppose that we consider the problem

$$\begin{cases} \eta\Delta\mathbf{u} = \nabla p, \quad \nabla\cdot\mathbf{u} = 0 & \text{in the domain } [0,1]^3, \\ \mathbf{M}\dfrac{\partial\mathbf{u}}{\partial z} = \mathbf{u} & \text{on } z = 0, \\ \mathbf{u} & \text{subject to appropriate boundary conditions} \\ & \text{on the rest boundary,} \end{cases} \tag{3.2}$$

where $\mathbf{M} = \mathbf{M}(x/L, y/L)$ is a positive-definite 2×2 matrix with entries being periodic functions of x/L and y/L. If we assume that the eigenvalues of $\mathbf{M}$ have some positive lower bounds which are independent of L, then the same ideas as before can give us a proof of the following theorem.

THEOREM 3.2. *If the eigenvalues of* $\mathbf{M}$ *have some positive lower bounds which are independent of L, then*

$$\widetilde{\mathbf{M}} = \langle \mathbf{M}^{-1} \rangle^{-1}, \tag{3.3}$$

where $\mathbf{A}^{-1}$ *is the inverse of the matrix* $\mathbf{A}$ *and* $\langle \mathbf{A} \rangle$ *is a matrix obtained by averaging each entry of* $\mathbf{A}$.

Here the key ingredients are the same as before. In the weak formulation of the problem, we can prove that the solution is strongly convergent to the limit in L^2, and the periodic function matrix $M^{-1}(x/L, y/L)$ is weakly convergent to its limit and the average $\langle M^{-1} \rangle$, which yields the effective slip matrix is given by (3.3).

3.2.2. *The numerical results.* At this stage, we want to present some numerical results with two goals: the first is to demonstrate how to find the effective slip matrix numerically, which is not obvious to us; the other is to show some numerical evidence to back up our derivation. Suppose that $\mathbf{u}$ is the x-direction flow solution to (3.2) with

TABLE 1. The computed values for the effective slip matrix.

k	$\widetilde{\mathbf{M}}_{11}$	$\widetilde{\mathbf{M}}_{12}$	$\widetilde{\mathbf{M}}_{21}$	$\widetilde{\mathbf{M}}_{22}$
1	1.826	0.685	0.659	1.766
2	1.812	0.639	0.627	1.750
3	1.809	0.624	0.621	1.744
4	1.808	0.614	0.614	1.741
5	1.807	0.610	0.610	1.739
6	1.806	0.607	0.607	1.739
7	1.806	0.604	0.604	1.7389
8	1.805	0.603	0.603	1.7376

periodic boundary conditions on the four lateral faces and the Neumann boundary condition on the top boundary

$$\frac{\partial \mathbf{u}}{\partial \mathbf{n}} = k\mathbf{e_1}.$$

Then we expect that as z becomes large,

$$\mathbf{u} \sim k\Big(z\mathbf{e_1} + \frac{\alpha}{k}\mathbf{e_1} + \frac{\beta}{k}\mathbf{e_2}\Big).$$

This is not sufficient to give the four elements of the effective slip matrix. In order to compute the effective slip matrix, we have to consider the y-direction flow as well. This means that we need to consider the solution $\mathbf{v}$ to (3.2) with Neumann condition on the top boundary

$$\frac{\partial \mathbf{v}}{\partial \mathbf{n}} = k\mathbf{e_2},$$

and express the solution $[\mathbf{u}, \mathbf{v}]^T$, when z is large, as

$$\begin{bmatrix}\mathbf{u}\\ \mathbf{v}\end{bmatrix} = k\left(z\begin{bmatrix}\mathbf{e_1}\\ \mathbf{e_2}\end{bmatrix} + \widetilde{\mathbf{M}}\begin{bmatrix}\mathbf{e_1}\\ \mathbf{e_2}\end{bmatrix}\right).$$

We take the local slip as a matrix M:

$$M_{11} = 2 + \sin(2k\pi x)\sin(2k\pi y), \quad M_{22} = 2 - \sin(2k\pi x)\cos(2k\pi y) \quad \text{and}$$
$$M_{12} = M_{21} = 0.5 * (1 + \sin(2k\pi x)\cos(2k\pi y)).$$

The effective slip matrix calculated by our formula (3.3) is

$$\widetilde{\mathbf{M}} = \begin{pmatrix}1.8050 & 0.5969\\ 0.5969 & 1.7375\end{pmatrix}.$$

We used the free finite element package Freefem++ [6] to solve our problem and the numerical results are shown in Table 1.

REMARK 3.3. From the numerical results, we observe that the formula we present does give a good approximation for a thick channel (which corresponds to a large k here). But, for a thin channel, the formula is not very satisfactory.

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4. Conclusion

We have proposed a new approach to studying the effective slip problem: the homogenization method. We have shown that the effective slip obtained by homogenization is consistent with the current results in the literature for the Stokes flow. Then, with this new approach, we have demonstrated how to obtain the effective slip length and the effective slip matrix in two examples, which seem quite hard to tackle with other analytical methods. We have also included some numerical results (obtained by the nonhomogenization approach) on the anisotropic slip problem to validate our analytical results.

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