The Vibrations of a Particle about a Position of Equilibrium—Part 2.

The Relation between the Elliptic Function and Series Solutions.

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1. In a previous paper, entitled the "Vibrations of a Particle about a Position of Equilibrium," by the author in collaboration with Professor E. B. Ross (*Proc. Edin. Math. Soc.*, XXXIX, 1921, pp. 34–57), a particular dynamical system having two degrees of freedom was chosen and solutions of the corresponding differential equations were obtained in terms of periodic series and also in terms of elliptic functions. It was shown that for certain values of the frequencies of the principal vibrations, the periodic series become divergent, whereas the elliptic function solution continues to give finite results.

In the present paper it is shown that the solution in terms of periodic series may be deduced from that in terms of elliptic functions by certain transformations. It is thus possible to discuss the cause of divergence of the periodic series and to define the region in which they are convergent. It is further shown that in the remaining region in which real solutions of the problem exist, the solution in terms of periodic series derived from the elliptic function solution takes a quite different form.

2. ϕ_1 and ϕ_2 were taken to represent the normal coordinates of the system, ψ_1 and ψ_2 the corresponding momenta and $\frac{s_1}{2\pi}$, $\frac{s_2}{2\pi}$ the frequencies of the principal vibrations. A contact transformation to another system of coordinates p_1, p_2, q_1, q_2 was applied, defined by the equations

$$\left.\begin{aligned} \phi_1 &= \sqrt{\frac{2q_1}{s_1}} \cdot \cos p_1 ; \quad \psi_1 = \sqrt{2s_1 q_1} \, . \sin p_1 ; \\ \phi_2 &= \sqrt{\frac{2q_2}{s_2}} \, . \cos p_2 ; \quad \psi_2 = \sqrt{2s_2 q_2} \, . \sin p_2 . \end{aligned}\right\} \qquad (1)$$

Hamilton's function (H) was assumed to be

$$H = s_1 q_1 + s_2 q_2 + \alpha q_1 q_2^{\frac{1}{2}} . \cos(2p_1 - p_2). \quad \ldots\ldots\ldots\ldots(2)$$

The equations of motion of the system then become

$$\left.\begin{aligned} \dot{q}_1 &= -2\alpha q_1 q_2^{\frac{1}{2}} . \sin(2p_1 - p_2), \\ \dot{q}_2 &= \alpha q_1 q_2^{\frac{1}{2}} . \sin(2p_1 - p_2), \\ \dot{p}_1 &= -s_1 - \alpha q_2^{\frac{1}{2}} . \cos(2p_1 - p_2), \\ \dot{p}_2 &= -s_2 - \tfrac{1}{2}\alpha \frac{q_1}{q_2^{\frac{1}{2}}} . \cos(2p_1 - p_2). \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

From these it follows that $q_1 + 2q_2 = c$..................................(4)
where c is a constant. Since the energy of the system is a constant, we have further $s_1 q_1 + s_2 q_2 + \alpha q_1 q_2^{\frac{1}{2}} \cos(2p_1 - p_2) = h$ ....(5)
where h is a constant.

We assumed, without loss of generality, that c and s_1 were both equal to unity, and wrote $1 - h = g$ and $2s_1 - s_2 = s$; we then deduced solutions in series in the form

$$q_2 = a_2 + a_1 a_2^{\frac{1}{2}} \frac{\alpha}{s} \cos M + \tfrac{1}{4} a_1 (a_1 - 4a_2) \frac{\alpha^2}{s^2} - 2a_1 a_2^{\frac{1}{2}} (a_1 - a_2) \frac{\alpha^3}{s^3} \cos M$$

$$- \left\{\tfrac{3}{4} a_1 (a_1^2 - 8a_1 a_2 + 4a_2^2) + a_1^2 a_2 \cos 2M\right\} \frac{\alpha^4}{s^4}, * \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

with corresponding expressions for q_1, p_1 and p_2; where

$$\left.\begin{aligned} a_1 &= k_1^2 - \tfrac{1}{2}k_1^2(k_1^2 - 4k_2^2)\frac{\alpha^2}{s^2} + \tfrac{3}{2}k_1^2(k_1^4 - 6k_1^2 k_2^2 + 4k_2^4)\frac{\alpha^4}{s^4}, \\ a_2 &= k_2^2 + \tfrac{1}{4}k_1^2(k_1^2 - 4k_2^2)\frac{\alpha^2}{s^2} - \tfrac{3}{4}k_1^2(k_1^4 - 6k_1^2 k_2^2 + 4k_2^4)\frac{\alpha^4}{s^4}; \end{aligned}\right\} \ldots\ldots(7)$$

$$\left.\begin{aligned} k_1^2 &= 1 - \frac{2g}{s}, \\ k_2^2 &= \frac{g}{s}; \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(8)$$

$$M = 2\epsilon_1 - \epsilon_2 - t\left\{s + 2(a_1 - a_2)\frac{\alpha^2}{s} - \tfrac{1}{2}(7a_1^2 - 20a_1 a_2 + 4a_2^2)\frac{\alpha^4}{s^3}\right\}; \ \ldots\ldots(9)$$

ϵ_1 and ϵ_2 are arbitrary constants and t is the time.

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* In the previous paper by Baker and Ross, *loc. cit.*, this equation, given on page 40, contains a misprint in the last term.

This solution is obviously divergent when s is very small, *i.e.* when the frequencies of the principal vibrations are such that s_2 is nearly equal to $2s_1$.

3. The solution in terms of elliptic functions was shown to depend on the cubic equation

$$4\alpha^2 x^3 - (4\alpha^2 + s^2)x^2 + (\alpha^2 + 2sg)x - g^2 = 0 \; ; \quad \ldots\ldots\ldots\ldots(10)$$

since, when p_1, p_2 and q_1 are eliminated between the second of equations (3) and equations (4) and (5) we obtain

$$\dot{q}_2^2 = 4\alpha^2 q_2^3 - (4\alpha^2 + s^2) q_2^2 + (\alpha^2 + 2sg) q_2 - g^2.$$

Denoting the roots of equation (10) by λ, μ, ν, where $\lambda \leqq \mu \leqq \nu$, and defining quantities l, m, k by the equations

$$\nu = \frac{l+m}{2}, \; \mu = \frac{l+mk}{1+k}, \; \lambda = \frac{l-mk}{1-k}, \quad \ldots\ldots\ldots\ldots(11)$$

where $0 < km < l < m$ and $0 < \lambda < l < \mu < \nu < m$, it follows that

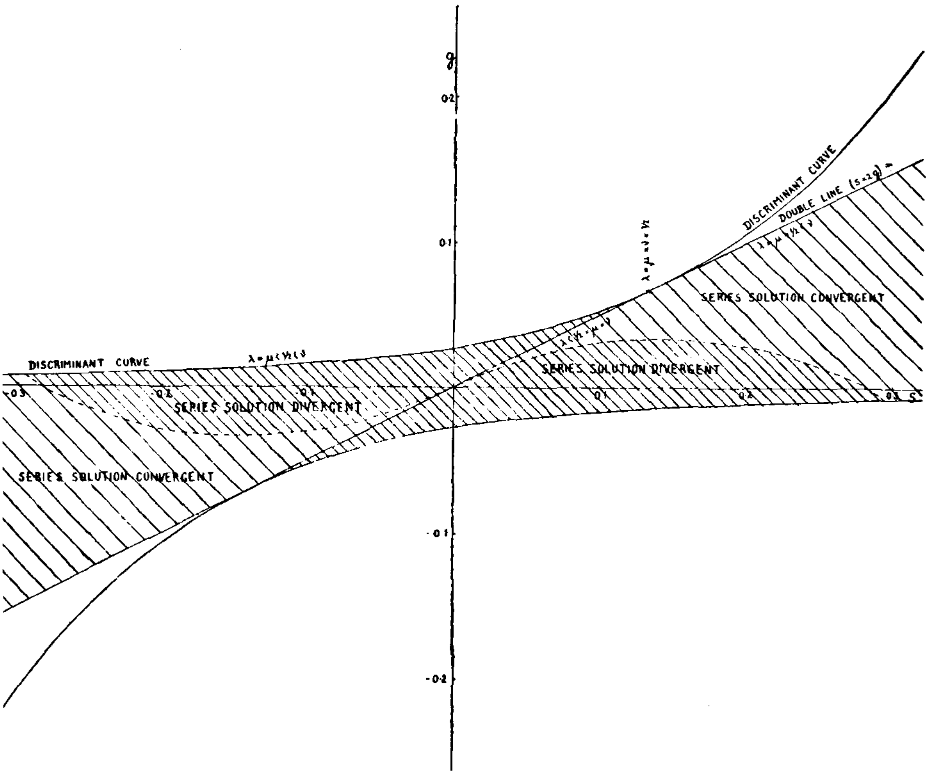
$$q_2 = \frac{mk \operatorname{sn}(u, k) + l}{1 + k \operatorname{sn}(u, k)}, \quad \ldots\ldots\ldots\ldots(12)$$

where
$$\dot{u}^2 = \frac{4\alpha^2 (l-\lambda)(l-\mu)(l-\nu)}{k^2 (m-l)^2}, \quad \ldots\ldots\ldots\ldots(13)$$

with corresponding expressions for q_1, p_1 and p_2.

By means of this elliptic function solution it was shown that real solutions of the problem can only occur for a certain range of values of s and g, namely, when all the roots of the cubic (10) are real and two of the roots are less than $\frac{1}{2}$; if s and g are taken as variables the range of permissible values is limited by the curve obtained by plotting the discriminant of the cubic equated to zero, and was shown to be the shaded area in the figure (in obtaining the figure α was taken equal to 0·1). The elliptic function solution is valid, and admits of calculation in any particular case, throughout the whole of this permissible area except on the double line $s = 2g$ (so called since the factor $s - 2g$ occurs squared in the expression for the discriminant). On this double line the modulus k of the elliptic functions is unity, and the solution is obtainable in a particular form which was given (*loc. cit.*, p. 52).

We have therefore the apparent contradiction that the elliptic function solution is valid whenever a real solution exists (except upon the double line), whereas the series solution certainly breaks down for some such values. The problem is therefore to discover why the series solution is limited in its application.



4. To investigate this we first reduce the elliptic function solution to the series solution. It is convenient to concentrate attention on one coordinate, say q_2, the work being obviously similar for the other coordinates.

Equation (12) may be written

$$q_2 = l + (m - l)\left[k \operatorname{sn} u - k^2 \operatorname{sn}^2 u + k^3 \operatorname{sn}^3 u - k^4 \operatorname{sn}^4 u + \ldots\right],$$

provided $k < 1$. Putting $u = \dfrac{2Kx}{\pi}$, where $4K$ is the real period of the

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elliptic functions, and using the formula

$$\operatorname{sn}\frac{2Kx}{\pi}=\frac{1}{k^{\frac{1}{2}}}\left[\frac{2q^{\frac{1}{4}}\sin x-2q^{\frac{9}{4}}\sin 3x+2q^{\frac{25}{4}}\sin 5x-\dots}{1-2q\cos 2x+2q^4\cos 4x-\dots}\right],$$

we obtain, after some reduction,

$$q_2=l+(m-l)\left[2k^{\frac{1}{2}}\,.\sin x\,.\,q^{\frac{1}{4}}-2k\,(1-\cos 2x)\,q^{\frac{1}{2}}\right.$$
$$\left.+2k^{\frac{3}{2}}\,.(3\sin x-\sin 3x)q^{\frac{3}{4}}-2k^2(3-4\cos 2x+\cos 4x)q+\dots\right]\dots(14)$$

This expansion is valid throughout the whole region in which a real solution exists except in the immediate neighbourhood of that part of the double line, $s=2g$, for which $k=1$. From equations (11) we obtain

$$\left.\begin{aligned} m&=\nu+\sqrt{(\nu-\mu)(\nu-\lambda)},\\ l&=\nu-\sqrt{(\nu-\mu)(\nu-\lambda)},\\ k&=\frac{2\nu-\mu-\lambda-2\sqrt{(\nu-\mu)(\nu-\lambda)}}{\mu-\lambda},\end{aligned}\right\}\dots\dots\dots\dots(15)$$

the ambiguity of sign being removed by the condition $m>l$.

The series (14) reduces to the particular forms given in the previous paper (*loc. cit.*, pp. 53 and 54) on all boundaries of the permissible area: for on the part of the double line, $s=2g$, which is a boundary of the permissible area we have $\lambda=\mu=\frac{1}{2}<\nu$ and therefore $l=\frac{1}{2}$ and $k=0$, $\therefore$ q_2 reduces to $\frac{1}{2}$, which agrees with the result previously obtained; on the curved part of the discriminant curve which is a boundary of the permissible area we have $\lambda=\mu<\frac{1}{2}<\nu$ and therefore $l=\lambda$ and $k=0$, giving $q_2=\lambda$ which agrees with the previous result.

5. To reduce q_2 to the form of an expansion in powers of $\dfrac{\alpha}{s}$ we require to obtain λ, μ and ν in series of powers of $\dfrac{\alpha}{s}$.

Writing $g=k_2^2 s$, so that k_2 is the same as the constant introduced in the series solution and defined by equation (8), the cubic (10) takes the form

$$x(1-2x)^2=\frac{s^2}{\alpha^2}(k_2^2-x)^2.\ \dots\dots\dots\dots(16)$$

Assuming α to be small in comparison with s, the two smallest roots of the cubic are given by

$$x=k_2^2\pm\frac{\alpha}{s}\sqrt{x}(1-2x).\ \dots\dots\dots\dots(17)$$

Solving this equation by successive approximations we obtain

$$\left.\begin{aligned} \mu &= A_0 + A_1\frac{\alpha}{s} + A_2\frac{\alpha^2}{s^2} + A_3\frac{\alpha^3}{s^3} + A_4\frac{\alpha^4}{s^4} + A_5\frac{\alpha^5}{s^5} + A_6\frac{\alpha^6}{s^6} + \dots, \\ \lambda &= A_0 - A_1\frac{a}{s} + A_2\frac{\alpha^2}{s^2} - A_3\frac{\alpha^3}{s^3} + A_4\frac{\alpha^4}{s_4} - A_5\frac{\alpha^5}{s^5} + A_6\frac{\alpha^6}{s^6} - \dots, \end{aligned}\right\} \quad \dots\dots(18)$$

where

$$\left.\begin{aligned} A_0 &= k_2^2, \\ A_1 &= k_2(1-2k_2^2), \\ A_2 &= \tfrac{1}{2}(1-2k_2^2)(1-6k_2^2), \\ A_3 &= \frac{1}{8k_2}(1-2k_2^2)(1-28k_2^2+84k_2^4). \\ A_4 &= -2(1-2k_2^2)(1-10k_2^2+20k_2^4), \\ A_5 &= -\frac{(1-2k_2^2)}{128k_2^3}(1+72k_2^2-2376k_2^4+14028k_2^6-20592k_2^8), \\ A_6 &= 2(1-2k_2^2)(5-70k_2^2+280k_2^4-336k_2^6). \end{aligned}\right\} \quad \dots\dots(19)$$

also, since $\lambda + \mu + \nu = 1 + \dfrac{s^2}{4\alpha^2}$, we have

$$\nu = \frac{s^2}{4\alpha^2} + (1-2A_0) - 2\left[A_2\frac{\alpha^2}{s^2} + A_4\frac{\alpha^4}{s^4} + A_6\frac{\alpha^6}{s^6} + \dots\dots\right] \dots(20)$$

Neglecting higher powers of $\dfrac{\alpha}{s}$ and substituting these values in equations (15) we obtain

$$k = 2k_2(1-2k_2^2)\frac{\alpha^3}{s^3}\left[\begin{aligned} &1 + \frac{1}{8k_2^2}(1-60k_2^2+180k_2^4)\frac{\alpha^2}{s^2} \\ &- \frac{1}{128k_2^4}(1+136k_2^2-7176k_2^4+43212k_2^6 \\ &\qquad\qquad\qquad - 64368k_2^8)\frac{\alpha^4}{s^4} \end{aligned}\right];$$

$$\therefore\ k' = \sqrt{1-k^2} = 1 - 2k_2^2(1-2k_2^2)\frac{\alpha^6}{s^6} - \tfrac{1}{2}(1-2k_2^2)^2(1-60k_2^2+180k_4^2)\frac{\alpha^8}{s^8}$$
$$+ \tfrac{1}{8}(1-2k_2^2)^2(64-2784k_2^2+16203k_2^4-24192k_2^6)\frac{\alpha^{10}}{s^{10}}.$$

To this approximation

$$q=\tfrac{1}{2}\cdot\frac{1-\sqrt{k'}}{1+\sqrt{k'}}=\tfrac{1}{4}k_2^2(1-2k_2^2)^2\frac{\alpha^6}{s^6}\left[\begin{array}{l}1+\dfrac{1}{4k_2^2}(1-60k_2^2+180k_2^4)\dfrac{\alpha^2}{s^2}\\ -\dfrac{1}{16k_2^2}(64-2784k_2^2+16203k_2^4\\ \qquad\qquad -24192k_2^6)\dfrac{\alpha^4}{s^4}\end{array}\right],$$

$$\therefore\ q^{\frac{1}{4}}=\frac{1}{\sqrt{2}}\sqrt{k_2(1-2k_2^2)}\left(\frac{\alpha}{s}\right)^{\frac{3}{2}}\left[\begin{array}{l}1+\dfrac{1}{16k_2^2}(1-60k_2^2+180k_2^4)\dfrac{\alpha^2}{s_2}\\ -\dfrac{1}{128k_2^4}(3-232k_2^2+6312k_2^4\\ \qquad\qquad -32394k_2^6+48816k_2^8)\dfrac{\alpha^4}{s^4}\end{array}\right],$$

and $q^{\frac{1}{3}}=\tfrac{1}{2}k_2(1-2k_2^2)\left(\dfrac{\alpha}{s}\right)\left[1+\dfrac{1}{8k_2}(1-60k_2^2+180k_2^4)\dfrac{\alpha^2}{s^2}\right].$

Further

$$m=\frac{s^2}{2\alpha^2}\left[1+2(2-5k_2^2)\frac{\alpha^2}{s^2}-5(1-2k_2^2)(1-6k_2^2)\frac{\alpha^4}{s^4}\right.$$
$$\left.+4(1-2k_2^2)(5-51k_2^2+102k_2^4)\frac{\alpha^6}{s^6}\right],$$

$$l=k_2^2+\tfrac{1}{2}(1-2k_2^2)(1-6k_2^2)\frac{\alpha^2}{s^2}-2(1-2k_2^2)(1-11k_2^2+22k_2^4)\frac{\alpha^4}{s^4}.$$

We deduce

$$k^{\frac{1}{2}}q^{\frac{1}{4}}=k_2(1-2k_2^2)\left(\frac{\alpha}{s}\right)^3\left[1+\frac{1}{8k_2}(1-60k_2^2+180k_2^4)\frac{\alpha^2}{s^2}\right],$$

and $\quad kq^{\frac{1}{2}}=k_2^2(1-2k_2^2)^2\dfrac{\alpha^6}{s^6}.$

Substituting these values in equation (14) we obtain

$$q_2=k_2^2+k_2(1-2k_2^2)\sin x\,.\frac{\alpha}{s}+\tfrac{1}{2}(1-2k_2^2)(1-6k_2^2)\frac{\alpha^2}{s^2}$$
$$+\frac{1}{8k_2}(1-2k_2^2)(1-28k_2^2+84k_2^4)\sin x\,.\frac{\alpha^3}{s^3}$$
$$-\left[(1-2k_2^2)(2-21k_2^2+42k_2^4)-k_2^2(1-2k_2^2)^2\cos 2x\right]\frac{\alpha^4}{s^4}\ \ldots\ldots\ldots\ldots(21)$$

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6. Now using the series solution for q_2, equation (6), substituting for k_1 in terms of k_2 and arranging in powers of $\frac{\alpha}{s}$, we have

$$q_2 = k_2^2 + k_2(1-2k_2^2)\cos M \cdot \frac{\alpha}{s} + \tfrac{1}{2}(1-2k_2^2)(1-6k_2^2)\frac{\alpha^2}{s^2}$$
$$+\frac{1}{8k_2}(1-2k_2^2)(1-28k_2^2+84k_2^4)\cos M \cdot \frac{\alpha^3}{s^3}$$
$$-\left[(1-2k_2^2)(2-21k_2^2+42k_2^4)+k_2^2(1-2k_2^2)^2\cos 2M\right]\frac{\alpha^4}{s^4} \quad \ldots\ldots\ldots\ldots(22)$$

Thus if we can identify M with $x+\frac{\pi}{2}$ the two expansions (21) and (22) will be the same.

Integrating equation (13) we obtain

$$u = \pm\frac{2\alpha}{k(m-l)}\sqrt{(l-\lambda)(l-\mu)(l-\nu)}\,.\,t + C, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(23)$$

where C is an arbitrary constant of integration. Taking the negative sign and using the values already obtained we find

$$x = C - t\left[s + 2(1-3k_2^2)\frac{\alpha^2}{s} - (5-36k_2^2+54k_2^4)\frac{\alpha^4}{s^3}\right], \quad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

since, to this approximation $\frac{2K}{\pi}=1$ and therefore $x=u$.

Now M is given by

$$M = 2\epsilon_1 - \epsilon_2 - t\left[s + 2(a_1-a_2)\frac{\alpha^2}{s} - \tfrac{1}{2}(7a_1^2 - 20a_1a_2 + 4a_2^2)\frac{\alpha^4}{s^3}\right]$$
$$= 2\epsilon_1 - \epsilon_2 - t\left[s + 2(1-3k_2^2)\frac{\alpha^2}{s} - (5-36k_2^2+54k_2^4)\frac{\alpha^4}{s^3}\right] \ldots(25)$$

Comparing (24) and (25) it follows that x and M differ only by an arbitrary constant, which may be taken to be $\frac{\pi}{2}$.

The series solution is thus completely identified with a series solution obtained from the elliptic function solution.

7. In obtaining this result it has been assumed that α is small in comparison with s. Now let us suppose s to be small compared

with α. In this case two of the roots of the cubic are given by

$$x = \tfrac{1}{2} \pm \tfrac{1}{2}\,\frac{s}{\alpha}\,\frac{(k_2^2 - x)}{\sqrt{x}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(26)$$

Solving this equation by successive approximations we find the two largest roots of the cubic to be

$$\left.\begin{aligned} \nu = \frac{1}{2} &+ \frac{1}{2\sqrt{2}}(1-2k_2^2)\frac{s}{\alpha} + \frac{1}{8}(1-2k_2^2)(1+2k_2^2)\frac{s^2}{\alpha^2} \\ &+ \frac{1}{32\sqrt{2}}(1-2k_2^2)(1+4k_2^2+20k_2^4)\frac{s^3}{\alpha^3} \\ &\qquad + \frac{1}{4}(1-2k_2^2)\,k_2^4\,\frac{s^4}{\alpha^4} + \ldots, \\ \mu = \frac{1}{2} &- \frac{1}{2\sqrt{2}}(1-2k_2^2)\frac{s}{\alpha} + \frac{1}{8}(1-2k_2^2)(1+2k_2^2)\frac{s^2}{\alpha^2} \\ &- \frac{1}{32\sqrt{2}}(1-2k_2^2)(1+4k_2^2+20k_2^4)\frac{s^3}{\alpha^3} \\ &\qquad + \frac{1}{4}(1-2k_2^2)\,k_2^4\,\frac{s^4}{\alpha^4} - \ldots; \end{aligned}\right\} \ldots(27)$$

the remaining root is then

$$\lambda = k_2^4\frac{s^2}{\alpha^2} - \tfrac{1}{2}(1-2k_2^2)\,k_2^4\,\frac{s^4}{\alpha^4} + \ldots(27)$$

This gives

$$\sqrt{(\nu-\mu)(\nu-\lambda)} = C_1\left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + C_2\left(\frac{s}{\alpha}\right)^{\frac{3}{2}} + C_3\left(\frac{s}{\alpha}\right)^{\frac{5}{2}} + C_4\left(\frac{s}{\alpha}\right)^{\frac{7}{2}}$$

where

$$\left.\begin{aligned} C_1 &= \frac{1}{\sqrt[4]{8}}\sqrt{1-2k_2^2} \\ C_2 &= \frac{1}{2\sqrt{2}\,.\,\sqrt[4]{8}}(1-2k_2^2)^{\frac{3}{2}} \\ C_3 &= \frac{3}{32\,.\,\sqrt[4]{8}}(1-2k_2^2)^{\frac{3}{2}}(1+6k_2^2) \\ C_4 &= \frac{1}{64\sqrt{2}\,.\,\sqrt[4]{8}}(1-2k_2^2)^{\frac{3}{2}}(1+4k_2^2+116k_2^4) \end{aligned}\right\} \ldots(28)$$

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Writing

$$\left.\begin{aligned}
B_1 &= \frac{1}{2\sqrt{2}}(1-2k_2^2), \\
B_2 &= \tfrac{1}{8}(1-2k_2^2)(1+2k_2^2), \\
B_3 &= \frac{1}{32\sqrt{2}}(1-2k_2^2)(1+4k_2^2+20k_2^4), \\
B_4 &= \tfrac{1}{4}k_2^4(1-2k_2^2),
\end{aligned}\right\} \quad \ldots\ldots\ldots(29)$$

we obtain

$$k = 1 - 4C_1\left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + 8B_1\frac{s}{\alpha} - (4C_2+8B_1C_1)\left(\frac{s}{\alpha}\right)^{\frac{3}{2}} + 16B_1^2\left(\frac{s}{\alpha}\right)^2;$$

$$\therefore\quad q = \tfrac{1}{2}\left[1 - 2\sqrt[4]{8C_1}.\left(\frac{s}{\alpha}\right)^{\frac{1}{8}} + 2.\sqrt{8C_1}\left(\frac{s}{\alpha}\right)^{\frac{1}{4}}\right].$$

Also

$$l = \tfrac{1}{2} + B_1\left(\frac{s}{\alpha}\right) + B_2\left(\frac{s}{\alpha}\right)^2 + B_3\left(\frac{s}{\alpha}\right)^3 + B_4\left(\frac{s}{\alpha}\right)^4$$
$$- \left[C_1\left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + C_2\left(\frac{s}{\alpha}\right)^{\frac{3}{2}} + C_3\left(\frac{s}{\alpha}\right)^{\frac{5}{2}} + C_4\left(\frac{s}{\alpha}\right)^{\frac{7}{2}}\right],$$

$$m - l = 2\left[C_1\left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + C_2\left(\frac{s}{\alpha}\right)^{\frac{3}{2}} + C_3\left(\frac{s}{\alpha}\right)^{\frac{5}{2}} + C_4\left(\frac{s}{\alpha}\right)^{\frac{7}{2}}\right].$$

Substituting in equation (14) we deduce

$$q_2 = \tfrac{1}{2} + B_1\frac{s}{\alpha} + B_2\left(\frac{s}{\alpha}\right)^2 + \ldots - \left\{C_1\left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + C_2\left(\frac{s}{\alpha}\right)^{\frac{3}{2}} + \ldots\right\}$$
$$+ 2\left\{C_1\left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + C_2\left(\frac{s}{\alpha}\right)^{\frac{3}{2}} + \ldots\right\} \times$$

$$\left[\begin{aligned}
&2\left\{1 - \tfrac{1}{2}C_1\left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + (4B_1 - 2C_1^2)\frac{s}{\alpha} + \ldots\right\}\sin x \times \\
&\qquad\qquad \frac{1}{\sqrt[4]{2}}\left\{1 - \tfrac{1}{4}\sqrt[4]{8C_1}\left(\frac{s}{\alpha}\right)^{\frac{1}{8}} - \tfrac{1}{8}\sqrt{8C_1}\left(\frac{s}{\alpha}\right)^{\frac{1}{4}} + \ldots\right\} \\
&+ 2\left\{1 - \tfrac{1}{4}C_1\left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + 8B_1\frac{s}{\alpha} + \ldots\right\} \times \\
&\quad \frac{1}{\sqrt{2}}\left\{1 - \sqrt[4]{8C_1}\left(\frac{s}{\alpha}\right)^{\frac{1}{8}} + \tfrac{1}{2}\sqrt{8C_1}\left(\frac{s}{\alpha}\right)^{\frac{1}{4}} + \ldots\right\}(1 - \cos 2x) \\
&+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}\right]$$

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We thus obtain a series of the form

$$q_2 = \tfrac{1}{2} + D_1 \left(\frac{s}{\alpha}\right)^{\frac{1}{2}} + D_2 \left(\frac{s}{\alpha}\right)^{\frac{5}{8}} + D_3 \left(\frac{s}{\alpha}\right)^{\frac{3}{4}} + D_4 \left(\frac{s}{\alpha}\right)^{\frac{7}{8}} + D_5 \left(\frac{s}{\alpha}\right) + \dots, \quad (30)$$

the coefficients D_1, D_2, D_3, ... being infinite series involving x and k_2. A series of this form will therefore represent q_2 in that part of the permissible area in which the series (21) is divergent.

8. To summarise the results: we have found that the elliptic function solution may be expressed in the form of a series of powers of q, the coefficients in this series being functions of the roots of the cubic (10) and periodic functions of the quantity x. The roots of the cubic may be expressed in the form of series of positive powers of $\frac{\alpha}{s}$, provided that s is sufficiently large in comparison with α; in this case the series obtained from the elliptic function solution reduces to the series solution obtained in the previous paper, namely, to a periodic series of positive powers of $\frac{\alpha}{s}$. On the other hand, if s is small compared with α, the roots of the cubic may be expressed in the form of series of positive powers of $\frac{s}{\alpha}$, and the series obtained from the elliptic function solution then reduces to the form (30).

The convergence or divergence of the series solution therefore depends directly on the convergence or divergence of the series of positive powers of $\frac{\alpha}{s}$ expressing the roots of the cubic. The divergence of the series solution does not imply any discontinuity in the system but merely shows that this *form* of series is not capable of representing the functions over the whole range of values of s and g which correspond to real solutions.

A fundamental difference is thus to be observed between the series solution and the series (14) obtained from the elliptic function solution This latter series is divergent when $k = 1$, *i.e.* when $s = 2g$, and this divergence does imply a discontinuity of the system. This appears to bear out a remark made in the previous

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paper (*loc. cit.*, p. 57), namely, that the double line is probably the limiting case of a belt in the region of real solutions, in which no stable orbit can occur; such a belt would probably arise when other less important and suitably chosen terms were introduced into the fundamental expression for H, equation (2).

9. To determine the boundary of the region in which the series solution is convergent, we observe that the form (21) will be derived from the expansion (14) whenever the roots of the cubic are expressible in the form of infinite series of positive powers of $\frac{\alpha}{s}$; and that the form (30) will be derived whenever the roots are expressible in the form of infinite series of positive powers of $\frac{s}{\alpha}$.

Further, it is apparent that at least one of the roots of this cubic is an even function of $\frac{\alpha}{s}$, this root being obtained from equation (16) when it is put in the form

$$x = \frac{s^2}{4\alpha^2}\,\frac{(k_2^2 - x)^2}{(\tfrac{1}{2} - x)^2}, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(31)$$

this equation then being solved by successive approximations. If then $x > \frac{1}{2}$, equation (31) will take the form

$$x = \frac{s^2}{4\alpha^2}\left(1 - \frac{2k_2^2}{x} + \frac{k_2^4}{x^2}\right)\left(1 + \frac{1}{x} + \frac{3}{4x^2} + \ldots\right), \quad \ldots\ldots(32)$$

and we shall then obtain from this root an infinite series of positive powers of $\frac{\alpha}{s}$, and q_2 will be of the form (21). On the other hand, if $x < \frac{1}{2}$, equation (31) will take the form

$$x = \frac{s^2}{\alpha^2}\,(k_2^4 - 2k_2^2\,x + x^2)\,(1 + 4x + 12x^2 + \ldots)\,,\ldots\ldots\ldots\ldots(33)$$

and we shall obtain for this root an infinite series of positive powers of $\frac{s}{\alpha}$ so that q_2 will be of the form (30).

Now in the previous paper (*loc. cit.*, p. 48) it was shown that, in the permissible area, one, and only one, root of the cubic is greater than $\frac{1}{2}$; so that, when equation (31) reduces to the form (32), this equation must give the *greatest* root of the cubic. Thus a necessary condition that q_2 shall be represented by a series of the form (21) is that the greatest root of the cubic shall be an even function of $\dfrac{\alpha}{s}$.

The condition is also sufficient: for suppose the greatest root of the cubic to be an even function of $\dfrac{\alpha}{s}$; this root must be given either by equation (31) or by equation (26) or by equation (17). If it is given by equation (31) the series obtained will consist of positive powers of $\dfrac{\alpha}{s}$, and q_2 will be given by equation (21). If, however, the greatest root is given by either of equations (17) or (26), another root of the cubic must be obtained by changing the sign of $\dfrac{\alpha}{s}$. But we have supposed this root to be an even function of $\dfrac{\alpha}{s}$, and therefore in this case we must have two roots equal and greater than $\frac{1}{2}$, which is impossible. Thus if the greatest root is an even function of $\dfrac{\alpha}{s}$ it must be given by equation (31).

A necessary and sufficient condition for the convergence of the series (21) is therefore that the greatest of the roots of the cubic (10) should be an even function of $\dfrac{\alpha}{s}$.

This condition, though of interest theoretically, does not lend itself to the determination of the boundary of the region of convergence of the series solution. It will be seen, however, that it does define a certain range of values of s and g, for which the series (21) will be convergent.

10. In certain simple cases this range of values of s and g may be readily determined.

(i) On the double line, $s = 2g$, outside the points of contact with the discriminant curve, the roots of the cubic are

$$\lambda = \tfrac{1}{2},\ \mu = \tfrac{1}{2},\ \nu = \frac{s^2}{4\alpha^2} > \tfrac{1}{2}.$$

The condition is therefore satisfied all along this boundary of the permissible area.

EDINBURGH MATHEMATICAL SOCIETY

FORTY FIRST SESSION, 1922-23.

THE FOURTH ORDINARY MEETING for the Session will be held on *Friday, 9th February, at 7 p.m.* in the RESEARCH ROOM of the MATHEMATICAL INSTITUTE of EDINBURGH UNIVERSITY.

BUSINESS.

1. The Algebra of Geometrical Reciprocation,
   Professor H. W. TURNBULL.
2. Note on a Theorem of Barlow's regarding the representation of a number as a sum of four squares,
   Professor J. E. A. STEGGALL.
3. Motion with reference to Rule 6 (as below).

At the January Meeting the Committee gave notice of the following motion with reference to Rule 6, to be moved at the February Meeting :—

"That the Ordinary Meetings be held on the First Friday of the month instead of the Second Friday, except in the month of January."

Any Members to whom this alteration will cause inconvenience are requested to communicate with the Secretary before the Meeting.

Tea will be served in the Institute from 6 15 p.m.

The Subscription of **Ten Shillings** for the current Session is now due, and payable *to the Hon. Treasurer,*

E. M. HORSBURGH, M.A., D.Sc., A.M.I.C.E.,
11 GRANVILLE TERRACE, EDINBURGH.

BEVAN B. BAKER, *Hon. Sec.*,
Mathematical Institute, 16 Chambers Street, Edinburgh.

Committee Meeting at 6.40 p.m.

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(ii) When $g=0$, *i.e.* $k_2=0$, the roots of the cubic are given by

$$\nu=\frac{4\alpha^2+s^2+s\sqrt{8\alpha^2+s^2}}{8\alpha^2},$$

$$\mu=\frac{4\alpha^2+s^2-s\sqrt{8\alpha^2+s^2}}{8\alpha^2},$$

$$\lambda=0.$$

If therefore $s^2>8\alpha^2$, ν will be an even function of $\frac{\alpha}{s}$ and the series obtained will be

$$\nu=\frac{s^2}{4\alpha^2}\left\{1+4\frac{\alpha^2}{s^2}-4\frac{\alpha^4}{s^4}+16\frac{\alpha^6}{s^6}-80\frac{\alpha^8}{s^8}+448\frac{\alpha^{10}}{s^{10}}-2688\frac{\alpha^{12}}{s^{12}}+\dots\right\},$$

$$\mu=\frac{\alpha^2}{s^2}\left\{1-4\frac{\alpha^2}{s^2}+20\frac{\alpha^4}{s^4}-112\frac{\alpha^6}{s^6}+672\frac{\alpha^8}{s^8}-\dots\right\},$$

$$\lambda=0.$$

We obtain then, from equation (14),

$$q_2=\left(\frac{\alpha^2}{2s^2}-\frac{2\alpha^4}{s^4}\right)(1+\sin x)+\dots,$$

and the series solution in this case reduces to

$$q_2=\left(\frac{\alpha^2}{2s^2}-\frac{2\alpha^4}{s^4}\right)(1+\cos M)+\dots;$$

it may be verified, as before, that $M=x+\frac{\pi}{2}$.

If, however, $s^2<8\alpha^2$, ν will be an odd function of $\frac{s}{\alpha}$ and the series obtained will be

$$\nu=\tfrac{1}{2}+\frac{1}{\sqrt{8}}\frac{s}{\alpha}+\frac{1}{8}\frac{s^2}{\alpha^2}+\frac{1}{16\sqrt{8}}\frac{s^3}{\alpha^3}-\frac{1}{512\sqrt{8}}\frac{s^5}{\alpha^5}+\dots,$$

$$\mu=\tfrac{1}{2}-\frac{1}{\sqrt{8}}\frac{s}{\alpha}+\frac{1}{8}\frac{s^2}{\alpha^2}-\frac{1}{16\sqrt{8}}\frac{s^3}{\alpha^3}+\frac{1}{512\sqrt{8}}\frac{s^5}{\alpha^5}-\dots,$$

$$\lambda=0;$$

in this case we shall reach a solution of the form (30).

The demarcation between the two forms of series solution thus occurs when $s^2=8\alpha^2$.

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11. To determine the boundary of the region of convergence in general, suppose the roots of the cubic to be diminished by λ, the smallest root. The cubic then becomes

$$z\left\{z^2-(\nu+\mu-2\lambda)z+(\nu-\lambda)(\mu-\lambda)\right\}=0, \quad\ldots\ldots\ldots\ldots(34)$$

having roots

$$z=0,\quad z=\frac{\nu+\mu-2\lambda\pm\sqrt{(\nu+\mu-2\lambda)^2-4(\nu-\lambda)(\mu-\lambda)}}{2}.$$

We thus get the following relations between the roots of the original cubic (10)

$$\nu=\frac{s^2}{8\alpha^2}+\tfrac{1}{2}-\frac{\lambda}{2}+\tfrac{1}{2}\sqrt{\frac{s^4}{16\alpha^4}+\frac{s^2}{\alpha^2}\left(\frac{1+\lambda}{2}-2k_2^2\right)+2\lambda-3\lambda^2},$$

$$\mu=\frac{s^2}{8\alpha^2}+\tfrac{1}{2}-\frac{\lambda}{2}-\tfrac{1}{2}\sqrt{\frac{s^4}{16\alpha^4}+\frac{s^2}{\alpha^2}\left(\frac{1+\lambda}{2}-2k_2^2\right)+2\lambda-3\lambda^2}.$$

These will develope into series of positive powers of $\frac{\alpha}{s}$ only if $\frac{s^4}{16\alpha^4}$ is the dominant term under the radical, *i.e.* if

$$\frac{s^4}{16\alpha^4}>\left|\frac{s^2}{\alpha^2}\left(\frac{1+\lambda}{2}-2k_2^2\right)+2\lambda-3\lambda^2\right|$$

or

$$\left|\frac{8\alpha^2}{s^2}(1+\lambda-4k_2^2)+\frac{16\alpha^4}{s^4}(2\lambda-3\lambda^2)\right|>1\ldots\ldots(35)$$

This condition will define a certain range of values of s and g for which the series solution will be convergent. The limit between the two forms of series solution will occur when

$$\left|\frac{8\alpha^2}{s^2}(1+\lambda-4k_2^2)+\frac{16\alpha^4}{s^4}(2\lambda-3\lambda^2)\right|=1\ldots\ldots\ldots\ldots(36).$$

By eliminating λ between equation (36) and the equation of the cubic (10), when x is replaced by λ, the equation of this boundary curve could be obtained.

In practice it is simplest to take trial values of s and g, solve the cubic and test whether the condition (35) is satisfied. Proceeding in this way, the boundary curve is found to consist of that part of the double line which lies between its points of contact with the curved branches of the discriminant curve, together with the curve indicated by a broken line in the figure. For values of

s and g corresponding to points on that side of this curve which is remote from the origin and which lie in the permissible area, the series solution (21) will be convergent. For other points in the permissible area the series solution will be divergent, and the solution will be represented by series of the form (30).

13. The discussion has been only concerned with the one coordinate q_2, but it is seen to apply also to the coordinate q_1, since it is related to q_2 by the equation

$$q_1 + 2q_2 = 1.$$

The reduction of the coordinates p_1 and p_2 would be more difficult on account of the complicated form of their expression in terms of elliptic functions (see Baker and Ross, *loc. cit*, p. 56), but there is no doubt that agreement could be obtained in the same manner.

The argument has shown that the divergence of the series solution arises in a natural manner from the divergence of the series representing the roots of a certain cubic, and it demonstrates fully the failure of the series solution to give a complete representation of the system, just as one form of the expansions for the roots of the cubic represents them only for a limited range of the coefficients.

This is believed to be the first case in which it has been found possible to determine the conditions of convergence of series of the form of those discussed, and is of importance on account of the frequency with which they occur in the problems of Mathematical Physics. Although the argument applies only to this particular problem, there seems some justification for believing that the conditions obtained indicate the form of the conditions which govern the convergence of such series in general.

In conclusion, I would express my continued obligation to Professor E. T. Whittaker, who originated the problem, for repeated advice and encouragement.