CONICAL DIFFERENTIATION

N. D. LANE AND K. D. SINGH

1. Introduction.

1.1. This paper follows naturally a note on parabolic differentiation **(2)** in which the parabolically differentiable points in the real affine plane were discussed. In the parabolic case, the four-parameter family of parabolas in the affine plane led to three differentiability conditions. In the present paper, the five-parameter family of conics in the real projective plane gives rise to four differentiability conditions and a point of an arc in the projective plane is called *conically differentiable* if these four conditions are satisfied. The differentiable points are classified by the nature of their families of osculating conics, superosculating conics, and their ultraosculating conics.

In § 2, two definitions of the convergence of a sequence of conics to a conic are given and it is observed (Theorem 1) that these definitions are equivalent. Certain families of tangent conics, which are useful later on, are discussed in § 3. In § 4, an arc A is defined to be differentiable at a point p if it has an ordinary tangent at p (Condition I). Tangent conics at the differentiable point p of A are defined and it is proved (Theorem 2) that the non-degenerate, non-tangent conics of A through an interior point p all intersect A at p or all of them support. Osculating conics and a second condition for conical differentiability are introduced in § 5. The nature of the family of osculating conics of an arc at a twice-differentiable point is discussed in Theorem 5. Theorems 3, 4, 6, and 8 are concerned with properties of the osculating conics. In Theorem 7, it is proved that the non-degenerate, non-osculating tangent conics of A at an interior point p all support A at p except when the osculating conics are pairs of distinct lines through p and A crosses its tangent at p, in which case they all intersect A at p.

In § 6, superosculating conics of A at p are defined by means of a third condition for conical differentiability. The types of superosculating conics are described in Theorem 10, and discussed in Theorems 9 and 12. It is shown (Theorem 11) that if an arc A is three times conically differentiable at an interior point p, then the non-superosculating osculating conics all intersect A at p or all of them support. If the superosculating conics are non-degenerate,

Received March 15, 1963. A part of this research was done while the first author was a Fellow of the Summer Research Institute of the Canadian Mathematical Congress in 1958. The second author's research was supported by the National Research Council of Canada. The authors gratefully acknowledge the valuable suggestions of Peter Scherk.

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

then the other osculating conics all support or all intersect according as A has or has not a cusp at p.

In § 7, a fourth condition for conical differentiability and the definition of an ultraosculating conic are introduced and it is observed (Theorem 13) that the non-ultraosculating superosculating conics of A at an interior point p all support A at p. The various types of conically differentiable points are illustrated by some examples.

The present discussion is somewhat simpler than that in **(2)** for the following reasons: the projective plane is compact, while the affine plane is not; also, five points in the projective plane, no three of which are collinear, determine a unique conic, while through the vertices of a strictly convex quadrangle in the affine plane, there always pass two distinct parabolas.

The direct synthetic approach which is used in this paper contrasts with the classical analytic approach to projective differential geometry used, for example, by Bol. Our differentiability conditions are weaker than Bol's, however, and provide the basis of a more general theory of arcs. In particular, although Bol's *Schmiegkegelschnitt* corresponds to our ultraosculating conic, his *Schmiegkegelschnitt* can never be degenerate.

1.2. Notation. The letters $p, s, Q, \ldots$ denote points in the real projective plane. Gothic letters $\mathfrak{L}, \ldots$ denote lines. A conic, which may be a pair of lines, a double line (i.e., a line counted twice), a double line segment (i.e., a line segment counted twice), or a point, will be denoted by γ.

A non-degenerate conic γ has a well-defined *interior* γ_* and an *exterior* γ^*.

Two distinct lines define a conic γ, which decomposes the projective plane into two homeomorphic disjoint regions, which we may denote by γ_* and γ^*.

Two points are said to be separated by a non-degenerate conic or a pair of distinct lines γ if and only if one of the points lies in γ_* and the other in γ^*.

A line or a line segment γ has an exterior γ^* but no interior.

1.3. In real projective co-ordinates, the equation $x^2 - y^2 = a^2t^2$ represents a family of conics through the points $(\pm 1, 1, 0)$. As $a \to 0$ the conics tend to the pair of lines $x^2 - y^2 = 0$.

$y^2 = a^2(t^2 - x^2)$ represents a family of conics through the points $(\pm 1, 0, 1)$. These conics tend to the double line segment $y^2 = 0$, $|x| \leqslant 1$, $t = 1$, as $a \to 0$.

$y^2 = a^2(x^2 - a^2t^2)$ represents a family of conics which tend to the double line $y^2 = 0$ as $a \to 0$.

The equation $x^2 + y^2 = a^2t^2$ represents a family of conics which tend to the point $(0, 0, 1)$ as $a \to 0$.

1.4. Five distinct points, no three of which are collinear, determine a unique non-degenerate conic.

If three of the five points lie on a line $\mathfrak{L}$, which does not pass through the

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

remaining two, there is a unique pair of lines through them, viz., the line $\mathfrak{L}$ and the line joining the other two points.

If exactly four of the five points lie on a line $\mathfrak{L}$, which does not contain the fifth point, there are infinitely many conics through these five points, viz., $\mathfrak{L}$ and any other line through the fifth point.

If all the five points lie on a line $\mathfrak{L}$, there are infinitely many conics through them, viz., $\mathfrak{L}$ together with any other line, the double line coincident with $\mathfrak{L}$, and any double line segment on $\mathfrak{L}$ containing the five points.

2. Convergence.

2.1. A *neighbourhood* of a point P is the interior of a non-degenerate conic which contains P in its interior.

A sequence of points $\{P_n\}$ is defined to be *convergent* to a point P if every neighbourhood of P contains P_n for all but a finite number of n.

A point P is defined to be an *accumulation point* [*limit point*] of a sequence of conics $\{\gamma_n\}$ if every neighbourhood of P contains points of γ_n for infinitely many n [for all but a finite number of n].

A sequence of conics $\{\gamma_n\}$ is defined to be (*pointwise*) *convergent* if every accumulation point of $\{\gamma_n\}$ is a limit point.

Every infinite sequence of conics in the projective plane has a convergent sub-sequence. Furthermore, the set of limit points of a convergent sequence of conics is a conic, i.e., either a non-degenerate conic or one of the following: a pair of lines, a single line, a line segment, or a point. In other words, the set of conics (including the degenerate ones) in the projective plane is countably compact.

2.2. Let P, Q, R, S, T be any five points on a non-degenerate conic γ. Thus, no three of the five points P, Q, R, S, T are collinear. Let P_n, Q_n, R_n, S_n, T_n converge to P, Q, R, S, T respectively. Thus no three of the five points P_n, Q_n, R_n, S_n, T_n are collinear if n is sufficiently large and there is a unique non-degenerate conic γ_n through them. Any limit conic of $\{\gamma_n\}$ passes through P, Q, R, S, T and thus is equal to γ.

Suppose that P, Q, R, lie on a line not incident with S or T. Let γ be the pair of lines through these five points. As above, one readily verifies that if n is sufficiently large, there is a unique conic γ_n through P_n, Q_n, R_n, S_n, T_n which converges to γ as P_n, Q_n, R_n, S_n, T_n converge to P, Q, R, S, T respectively.

2.3. Suppose that a sequence of non-degenerate conics $\{\gamma_n\}$ converges to a pair of lines γ, which meets a line $\mathfrak{L}$ at two distinct points P and Q. If n is sufficiently large, then γ_n meets $\mathfrak{L}$ at two distinct points P_n and Q_n which converge respectively to P and Q. If γ meets $\mathfrak{L}$ at exactly one point P, then γ_n need not meet $\mathfrak{L}$. However, if γ_n meets $\mathfrak{L}$, then γ_n meets $\mathfrak{L}$ at one or two points which converge to P.

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

Let $\{\gamma_n\}$ be a sequence of conics which converge to a line, say $\mathfrak{M}$. Let two lines $\mathfrak{L}$ and $\mathfrak{K}$ meet $\mathfrak{M}$ at two distinct points. Let n be sufficiently large.

1. If γ_n is non-degenerate, then γ_n will meet at least one of the lines $\mathfrak{L}$ and $\mathfrak{K}$ at two distinct points. If, for example, γ_n meets $\mathfrak{L}$ at two distinct points for infinitely many n, then these points converge to $\mathfrak{L} \cap \mathfrak{M}$. (Both $\mathfrak{L} \cap \mathfrak{M}$ and $\mathfrak{K} \cap \mathfrak{M}$ are limit points of γ_n.) This justifies our calling the line $\mathfrak{M}$ a *double line*.

2. If γ_n is a pair of lines, then γ_n meets $\mathfrak{L}$ at one or two points which converge to $\mathfrak{L} \cap \mathfrak{M}$.

3. If γ_n is a double line, then γ_n meets $\mathfrak{L}$ at exactly one point which converges to $\mathfrak{L} \cap \mathfrak{M}$.

4. If γ_n is a line segment, then γ_n meets at least one of the lines $\mathfrak{L}$ and $\mathfrak{K}$. Hence γ_n meets, say, $\mathfrak{L}$ at exactly one point for infinitely many n. This point will converge to $\mathfrak{L} \cap \mathfrak{M}$.

Let $\{\gamma_n\}$ be a sequence of non-degenerate conics which converges to a line segment γ. If a line $\mathfrak{L}$ meets γ at an interior point, then γ_n will meet $\mathfrak{L}$ at two distinct points which converge to $\mathfrak{L} \cap \gamma$.

This justifies our calling γ a *double segment.*

2.4. Neighbourhood of conics. A *neighbourhood* of a non-degenerate conic γ is the region which lies outside a non-degenerate conic $\epsilon \subset \gamma_*$ and inside a non-degenerate conic $\eta \subset \gamma^*$, i.e., the neighbourhood is $\eta_* \cap \epsilon^*$.

A *neighbourhood* of a pair of distinct lines γ is the common exterior of two non-degenerate conics which are separated by γ. Thus, one of the conics lies in γ_* and the other in γ^*.

A *neighbourhood* of a double line γ is the exterior of a non-degenerate conic which does not meet γ.

A *neighbourhood* of a double segment γ [a point conic γ] is the interior of a non-degenerate conic which contains γ in its interior.

2.5. A sequence of non-degenerate conics $\{\gamma_n\}$ is called *globally convergent* to a conic γ if every neighbourhood of γ contains γ_n for all but a finite number of n and, if γ is not a point conic, γ_n satisfies an additional assumption for all large values of n:

(i) If γ is non-degenerate and its neighbourhood is $\eta_* \cap \epsilon^*$, then $\epsilon \subset \gamma_{n*}$.

(ii) If γ is a pair of lines, then γ_n contains exactly one of the conics of the neighbourhood of γ in its interior.

(iii) If γ is a double line, then γ_n intersects at least one of every pair of lines which does not contain γ and which passes through distinct points of γ.

(iv) If γ is a double segment, then γ_n intersects every line which intersects γ. (It can be shown that the end-points of γ, as well as the interior points, are limit points of γ_n.)

2.6. If every neighbourhood of a non-degenerate conic γ [a pair of lines $\gamma = \mathfrak{L} \cup \mathfrak{M}$] contains γ_n for all but a finite number of n and $\{\gamma_n\}$ has two

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

distinct limit points on γ [$\{\gamma_n\}$ has a limit point $\neq \mathfrak{L} \cap \mathfrak{M}$ on each of the lines $\mathfrak{L}$ and $\mathfrak{M}$], then $\{\gamma_n\}$ is globally convergent to γ.

In this paper, we shall be mainly concerned with non-degenerate conics through distinct fixed points p and Q. Thus, the initial assumption will ensure the global convergence of $\{\gamma_n\}$ to γ.

2.7. Suppose that $\{\gamma_n\}$ is globally convergent to γ. For all large values of n, we have:

(i) If γ is non-degenerate, then γ_n is non-degenerate.

(ii) If γ is a pair of lines, then γ_n is either non-degenerate or a pair of lines.

(iii) If γ is a double line, then γ_n is not a point.

(iv) If γ is a double segment, then γ_n is either non-degenerate or a double segment.

(v) If γ is a point, then γ_n is non-degenerate, a double segment, or a point.

2.8. Let γ be a non-degenerate conic and let $\eta_* \cap \epsilon^*$ be a neighbourhood of γ. Then any continuous arc joining η to ϵ will meet γ.

2.9. THEOREM 1. *A sequence of conics $\{\gamma_n\}$ is globally convergent to a conic γ if and only if $\{\gamma_n\}$ is pointwise convergent to γ.*

For an indication of the proof of Theorem 1, we refer the reader to the corresponding Theorem 1 in **(2)**.

From now on, a sequence $\{\gamma_n\}$ which is either globally or pointwise convergent to γ (and thus both globally and pointwise convergent to γ), will be called *convergent* to γ.

2.10. It may occur to the reader that the neighbourhood system which has been introduced earlier in § 2 could be replaced by the topology defined by regarding the conics

$$ax^2 + bxy + cy^2 + dxt + eyt + ft^2 = 0$$

as points of a projective 5-space (a, b, c, d, e, f). This correspondence does not take care of the double segments, however, and the induced topology is not identical with ours. In particular, as $a \to 0$ the points $(a, 0, 1, 0, 0, -a)$ in the projective 5-space converge to the unique point $(0, 0, 1, 0, 0, 0)$, which we have to associate with the double line $y^2 = 0$. On the other hand, if $a \to 0$, $a > 0$, the conics $ax^2 + y^2 - at^2 = 0$ through $(\pm 1, 0, 1)$ converge to the double segment $y^2 = 0$, $|x| \leqslant |t|$, but if $a \to 0$, $a < 0$, they converge to the double segment $y^2 = 0$, $|x| \geqslant |t|$.

We observe that the point-conics do not correspond to a unique point of the 5-space. Again, as $a \to 0$, the points $(a^2, 0, 1, -2a^3, 0, 0)$ converge to $(0, 0, 1, 0, 0, 0)$ which we have associated with the double line $y^2 = 0$, while the conics $a^2x^2 + y^2 - 2a^3xt = 0$ converge to a unique point-conic. The two

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

topologies differ, however, only when a double segment, a point, or a double line, is involved.

To summarize, the induced topologies in the subspace consisting of the non-degenerate conics and the pairs of lines are identical and we have merely distinct compactifications of this subspace.

2.11. Let P be a point of γ ($P \neq \mathfrak{L} \cap \mathfrak{M}$ if $\gamma = \mathfrak{L} \cup \mathfrak{M}$). A small conical neighbourhood N of P is decomposed by γ into two regions, unless P is an end-point of γ. Let γ' be any conic through P ($P \neq \mathfrak{L}' \cap \mathfrak{M}'$ if $\gamma' = \mathfrak{L}' \cup \mathfrak{M}'$). If $\gamma' \cap N - P$ meets both regions [exactly one region], then P is called a *point of intersection* [*support*] of γ and γ'.

This relation is symmetric in γ and γ'.

The methods used in the proof of the next five lemmas are similar to those given for Lemma 4–7 in **(2)**.

LEMMA 1. *Let $\{\gamma_n\}$ converge to a non-degenerate conic γ and let $\{\gamma_n'\}$ converge to γ', where γ' is non-degenerate or a pair of lines $\mathfrak{L} \cup \mathfrak{M}$. If γ and γ' intersect at a point Q, where $Q \neq \mathfrak{L} \cap \mathfrak{M}$ in case γ' is degenerate, then γ_n and γ_n' intersect at a point Q_n close to Q when n is large.*

LEMMA 2. *Let $\{\gamma_n\}$ and $\{\gamma_n'\}$ converge to a non-degenerate conic γ and a double segment γ' respectively. Let γ and γ' intersect at a point Q. If γ_n' is non-degenerate [γ_n' is a double segment], then γ_n and γ_n' intersect at two distinct points [at exactly one point] near Q, when n is large.*

LEMMA 3. *Let $\{\gamma_n\}$ and $\{\gamma_n'\}$ converge to a non-degenerate conic γ and a double line γ' respectively. Let γ and γ' intersect at a point Q. If n is large and*

(i) *γ_n' is non-degenerate [a double segment], then near Q, γ_n and γ_n' meet at a pair of points [intersect at a point], or support at a point, or do not meet at all;*

(ii) *γ_n' is a double line [a pair of lines], then γ_n and γ_n' meet at exactly one point [at two distinct points or exactly one point] near Q.*

LEMMA 4. *Let $\{\gamma_n\}$ converge to a non-degenerate conic γ and let $\{\gamma_n'\}$ converge to a non-degenerate conic γ' [a double segment γ']. Suppose that γ and γ' support at a point P. If n is large and γ_n' is non-degenerate [degenerate], then inside a small neighbourhood N of P, γ_n and γ_n' intersect at two points [one point], or support at a point, or do not meet at all.*

LEMMA 5. *Let $\{\gamma_n\}$ converge to a pair of distinct lines $\gamma = \mathfrak{L} \cup \mathfrak{M}$, and let the sequence of lines $\{\mathfrak{K}_n\}$ converge to a line $\mathfrak{K}$. If $\mathfrak{K}$ intersects γ at a point $P \neq \mathfrak{L} \cap \mathfrak{M}$, then $\mathfrak{K}_n$ intersects γ_n at a point close to P, for all large n.*

2.12. LEMMA 6. *Let $\{\gamma_n\}$ converge to a non-degenerate conic γ. Let P_n and Q_n converge to P; $P_n \in \gamma_n$, $Q_n \in \gamma_n$, $P_n \neq Q_n$ (thus $P \in \gamma$). Then the line P_nQ_n converges to the tangent of γ at P.*

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

Lemma 6 can be extended to the case where γ is a pair of distinct lines $\mathfrak{L}$ and $\mathfrak{M}$ and $P \neq \mathfrak{L} \cap \mathfrak{M}$. Thus, if P lies on $\mathfrak{L}$, the line $P_n Q_n$ converges to $\mathfrak{L}$.

3. Families of tangent conics.

3.1. Let τ be the three-parameter family of non-degenerate conics which touch a line $\mathfrak{T}$ at P. Thus $\tau = \tau(P, \mathfrak{T})$.

The compactified family $\bar{\tau}$ is obtained by adding to τ all its limit conics.

Suppose that a degenerate conic γ is the limit of the sequence $\{\gamma_n\}$ of conics of τ. By the remark at the end of 2.3, a double segment which has P as an interior point and does not lie on $\mathfrak{T}$ is not a member of $\bar{\tau}$. Thus, the degenerate limit conic γ is a pair of lines one of which is $\mathfrak{T}$, a pair of lines which intersect at P, any double line through P, a double segment on $\mathfrak{T}$ through P, a double segment with one vertex P, or the point P.

Suppose, next, that the degenerate conic γ is the limit of these degenerate conics γ_n of $\bar{\tau}$. Each γ_n is the limit of conics of τ. Thus, γ will also be the limit of conics of τ. This verifies that every degenerate limit conic of $\bar{\tau}$ can be obtained by the above step and the family $\bar{\tau}$ described above is already compact.

3.2. LEMMA 7. *Any two conics of τ having three distinct points in common outside P coincide.*

The conic of τ through Q, R, S will be denoted by $\gamma(\tau; Q, R, S)$.

Let P, Q, R, S be mutually distinct. If exactly two of the three points Q, R, S, say Q and R, are collinear with P, but do not lie on $\mathfrak{T}$, then there is exactly one conic of $\bar{\tau}$ through them, viz., the pair of lines PQ and PS.

If exactly one of the points Q, R, S lies on $\mathfrak{T}$, there is a unique conic of $\bar{\tau}$ through them, viz., $\mathfrak{T}$ and the line through the other two.

If $S \notin \mathfrak{T} = QR$, there are infinitely many conics of $\bar{\tau}$ through Q, R, S, namely, $\mathfrak{T}$ and any other line through S.

If P, Q, R, S lie on a line $\mathfrak{L}$, $\mathfrak{L} = \mathfrak{T}$ [$\mathfrak{L} \neq \mathfrak{T}$], there are infinitely many conics of $\bar{\tau}$ through Q, R, S, namely, any pair of lines [any pair of lines through P] one of which is $\mathfrak{L}$; the double line $\mathfrak{L}$; any double segment through P, Q, R, S [through P, Q, R, S with one end-point P].

3.3. The one-parameter family ψ. Let Q and R be any two points not collinear with P, $Q \notin \mathfrak{T}$, $R \notin \mathfrak{T}$. Let $\psi = \psi\ (Q, R)$ denote the one-parametric subfamily of τ which consists of its conics through Q and R. Thus, any two members of ψ support at P and intersect at Q and R. The compactified family $\bar{\psi}$ is obtained by adding to ψ its two degenerate limit conics, viz., the pair of lines PQ and PR and the pair QR and $\mathfrak{T}$.

The family ψ is decomposed by the two degenerate members of $\bar{\psi}$ into two subfamilies such that any two members of one subfamily pass through the same three of the six regions defined by the lines PQ, PR, QR, and $\mathfrak{T}$, while the members of the other subfamily pass through the other three regions.

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

Let γ be the pair of lines PQ and PR, and let γ' be the pair QR and $\mathfrak{T}$. Then the conics of one subfamily of ψ lie in

$$(\gamma^* \cap \gamma'_*) \cup (\gamma_* \cap \gamma'^*) \cup \{P, Q, R\}$$

and the conics of the other subfamily of ψ lie in

$$(\gamma^* \cap \gamma'^*) \cup (\gamma_* \cap \gamma'_*) \cup \{P, Q, R\}.$$

3.4. Let $\mathfrak{L}$ be a line, $P \notin \mathfrak{L}$. We may consider the one-parameter family $\psi = \psi(\mathfrak{L}, Q)$ of conics of τ which touch $\mathfrak{L}$ at a point $Q \neq \mathfrak{L} \cap \mathfrak{T}$, as a limit case of the family ψ of 3.3. There are three degenerate members in the closure $\bar{\psi}$ of this family, viz., $\mathfrak{L} \cup \mathfrak{T}$ and the two double segments with the end-points P and Q.

The conic $\gamma = \mathfrak{L} \cup \mathfrak{T}$ decomposes ψ into two subfamilies such that one of them lies in $\gamma_* \cup P \cup Q$ and the other lies in $\gamma^* \cup P \cup Q$.

3.5. The family ϕ. Each conic $\gamma_0 \in \tau$ determines a two-parametric family $\phi = \phi(\gamma_0)$ of τ which consists of γ_0 and those conics of τ which have at least three-point contact with γ_0 at P. If P, Q, R are not collinear and $Q \notin \mathfrak{T}$, $R \notin \mathfrak{T}$, there exists one and only one conic $\gamma \in \phi$ through Q and R. We denote it by $\gamma(\phi; Q, R)$. In particular, if $Q \subset \gamma_{0*} \cup \gamma_0$ and $R \subset \gamma_0{}^*$, then $\gamma(\phi; Q, R)$ and γ_0 have exactly three-point contact at P. Thus $\gamma(\phi; Q, R)$ and γ_0 intersect at P and at exactly one other point $\neq P$.

The compactification $\bar{\phi}$ of ϕ includes the point-conic P, the double line on $\mathfrak{T}$, the pairs of lines through P one of which is $\mathfrak{T}$, and the double segments on $\mathfrak{T}$ with an end-point P.

In the rest of § 3 we discuss subfamilies of ϕ.

3.6. The family ϕ_R. Let ϕ_R be the one-parameter subfamily of ϕ consisting of those conics of ϕ which pass through a fixed point $R \notin \mathfrak{T}$. Thus, the conic in ϕ_R through Q is $\gamma(\phi; Q, R)$. Any member of ϕ_R determines the whole family uniquely. Any two members of ϕ_R intersect at P and R and meet at no other points.

The only degenerate conic of the compactified family $\bar{\phi}_R$ of ϕ_R is the pair of lines $\mathfrak{T}$ and PR. We note that the same one of the two line segments with the end-points P and R lies in the interior of every conic of ϕ_R.

3.7. The family ϕ_P. Let $\phi = \phi(\gamma_0)$, as in 3.5. If $Q \notin \gamma_0$, there is a unique conic γ of τ through Q which has four-point contact with γ_0 at P. By our definition of ϕ, $\gamma \in \phi$. We shall denote the subfamily consisting of γ_0 and all such conics of ϕ by ϕ_P. The conic of ϕ_P through Q will be denoted by $\gamma(\phi_P; Q)$. It does not meet γ_0 outside P; however, it intersects every conic of ϕ which intersects γ_0.

The compactified family $\bar{\phi}_P$ of ϕ_P includes the point conic P and the double line on $\mathfrak{T}$.

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

3.8. Let P, Q, R lie on a line $\mathfrak{L}$; $Q_n \to Q$, $R_n \to R$; P, Q_n, R_n not collinear. If $\mathfrak{L} \neq \mathfrak{T}$ [$\mathfrak{L} = \mathfrak{T}$ and $Q_n \notin \mathfrak{T}$, $R_n \notin \mathfrak{T}$], then the conic $\gamma(\phi; Q_n, R_n)$ converges to the pair of lines $\mathfrak{L} \cup \mathfrak{T}$ [a pair of lines through p one of which is $\mathfrak{T}$, or a double segment on $\mathfrak{T}$ with p as a vertex, or the double line $\mathfrak{T}$].

Let $Q \in \mathfrak{T}$, $R \notin \mathfrak{T}$, $Q_n \to Q$, $Q_n \notin \mathfrak{T}$, $R_n \to R$. Then $\gamma(\phi; Q_n, R_n)$ converges to the pair of lines $PR \cup \mathfrak{T}$.

Thus, the family ϕ depends continuously on Q and R when Q and R converge to distinct points not both of which lie on $\mathfrak{T}$.

3.9. It may be of interest to give the interpretations of the families τ, $\psi(Q, R)$, $\psi(\mathfrak{L}, R)$, ϕ, ϕ_R, ϕ_P in the affine plane obtained by removing the line $\mathfrak{T}$ from the projective plane.

Let $\mathfrak{M} = PR$ and let μ denote the family of the non-degenerate parabolas having diameters parallel to $\mathfrak{M}$, together with the pairs of lines, double lines, and double rays, all parallel to $\mathfrak{M}$, and all the single lines. Dashes will indicate the images in the affine plane.

The family τ' is then the family of non-degenerate parabolas of μ: $\bar{\tau}' = \mu$.

$\psi'(Q, R)$ is the family of parabolas of τ' through Q and R.

$$\overline{\psi(Q, R)'} = \overline{\psi'(Q, R)}$$

contains, in addition, the line QR and the pair of lines through Q and R parallel to $\mathfrak{M}$.

$\psi'(\mathfrak{L}, R)$ is the pencil of the parabolas of τ' which touch $\mathfrak{L}$ at R.

$$\overline{\psi(\mathfrak{L}, R)'} = \overline{\psi'(\mathfrak{L}, R)}$$

contains, in addition, $\mathfrak{L}$ and the two double rays of μ with the vertex R.

ϕ yields the family ϕ' of all the parabolas obtained by translations from a single parabola $\gamma' \in \tau'$. $\bar{\phi}'$ contains, in addition, the single lines parallel to $\mathfrak{M}$.

ϕ_R' consists of all the parabolas of ϕ' through R. $\overline{\phi_R'} = \overline{\phi_R}'$ contains, in addition, the single line through R parallel to $\mathfrak{M}$.

ϕ_P' consists of the parabolas obtained from a single parabola $\gamma' \in \tau'$, by translations parallel to $\mathfrak{M}$.

4. Arcs.

4.1. An *arc* is defined as the continuous image in the real projective plane of a real parameter interval. If a sequence of points of the parameter interval converges to a point p, the corresponding sequence of the image points is defined to be convergent to the image of p. Thus the definition of convergence of a sequence of points of an arc is stronger than the definition of convergence of a sequence of points; cf. 2.1. The same letters p, s denote points of the parameter interval and their images on A. *The parameters s and p are supposed to be distinct and s will always be "sufficiently close" to p.* The *end-points* [*interior points*] of A are the respective images of the end-points [interior points] of the parameter interval.

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

A *neighbourhood* of p on A is the image of a neighbourhood of the parameter p on the parameter interval. If p is an interior point of A, this neighbourhood is decomposed by p into two (open) one-sided neighbourhoods. The images of distinct points of the parameter interval are to be considered to be different points of A even though they may coincide in the plane. Nevertheless, the notation $Q \neq R$ will indicate that the points Q and R do not coincide.

4.2. Let p be a given point on an arc A. We call A *differentiable* at p if the following condition is satisfied.

CONDITION I. *If the parameter s is sufficiently close to the parameter p, $s \neq p$, the line ps is uniquely determined. It converges as s tends to p.*

The limit straight line $\mathfrak{T}$ is the ordinary tangent of A at p. *From now on, we assume that A satisfies Condition I at p.*

We denote the family of non-degenerate conics which touch $\mathfrak{T}$ at p by τ and its compactification by $\bar{\tau}$, as in 3.1.

4.3. Let the points p, O, Q, R be distinct. Then there are conics through s, p, O, Q, R; cf. 4.1. These conics have accumulation conics as s tends to p; cf. 2.1. Any such limit conic γ will pass through p, O, Q, R and will touch the line $\mathfrak{T}$. Thus γ will belong to $\bar{\tau}$. We then call γ a *tangent conic* of A at p.

The next five sections show that every conic of τ and some of the degenerate ones of $\bar{\tau}$ are tangent conics of A at p according to the above definition. It will be convenient to designate also the remaining conics of $\bar{\tau}$ as tangent conics.

4.4. If no three of p, O, Q, R are collinear and none of the points O, Q, R lies on $\mathfrak{T}$, then no three of the five points p, s, O, Q, R will be collinear. Hence there will be a unique non-degenerate conic through these five points. When s tends to p, any accumulation conic γ will pass through p, O, Q, R and, by Lemma 6, will touch the limit $\mathfrak{T}$ of the line ps. Hence γ belongs to τ, and, by Lemma 7, γ is the unique conic of τ which passes through O, Q, R.

4.5. If no three of p, O, Q, R are collinear and $O \in \mathfrak{T}$, there is a unique conic through the points p, s, O, Q, R. It is non-degenerate unless $s \in \mathfrak{T}$, in which case it is a pair of lines one of which is $\mathfrak{T}$. When s tends to p, any limit conic γ will pass through O, Q, R. Thus, γ is either non-degenerate or a pair of lines. By Lemma 6, γ will touch the limit $\mathfrak{T}$ of the line ps at p; thus $\gamma \in \bar{\tau}$. Since $O \in \gamma$, γ is a pair of lines. Thus γ is the unique conic of $\bar{\tau}$ consisting of $\mathfrak{T}$ and QR.

4.6. If the three points p, O, Q lie on a line $\mathfrak{L} \neq \mathfrak{T}$ and $R \notin \mathfrak{L}$, there is a unique degenerate conic through p, s, O, Q, R consisting of the lines $\mathfrak{L}$ and

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

Rs. As s tends to p, this pair of lines converges to the pair of lines $\mathfrak{L}$ and pR. (The line pR can coincide with $\mathfrak{T}$.)

If $\mathfrak{L} = \mathfrak{T}$ and $R \notin \mathfrak{T}$, the conic through p, s, O, Q, R is degenerate. It is unique if $s \notin \mathfrak{T}$ and converges to the pair of lines $\mathfrak{T}$ and pR as s tends to p. If, however, there is a sequence of points $s \in A \cap \mathfrak{T}$, $s \to p$, then the conic through s, p, O, Q, R is not uniquely defined. Thus, a unique tangent conic through O, Q, R is not defined in this case. Any accumulation conic consists of $\mathfrak{T}$ and a line through R.

4.7. If O, Q, R lie on a line $\mathfrak{L}$, $p \notin \mathfrak{L}$, then there is a unique (degenerate) conic through the five points p, s, O, Q, R, viz., $\mathfrak{L}$ and the line ps. As s converges to p this conic tends to the degenerate conic $\mathfrak{L} \cup \mathfrak{T}$.

4.8. If p, O, Q, R lie on a line $\mathfrak{L}$, and $s \notin \mathfrak{L}$ in case $\mathfrak{L} = \mathfrak{T}$, there are infinitely many degenerate conics through p, s, O, Q, R, consisting of $\mathfrak{L}$ and any line through s. As $s \to p$, any limit conic is degenerate and consists of the line $\mathfrak{L}$ and any line through p. Any such pair of lines will still be defined as a tangent conic, although, in this case, a unique tangent conic through O, Q, R is not defined.

If $\mathfrak{L} = \mathfrak{T}$, and there is a sequence of points $s \in A \cap \mathfrak{T}$, $s \to p$, again the conic through p, s, O, Q, R is not uniquely defined. The accumulation conics are: any pair of lines one of which is $\mathfrak{T}$; the double line on $\mathfrak{T}$; and any double segment containing p, O, Q, R.

4.9. Let p, O, Q, R be distinct points on a line $\mathfrak{L}$. Let O_n, Q_n, R_n converge to O, Q, and R respectively. If no three of the points p, O_n, Q_n, R_n are collinear, there is a unique conic γ_n of $\bar{\tau}$ through these points; cf. the final remarks in 4.4 and 4.5. It can happen that $\{\gamma_n\}$ converges to a double segment on $\mathfrak{L}$ as $O_n \to O$, $Q_n \to Q$, $R_n \to R$.

It is convenient to define these double segments as tangent conics of A at p, even though they cannot always be obtained as the limits of conics through s and the fixed points p, O, Q, R; cf. 4.8.

Thus, if p, O, Q, R are mutually distinct points, any accumulation conic of the tangent conics of A at p, through O_n, Q_n, R_n, is always a tangent conic of A at p through O, Q, R.

4.10. Non-tangent conics. Suppose that p is an interior point of the arc A. Then p is called a *point of support* [*intersection*] with respect to a non-degenerate conic γ if a sufficiently small neighbourhood of p on A is decomposed by p into two one-sided neighbourhoods which lie in the same region [in different regions] bounded by γ. The conic γ is then called a *supporting* [an *intersecting*] conic of A at p. It can happen that A neither supports nor intersects γ at p.

Let $\mathfrak{L}$ be a line through p. Choose any line $\mathfrak{M}$; $p \notin \mathfrak{M}$. Then, by using the

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

degenerate conic $\gamma = \mathfrak{L} \cup \mathfrak{M}$, the support and intersection of A at p by $\mathfrak{L}$ can be defined, as above. It is independent of $\mathfrak{M}$.

4.11. *Let p be an interior point of the arc A.* Suppose that A satisfies Condition I at p. It is well known that A has the following property; for a proof, cf. (**2**, Lemma 10).

LEMMA 8. *The lines $\neq \mathfrak{T}$ through p either all support A at p or they all intersect.*

Using the method of (**2**, Lemma 11) we can verify the next lemma.

LEMMA 9. *If a non-tangent line $\mathfrak{L}$ through p intersects [supports] A at p, then every non-degenerate conic γ which touches $\mathfrak{L}$ at p also intersects [supports] A at p.*

Lemmas 8 and 9 imply the following result.

THEOREM 2. *The non-degenerate, non-tangent conics of A through p all intersect A at p or all of them support.*

4.12. Let $\{\gamma(s)\}$ and $\{\gamma'(s)\}$ be two sequences of conics of τ through s which converge to γ and γ' respectively as s tends to p. Here, we let s range through a certain sequence of points.

LEMMA 10. *γ and γ' do not intersect at two distinct points outside p.*

Proof. Suppose that γ and γ' intersect at Q and R; $Q \neq R$, $Q \neq p$, $R \neq p$. By Lemma 1, $\gamma(s)$ and $\gamma'(s)$ will intersect at two points close to Q and R. Since $\gamma(s)$ and $\gamma'(s)$ belong to τ and also meet at s close to p, they coincide, by Lemma 7.

LEMMA 11. *If γ and γ' belong to τ and intersect at a point $Q \neq p$, then $\gamma' \in \phi(\gamma)$.*

Proof. Since γ and γ' cannot support or intersect at another point $\neq p, Q$, they also intersect at p. By 3.5, $\gamma' \in \phi(\gamma)$.

LEMMA 12. *If $\gamma \in \tau$, $\mathfrak{L} \cap \gamma = \emptyset$, $\gamma'' = \mathfrak{L} \cup \mathfrak{T}$, $\gamma \subset \gamma''_* \cup p$, then $s \subset \gamma''_* \cup p$.*

Proof. Since $\gamma \subset \gamma''_* \cup p$, we conclude that $\gamma(s) \subset \gamma''_* \cup p$ and hence $s \subset \gamma''_* \cup p$.

COROLLARY. *If γ and γ' belong to τ and do not intersect outside p, then $\gamma' \subset \gamma_* \cup \gamma$ or $\gamma \subset \gamma'_* \cup \gamma'$.*

LEMMA 13. *If γ and γ' belong to τ, then $\gamma' \in \phi(\gamma)$. In particular, γ and γ' do not support outside p.*

Proof. On account of Lemmas 10 and 11, we assume that γ and γ' do not

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

intersect outside p. By the preceding corollary, we may assume, for example, that $\gamma' \subset \gamma_* \cup \gamma$. Choose $Q \in \gamma'_*$, $R \in \gamma^*$; p, Q, R not collinear. Let γ'' be any accumulation conic of $\{\gamma(\tau; s, Q, R)\}$. Thus γ'' intersects γ and γ'. We restrict s to a sequence of parameters converging to p such that

$$\gamma'' = \lim \gamma(\tau; s, Q, R).$$

By Lemma 11, $\gamma'' \in \phi(\gamma)$, $\gamma'' \in \phi(\gamma')$. Hence $\phi(\gamma) = \phi(\gamma')$ and $\gamma' \in \phi(\gamma)$.

LEMMA 14. *Let $\gamma \in \tau$. Then $\gamma' = \mathfrak{L} \cup \mathfrak{T}$, $p \notin \mathfrak{L}$ is impossible.*

Proof. Let $\gamma \in \tau$ and suppose that $\gamma' = \mathfrak{L} \cup \mathfrak{T}$, $p \notin \mathfrak{L}$. From Lemma 10, we may assume that $\gamma \subset \gamma'_* \cup \gamma'$. Choose $Q \in \gamma_*$, $R \in \gamma'^*$; p, Q, R not collinear. Let γ'' be any accumulation conic of $\{\gamma(\tau; s, Q, R)\}$. Then γ'' intersects both γ and γ' outside p. We may assume that $\gamma'' = \lim \gamma(\tau; s, Q, R)$. By Lemma 10, γ'' intersects γ exactly once outside p. Thus $\gamma'' \in \tau$. But γ'' intersects γ' exactly once outside p. Hence $\gamma' \in \tau$.

LEMMA 15. *If $\gamma \in \tau$ $[\gamma = \mathfrak{L} \cup \mathfrak{T},\ p \notin \mathfrak{L}]$, then γ' is not a double segment $\not\subset \mathfrak{T}$.*

Proof. Suppose that γ' is a double segment, $\gamma' \not\subset \mathfrak{T}$.

(i) If γ intersects γ' at a point Q, then $\gamma(s)$ and $\gamma'(s)$ will intersect at two points close to Q; cf. Lemma 2. This is excluded by Lemma 7.

(ii) Suppose that $\gamma' \subset \gamma_* \cup \gamma$. Choose a point $R \subset \gamma^*$, and an interior point Q of γ'; p, Q, R not collinear. Let γ'' be an accumulation conic of the sequence $\{\gamma(\tau; s, Q, R)\}$. Thus γ'' intersects γ. We may assume that $\gamma'' = \lim \gamma(\tau; s, Q, R)$. By Lemma 10, if $\gamma \in \tau$ $[\gamma = \mathfrak{L} \cup \mathfrak{T}]$ then $\gamma'' \in \tau$ $[\gamma'' = QR \cup \mathfrak{T}]$, and hence γ'' intersects γ'. This is excluded by Case (i).

5. Osculating conics.

5.1. Let the arc A be differentiable at p. The point p may be either an interior point of A or an end-point.

CONDITION II. *Let Q and R be any fixed points, $Q \notin \mathfrak{T}$, $R \notin \mathfrak{T}$; p, Q, R not collinear. If s is close to p, $s \in A$, $s \neq p$, the unique tangent conic $\gamma(\bar{\tau}; s, Q, R)$ of A at p through Q, R, and s converges as s tends to p.*

The limiting *osculating conic* through p, Q, R will be denoted by $\gamma(\sigma; Q, R)$. The family of all the osculating conics will be denoted by σ.

If Condition II is satisfied, then A is called *twice conically differentiable at p.*

We observe that if A satisfies Condition II and there is a sequence of points $s \in A \cap \mathfrak{T}$, converging to p, then $\gamma(\sigma; Q, R)$ is the pair of lines QR and $\mathfrak{T}$; cf. 4.5.

5.2. Let $Q \notin \mathfrak{T}$, $R \in \mathfrak{T}$, $R \neq p$.

If $s \notin \mathfrak{T}$, there is a unique degenerate conic of $\bar{\tau}$ through s, Q, R, consisting

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

of the pair of lines Qs and $\mathfrak{T}$; cf. 4.5. As s tends to p, this conic converges to the pair of lines pQ and $\mathfrak{T}$.

If there is a sequence of points $s \in A \cap \mathfrak{T}$, converging to p, then the conic of $\bar{\tau}$ through s, Q, R is not uniquely defined; cf. 4.6. Any accumulation conic will consist of $\mathfrak{T}$ and a line through Q.

5.3. If p, Q, R lie on a line $\mathfrak{L} \neq \mathfrak{T}$, there is a unique conic of $\bar{\tau}$ through s, Q, R. It consists of $\mathfrak{L}$ and the line ps; cf. 4.6. As s tends to p, this conic tends to the pair of lines $\mathfrak{L} \cup \mathfrak{T}$.

Let Q and R lie on $\mathfrak{T}$. If $s \notin \mathfrak{T}$, the (unique) degenerate conic of $\bar{\tau}$ through s, Q, R consisting of $\mathfrak{T}$ and the line ps (cf. 4.6) converges to the double line on $\mathfrak{T}$ as s tends to p. On the other hand, if there are points $s \in A \cap \mathfrak{T}$, $s \to p$, then the conic of $\bar{\tau}$ through s, Q, R is no longer uniquely defined; cf. 4.8.

5.4. THEOREM 3. *If Condition II holds for two points Q and R such that p, Q, R are not collinear and $Q, R \notin \mathfrak{T}$, then it holds for every such pair of points.*

Proof. Let p, Q, R and p, Q, R', respectively, be not collinear; $Q, R, R' \notin \mathfrak{T}$. Put $\gamma(s) = \gamma(\tau; s, Q, R)$, $\gamma'(s) = \gamma(\tau; s, Q, R')$. Suppose that $\gamma = \lim \gamma(s)$ exists and let γ' be any accumulation conic of the $\gamma'(s)$, as s tends to p. We restrict s to a sequence of parameters converging to p, such that

$$\gamma' = \lim \gamma'(s).$$

Neither γ nor γ' is a double segment through p, a double line through p, or the point conic p, since none of these is the limit of conics of τ through two fixed points not collinear with p. Thus γ [γ'] is non-degenerate, a pair of lines through p neither of which is $\mathfrak{T}$, or a pair of lines one of which is $\mathfrak{T}$ and the other of which does not pass through p.

By Lemmas 10 and 14, if γ is non-degenerate [a pair of lines through p; a pair of lines one of which is $\mathfrak{T}$], then γ' is also non-degenerate [a pair of lines through p; a pair of lines one of which is $\mathfrak{T}$].

Thus, if γ is a pair of lines through p [a pair of lines one of which is $\mathfrak{T}$], then $\gamma' = pQ \cup pR'$ [$\gamma' = \mathfrak{T} \cup QR'$].

Next, suppose that γ and γ' belong to τ. By Lemma 13, $\gamma' \in \phi(\gamma)$.

Thus, γ' is the unique conic of ϕ through Q and R'; cf. 3.5.

5.5. For the remainder of § 5, we assume that A satisfies Condition II at p.

THEOREM 4. *If the arc A intersects $\mathfrak{T}$ at p, then the osculating conic of A at p through Q and R is degenerate, viz., either the pair of lines pQ and pR, or the pair of lines $\mathfrak{T}$ and QR.*

Proof. Lemma 12.

5.6. THEOREM 5. *If A is twice conically differentiable at p, the set σ of the osculating conics of A at p is one of the following three subsets of $\bar{\tau}$:*

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

(1) *σ is a family ϕ of the type described in* 3.5;

(2) *σ consists of the pairs of distinct lines through p, both of them different from $\mathfrak{T}$;*

(3) *σ consists of the pairs of lines one of which is $\mathfrak{T}$ while the other does not pass through p.*

Proof. Let $\gamma \in \sigma$, $\gamma' \in \sigma$. As in the proof of Theorem 3, we can first show that γ $[\gamma']$ belongs to one of the three classes 1, 2, or 3.

By Lemmas 10 and 14, if γ belongs to class 1 [class 2; class 3] then γ' belongs to class 1 [class 2; class 3].

If γ belongs to class 2 [class 3], choose Q and R on γ; p, Q, R not collinear. Then $\gamma = \gamma(\sigma; Q, R)$. Thus every member of class 2 [class 3] belongs to σ.

From now on, we may assume that $\sigma \subset \tau$. Let $\gamma \in \sigma$. By Lemma 13, every member γ' of σ belongs to $\phi(\gamma)$. Thus $\sigma \subset \phi(\gamma)$.

We finally show that $\phi(\gamma) \subset \sigma$. Let $\gamma' \in \phi = \phi(\gamma)$.

(i) If γ' intersects γ outside p, let $Q \in \gamma' \cap \gamma^*$, $R \in \gamma' \cap \gamma_*$. Then $\gamma(\sigma; Q, R)$ intersects γ at exactly one point outside p. By Lemma 11, $\gamma(\sigma; Q, R)$ is the unique conic of $\phi(\gamma)$ through Q and R, i.e.,

$$\gamma' = \gamma(\sigma; Q, R) \in \sigma.$$

(ii) Let $\gamma' \subset \gamma_* \cup p$, for example. Let $Q \in \gamma^*$, $R \in \gamma'_*$; p, Q, R not collinear. Then the unique conic $\gamma'' \in \phi(\gamma)$ through Q and R belongs to σ, by the above. Since γ' also intersects γ'', and $\gamma'' \in \sigma$, we have $\gamma' \in \sigma$.

We say that the point $p \in A$ is of type i if the family σ of the osculating conics of A at p is of class i; $i = 1, 2, 3$.

The proof of Theorem 5 implies the following lemma.

LEMMA 16. *Two osculating conics of A at p coincide if they have two distinct points in common which are not collinear with p and do not lie on $\mathfrak{T}$.*

5.7. The compactified family $\bar{\sigma}$. We compactify the family σ of type 1 by adding to σ the pairs of lines through p one of which is $\mathfrak{T}$, the double line on $\mathfrak{T}$, the double segments on $\mathfrak{T}$ with one end-point p, and the point conic p.

In the case of type 2, σ is compactified by the addition of all the pairs of lines through p, one of which is $\mathfrak{T}$, and all the double lines through p.

When σ is of type 3, we compactify it by adding the pairs of lines through p, one of which is $\mathfrak{T}$, and the double line coincident with $\mathfrak{T}$.

5.8. *Let p be an end-point of A.*

THEOREM 6. *Let $\psi = \psi(\mathfrak{L}, Q)$ be the pencil of conics of τ which touch a given line $\mathfrak{L}$ at Q, $p \notin \mathfrak{L}$;* cf. 3.4. *Let $\gamma(\bar{\psi}; s)$ be the unique conic of $\bar{\psi}$ through s. If σ is of type* 1 [*type* 2; *type* 3] *and s tends to p, then $\gamma(\bar{\psi}; s)$ converges to an osculating conic of A at p [one of the double segments with the end-points p and Q; the pair of lines $\mathfrak{T} \cup \mathfrak{L}$.*]

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

Proof. (i) Let σ be of type 1 and choose $\gamma \in \sigma$. Let γ' be any accumulation conic of the $\gamma(\psi; s)$. We restrict s to a sequence of parameters converging to p such that $\gamma' = \lim \gamma(\psi; s)$. By Lemma 14, $\gamma' \neq \gamma'' = \mathfrak{L} \cup \mathfrak{T}$ and, by Lemma 15, γ' is not a double segment. Hence $\gamma' \in \tau$. By Lemma 13, $\gamma' \in \phi(\gamma)$. Hence $\gamma' \in \sigma$.

(ii) If σ is of type 2, we apply Lemma 10 as above, to show that $\gamma' \notin \tau$ and $\gamma' \neq \gamma''$. Hence γ' must be a double segment. By the remark at the end of 5.1, we may assume that $A - p \subset \gamma''_*$. Thus $\gamma' \subset \gamma''_* \cup p \cup Q$.

(iii) If σ is of type 3, we apply Lemma 15, as in case (i), to show that γ' is not a double segment, and we use Lemma 14 to verify that $\gamma' \notin \tau$. Hence $\gamma' = \gamma''$ in this case.

5.9. We continue the previous discussion. Let σ be of type 2. Let A intersect $\mathfrak{T}$ at the interior point p. Thus p decomposes a neighbourhood of p on A into two disjoint neighbourhoods, N and N', say. As s tends to p on N, $\gamma(\psi; s)$ converges to one of the double segments with the end-points p and Q, while if $s' \to p$, $s' \in N'$, $\lim \gamma(\psi; s)$ is the other double segment with the end-points p and Q; cf. 3.4.

5.10. LEMMA 17. *If p is an interior point of A of type* 1, *then the conics of $\tau - \lambda$ all support A at p.*

Proof. Let $\gamma \in \tau$. Suppose that γ neither intersects nor supports A at p. Then there is a sequence of points $s \in (A - p) \cap \gamma$, $s \to p$. Let p, Q, R be distinct points on γ. Then $\gamma = \gamma(\tau; s, Q, R)$ for each s. Let $s \to p$. From Condition II, $\gamma \in \sigma$. Hence each γ of $\tau - \sigma$ either intersects or supports A at p.

Suppose that $\gamma \in \tau$ intersects A at p. Let p, Q, R be distinct points on γ. Let $\gamma' = \lim \gamma(\tau; s, Q, R) = \gamma(\sigma; Q, R)$ as $s \to p$. By Theorem 4, $\mathfrak{T}$ supports A at p. Let γ'' be the pair of lines $\mathfrak{T}$ and QR. We define γ''_* to be the region defined by γ'' which contains the points s. If s lies in γ_*, then

$$\gamma(\tau; s, Q, R) \subset (\gamma^* \cap \gamma''^*) \cup (\gamma_* \cap \gamma''_*) \cup \{p, Q, R\}$$

and hence γ' lies in the closure of this set. Similarly, letting $s \to p$ on A through γ^* we conclude that γ' lies in the closure of

$$(\gamma^* \cap \gamma''_*) \cup (\gamma_* \cap \gamma''^*) \cup \{p, Q, R\}.$$

Hence $\gamma' = \gamma$, and $\gamma \in \sigma$.

LEMMA 18. *Let p be an interior point of A of type* 2. *If A supports $\mathfrak{T}$ at p [intersects $\mathfrak{T}$ at p], then each conic γ of $\mathfrak{T}$ supports [intersects] A at p.*

LEMMA 19. *If p is an interior point of A of type* 3, *then each conic γ of τ supports A at p.*

The proofs of Lemmas 18 and 19 are similar to that of Lemma 17.

The preceding Lemmas are combined in the following theorem.

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

THEOREM 7. *The conics of $\tau - \sigma$ all support A at p, except when p is of type 2 and A intersects $\mathfrak{T}$ at p, in which case they all intersect A at p.*

Remark. Let p be an end-point of A of type 2 or type 3. Let $\gamma \in \tau$ and let p, Q, R be distinct points on γ. Let γ' be the pair of lines pQ and pR and let γ'' be the pair QR and T. If γ and A lie in the set

$$(\gamma'_* \cap \gamma''^*) \cup (\gamma'^* \cap \gamma''_*) \cup \{p, Q, R\}$$

as in 3.3 and $\gamma \cap A = p$, then A lies in $\gamma_* \cup p$ or in $\gamma^* \cup p$ according as p is of type 2 or type 3.

5.11. Let the arc A be twice conically differentiable at p.

Let ϕ be a subfamily of τ of the kind described in 3.5. Thus, if p, Q, R are not collinear, $Q \notin \mathfrak{T}$, $R \notin \mathfrak{T}$, then $\gamma(\phi; Q, R)$ denotes the conic of ϕ through Q and R.

THEOREM 8. *If $\phi \neq \sigma$ and $R \notin \mathfrak{T}$, then as $s \to p$, $s \notin \mathfrak{T}$,*

$$\lim \gamma(\phi; s, R) = \mathfrak{T} \cup pR; \text{ cf. } 3.6.$$

Proof. Suppose that γ is a non-degenerate accumulation conic of $\gamma(\phi; s, R)$. Thus $\gamma \notin \sigma$. We may assume that $\gamma = \lim \gamma(\phi; s, R)$. Let $Q \in \gamma$, $Q \neq R, p$. Then γ and $\gamma(\sigma; Q, R)$ intersect at Q and R. By Lemma 10, this is impossible.

Remark. If there is a sequence of points $s \in A \cap \mathfrak{T}$, converging to p, then $\gamma(\bar{\phi}; s, R) = \mathfrak{T} \cup pR$ still converges to $\mathfrak{T} \cup pR$.

5.12. Let p be of type 1. Let $\{\gamma(s)\}$ and $\{\gamma'(s)\}$ be sequences of conics of σ converging to γ and γ' respectively as s tends to p.

LEMMA 20. *γ and γ' do not intersect at a point outside $\mathfrak{T}$.*

Proof. Suppose that γ and γ' intersect at a point $Q \notin \mathfrak{T}$. Then, $\gamma(s)$ and $\gamma'(s)$ belong to σ, pass through the point s near p and, by Lemma 1, they intersect near Q. This is excluded by Lemma 16.

6. Superosculating conics.

6.1. Triple differentiability. Let A be twice conically differentiable at p. From now on, we shall assume that p is of type 1. In particular, by Theorem 4, A supports $\mathfrak{T}$ at p, and, by 5.1, $s \notin \mathfrak{T}$. The point p will be called *three times conically differentiable* if, in addition, the following condition is satisfied.

CONDITION III. *If $Q \notin \mathfrak{T}$, then $\gamma(\sigma; s, Q)$ converges as $s \to p$ on A.*

The limit conic will be called the *superosculating conic* of A at p through Q. It will be denoted by $\gamma(\rho; Q$. The family of all the superosculating conics of A at p will be denoted by ρ.

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

6.2. If $Q \in \mathfrak{T}$, $Q \neq p$, $s \notin \mathfrak{T}$, the unique conic of $\bar{\sigma}$ through Q and s consists of $\mathfrak{T}$ and ps. It converges to the double line on $\mathfrak{T}$ if s tends to p; cf. 5.2.

If $Q \in \mathfrak{T}$, $Q \neq p$, and there are points $s \in A \cap \mathfrak{T}$ converging to p, then the conic of $\bar{\sigma}$ through Q and s is not uniquely defined.

6.3. THEOREM 9. *If Condition III holds for a single point $Q \notin \mathfrak{T}$, then it holds for all such points.*

Proof. Let $Q, R \notin \mathfrak{T}$. Put $\gamma(s) = \gamma(\sigma; s, Q)$, $\gamma'(s) = \gamma(\sigma; s, R)$. Suppose that $\gamma = \lim \gamma(s)$ exists as s tends to p and let γ' be any accumulation conic of the $\gamma'(s)$ as s tends to p. We restrict s to a sequence of parameters converging to p, such that $\gamma' = \lim \gamma'(s)$.

Neither γ nor γ' is the double line on $\mathfrak{T}$, a double segment on $\mathfrak{T}$ with one end-point p, or the point conic p, since none of these is the limit of conics of σ through a fixed point outside $\mathfrak{T}$. Thus γ $[\gamma']$ is either non-degenerate, or a pair of lines through p one of which is $\mathfrak{T}$.

If one of γ and γ' is non-degenerate and the other is a pair of lines through p, one of which is $\mathfrak{T}$, then they intersect at a point outside $\mathfrak{T}$. This is excluded by Lemma 20.

Thus, if $\gamma = pQ \cup \mathfrak{T}$, then $\gamma' = pR \cup \mathfrak{T}$.

If $\gamma \in \sigma$, then γ' also belongs to σ and by Lemma 20, $\gamma \cap \gamma' = p$. Thus γ' is the unique conic of $\phi_p(\gamma)$ through R; cf. 3.7.

6.4. THEOREM 10. *If A is three times conically differentiable at p, then the family ρ of superosculating conics of A at p is one of the following subsets of $\bar{\sigma}$:*

Type 1(a). *ρ is a subfamily ϕ_p of σ of the kind described in* 3.7.

Type 1(b). *ρ consists of the pairs of lines through p one of which is $\mathfrak{T}$.*

Proof. Let $\gamma \in \rho$, $\gamma' \in \rho$. As in the proof of Theorem 9, we can first show that γ and γ' belong to the same one of the two classes 1(a) and 1(b) of Theorem 10.

Let γ be any conic of the class 1(b). Choose $Q \in \gamma$, $Q \notin \mathfrak{T}$. Then

$$\gamma = pQ \cup \mathfrak{T} = \gamma(\rho; Q) \in \rho.$$

Let γ be any conic of the class 1(a), say $\gamma \in \phi_p(\gamma_0)$, $\gamma_0 \in \rho$. Thus $\gamma \in \sigma$. Choose $Q \in \gamma$, $Q \neq p$. If $\gamma \notin \rho$, then the conics γ and $\gamma(\rho; Q)$ must intersect at Q. Hence $\gamma(\rho; Q)$ intersects γ_0. This is excluded by Lemma 20.

Theorem 10 implies the following result.

LEMMA 21. *Two superosculating conics of A at p which have a point $Q \notin \mathfrak{T}$ in common coincide.*

We compactify the family ρ of type 1(a) by the addition of the point conic p and the double line on $\mathfrak{T}$, as in 3.7. The double line on $\mathfrak{T}$ is added to compactify the family ρ of type 1(b). The compactified family ρ will be denoted by $\bar{\rho}$.

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

The example $y = x^2 + x^3 \sin(1/x)$ shows that Condition II does not imply Condition III.

6.5. THEOREM 11. *If A is three times conically differentiable at the interior point p, then the conics of $\sigma - \rho$ all support A at p or all of them intersect. If p is of type* 1(a), *they all support or all intersect according as A has or has not a cusp at p.*

Proof. Case (1): p is of type 1(a). It can first be shown by the method of Lemma 17 that each conic of $\sigma - \rho$ either supports or intersects A at p.

Suppose that A has no cusp [A has a cusp] at p.

Let γ be a conic of σ which supports [intersects] A at p. Let $Q \in \gamma$. Let γ' be the pair of lines $\mathfrak{T} \cup pQ$. If, for example, $s \subset \gamma_* \cap \gamma'_*$, then

$$\gamma(\sigma; s, Q) \subset (\gamma_* \cap \gamma'_*) \cup (\gamma^* \cap \gamma'^*) \cup \{p, Q\}.$$

Hence $\gamma(\rho; Q)$ lies in the closure of this set. Letting $s \to p$ through $\gamma_* \cap \gamma'^*$ [$\gamma^* \cap \gamma'_*$] we conclude that $\gamma(\rho; Q)$ lies in the closure of

$$(\gamma_* \cap \gamma'^*) \cup (\gamma^* \cap \gamma'_*) \cup \{p, Q\}.$$

Since $\gamma(\rho; Q) \neq \gamma'$, we conclude that $\gamma = \gamma(\rho; Q) \in \rho$.

Case 2: p is of type 1(b). Let γ and γ' be two conics of σ such that, for example, γ supports A at p and γ' intersects. Suppose that a small one-sided neighbourhood N of p on $A - p$ lies in $\gamma_* \cap \gamma'^*$.

(i) Let γ and γ' intersect at p and Q. If $s \in N$,

$$\gamma(\sigma; s, Q) \subset (\gamma_* \cap \gamma'^*) \cup (\gamma^* \cap \gamma'_*) \cup \{p, Q\}.$$

Hence $\gamma(\rho; Q)$ lies in the closure of this set. Since $\gamma(\rho; Q)$ is degenerate, this is impossible.

Symmetrically, no point s lies in $\gamma^* \cap \gamma'_*$. It follows that γ and γ' both support A at p, or both of them intersect.

(ii) Let γ and γ' have no point in common outside p. Let γ'' be a conic of σ which intersects both γ and γ'. By (i), γ and γ'', and also γ' and γ'', both intersect A at p or both support. Hence the same holds for γ and γ'.

The proof of Theorem 11 yields the following corollary.

COROLLARY. *If p is of type* 1(b) *and $\gamma \in \sigma$, $\gamma' \in \sigma$, then*

$$s \notin (\gamma_* \cap \gamma'^*) \cup (\gamma^* \cap \gamma'_*),$$

if s is sufficiently close to p.

6.6. *Let the arc A be three times conically differentiable at its end-point p.*

Let ϕ_p be a one-parameter family of osculating conics of the type described in 3.7. Thus, $\gamma(\phi_p; s)$ denotes the conic of ϕ_p through s.

THEOREM 12. *If $\phi_p \neq \rho$, then*

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

$$\lim_{s\to p} \gamma(\phi_p; s)$$

exists. It is either the double line on $\mathfrak{T}$ *or the point conic* p.

Proof. Suppose that γ is a non-degenerate accumulation conic of $\gamma(\phi_p; s)$. We restrict s to a sequence of parameters converging to p such that

$$\lim \gamma(\phi_p; s) = \gamma.$$

Let $Q \in \gamma$, $Q \neq p$. Since $\phi_p \neq \rho$, we have $\gamma \in \rho$. Thus γ and $\gamma(\rho; Q)$ intersect at Q. This is excluded by Lemma 20.

From the above, a small neighbourhood N of p on $A - p$ cannot lie between two conics of ϕ_p. Thus, if N lies inside [outside] any member of ϕ_p,

$$\lim_{s\to p} \gamma(\phi_p; s)$$

is the point conic p [the double line on $\mathfrak{T}$].

6.7. We continue the previous discussion.

If p is of type 1(b), then the Corollary of Theorem 11 implies that $\lim \gamma(\phi_p; s)$ is independent of the choice of $\phi_p \subset \sigma$, and depends only on the arc A.

Let p be of type 1(a). Choose $R \notin \mathfrak{T}$. Then, there is a 1–1 correspondence between the conics of ϕ_R and the families $\phi_p \subset \sigma$. Let $\gamma' = \gamma(\rho; R)$, $\gamma'' = pR \cup \mathfrak{T}$. Thus γ' and γ'' decompose the family $\phi_R - \gamma'$ into two sub-families. Let N be a sufficiently small neighbourhood of p on $A - p$; $N \subset \gamma''_*$, say. Let $\gamma \in \phi_R - \gamma'$. If

$$\gamma \subset (\gamma'_* \cap \gamma''^*) \cup (\gamma'^* \cap \gamma''_*) \cup \{p, R\},$$

then $N \subset \gamma_*$. Thus,

$$\lim_{s\to p} \gamma\{\phi_p(\gamma); s\}$$

is the point conic p. If, however,

$$\gamma \subset (\gamma'_* \cap \gamma''_*) \cup (\gamma'^* \cap \gamma''^*) \cup \{p, R\},$$

then $N \subset \gamma^*$. Thus, in this case,

$$\lim_{s\to p} \gamma\{\phi_p(\gamma); s\}$$

is the double line on $\mathfrak{T}$.

These remarks are independent of the choice of $R \notin \mathfrak{T}$.

We observe that if $\mathfrak{L}$ is the tangent of γ' at R, then the conics γ of ϕ_R through $\gamma'^* \cap \gamma''_*$ [$\gamma'_* \cap \gamma''_*$], and thus the associated families $\phi_p(\gamma)$, are in 1–1 correspondence with the points $(\gamma \cap \mathfrak{L}) - R$ on $\mathfrak{L} \cap \gamma''_*$ [$\mathfrak{L} \cap \gamma''^*$].

7. Conically differentiable points.

7.1. From now on, we shall assume that A is three times conically differentiable at p and p is of type 1(a).

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

We define p to be *conically differentiable* if it satisfies, in addition to the Conditions I, II, and III, the following one.

CONDITION IV. *The superosculating conic $\gamma(\rho; s)$ converges as s tends to p.*

The limit conic is called the *ultraosculating conic* of A at p. We denote it by $\gamma(p)$. By 6.4, $\gamma(p)$ is non-degenerate, or the point conic p, or the double line on $\mathfrak{T}$.

7.2. Let the arc A be conically differentiable at an interior point p.

THEOREM 13. *The conics of $\rho - \gamma(p)$ all support A at p.*

Proof. As in Lemma 17, it can be readily verified that every conic of $\rho - \gamma(p)$ either supports or intersects A at p.

Suppose that $\gamma \in \rho$ intersects A at p. Let M be a small neighbourhood of p on A which does not meet γ outside p; $M = N \cup p \cup N'$. We may assume that N and N' lie in γ_* and γ^* respectively. Thus, if $s \in N$, $\gamma(\rho; s)$ lies in $\gamma_* \cup p$. Letting s tend to p, we conclude that $\gamma(p)$ lies in the closed region $\gamma_* \cup \gamma$. Replacing N by N' we see that $\gamma(p) \subset \gamma^* \cup \gamma$. From Condition IV, $\gamma = \gamma(p)$.

COROLLARY. *If $\gamma(p) = p$ [$\gamma(p)$ is the double line on $\mathfrak{T}$] and $\gamma \in \rho$, then a small neighbourhood of p on A lies in $\gamma^* \cup p$ [$\gamma^* \cup p$].*

7.3. The various types of differentiable points are illustrated by the following examples ($0 \leqslant s \leqslant 1$). The relevant point is the origin given by $s = 0$; m and n are positive integers.

Type 1. $\quad x = s^n, \quad y = s^{2n} + ks^{2n+m}.$

Type 1(a)(i) $\quad$ [$\gamma(p)$ is non-degenerate], $\quad m = n$ or $m \geqslant 2n$.

Type 1(a)(ii) $\quad$ [$\gamma(p) = p$], $\quad n < m < 2n,\ k > 0$.

Type 1(a)(iii) $\quad$ [$\gamma(p)$ is the double line on $\mathfrak{T}$], $\quad n < m < 2n,\ k < 0$.

Type 1(b) $\quad m < n$.

Type 2. $\quad x = s^n, \quad y = s^m, \quad n < m < 2n$.

Type 3. $\quad x = s^n, \quad y = s^m, \quad 2n < m$.

In these examples, the tangent conics have the same family of equations

$$ax^2 + bxy + cy^2 + dy = 0.$$

The osculating conics have the equations

$$a(x^2 - y) + bxy + cy^2 = 0 \qquad \text{(type 1)},$$

$$ax^2 + bxy + cy^2 = 0 \qquad \text{(type 2)},$$

$$bxy + cy^2 + dy = 0 \qquad \text{(type 3)}.$$

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press

In type 1(a), the superosculating conics have the equations

$$a(x^2 - y + kxy) + cy^2 = 0 \qquad (m = n),$$

$$a(x^2 - y) + cy^2 = 0 \qquad (m > n).$$

In type 1(b), the superosculating conics have the equations

$$bxy + cy^2 = 0.$$

In type 1(a)(i), $\gamma(p)$ has the equation

$$x^2 - y + kxy - k^2y^2 = 0 \qquad (m = n),$$

$$x^2 - y + ky^2 = 0 \qquad (m = 2n),$$

$$x^2 - y = 0 \qquad (m > 2n).$$

REFERENCES

1. G. Bol, *Projektive Differentialgeometrie* (Göttingen, 1954).
2. N. D. Lane, *Parabolic differentiation*, Can. J. Math., *15* (1963), 546–562.

McMaster University, and
University of Toronto

https://doi.org/10.4153/CJM-1964-018-1 Published online by Cambridge University Press