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Abstract

In this paper we address the issue of existence of cusp forms by using an extension and refinement of a classic method involving (adelic) compactly supported Poincaré series. As a consequence of our adelic approach, we also deal with cusp forms for congruence subgroups.

1. Introduction

The existence and construction of cusp forms is a fundamental problem in the modern theory of automorphic forms; see [Art05, Gol06, LS, Mul08, Sel56]. In this paper we address the issue of existence of cusp forms by using an extension and refinement of a classic method involving (adelic) compactly supported Poincaré series [Hen84, Sha90, Vig86]. Our approach is based on spectral decomposition of compactly supported Poincaré series. This method has been successfully applied to the case of a cocompact discrete subgroup of a semisimple Lie group [Mui08], yielding some quantitative information on the decomposition of the corresponding L^2-space. The main result of this paper develops this idea further (see Theorem 7.2(iv)) using adelic language.

This is not the only application of our Theorem 7.2. Another application that we have in mind is the one with which we began this introduction. To explain it, let us first introduce some notation.

Let G be a semisimple algebraic group defined over a number field k. We write V_f (respectively, V_∞) for the set of finite (respectively, archimedean) places. For $v \in V_\infty \cup V_f$, we write k_v for the completion of k at v; if $v \in V_f$, then we let $\mathcal{O}_v$ be the ring of integers of k_v. Let $G_\infty = \prod_{v \in V_\infty} G(k_v)$. This is a semisimple Lie group with finite center; let K_∞ and $\mathfrak{g}_\infty$ be a maximal compact subgroup and the (real) Lie algebra of G_∞, respectively. Let $G(\mathbb{A}_f)$ be the restricted product of all $G(k_v)$ for $v \in V_f$. Let $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ be the space of K_∞-finite cusp forms for $G(\mathbb{A})$ (see [BJ79] or § 2). This is a $((\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f))$-module. In particular, it is a smooth $G(k_v)$-module for $v \in V_f$. This fact enables us to apply Bernstein's theory and decompose $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ according to the Bernstein classes $\mathfrak{M}_v$ (see § 5):

$$\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A})) = \bigoplus_{\mathfrak{M}_v} \mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))(\mathfrak{M}_v).$$

If $\mathfrak{M}_v$ is a Bernstein class of (M_v, ρ_v), where M_v is a Levi subgroup of $G(k_v)$ and ρ_v is an irreducible supercuspidal representation of M_v, then, by definition, $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))(\mathfrak{M}_v)$ is the largest $G(k_v)$-submodule of $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ such that every irreducible subquotient of it is a subquotient of $\mathrm{Ind}_{P_v}^{G(k_v)}(\chi_v \rho_v)$, for some unramified character χ_v of M_v. Here P_v is an

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arbitrary parabolic subgroup of $G(k_v)$ containing M_v as a Levi subgroup. Obviously, this is also a $((\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f))$-module decomposition. Further, we can iterate this for v ranging over a finite set of places, and as a result we arrive at the question of non-triviality of a $((\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f))$-module $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))(\mathfrak{M}_v; v \in T)$, where $T \subset V_f$ is a finite and non-empty set of places. The following theorem gives rather precise information on the structure of $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))(\mathfrak{M}_v; v \in T)$. The present nice formulation was suggested by the referee.

Theorem 1.1. *Let T be a finite set of places of k such that G is unramified over k_v for $v \in V_f - T$. For $v \in T$, let $\mathfrak{M}_v$ be a Bernstein class of $G(k_v)$ determined by (M_v, ρ_v). We define $\mathfrak{P}$ to be the set of all k-parabolic subgroups P such that a Levi factor of $P(k_v)$ contains a $G(k_v)$-conjugate of M_v for all $v \in T$. Then we have the following.*

(i) $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))(\mathfrak{M}_v; v \in T) \neq 0$.

(ii) *Assume that*

$$\mathfrak{P} = \{G\}. \tag{1.2}$$

Then, for a sufficiently small open-compact subgroup $L \subset G(\mathbb{A}_f)$ of the form

$$L = \prod_{v \in T} L_v \times \prod_{v \in V_f - T} G(\mathcal{O}_v),$$

there exist infinitely many K_∞-types δ which depend on L such that a $((\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f))$-module $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))(\mathfrak{M}_v; v \in T)$ contains infinitely many irreducible representations of the form $\pi^j_\infty \otimes_{v \in V_f} \pi^j_v$. Here, π^j_v is unramified for $v \in V_f - T$, belongs to the class $\mathfrak{M}_v$ and contains a non-trivial vector invariant under L_v for $v \in T$, while the irreducible unitarizable $(\mathfrak{g}_\infty, K_\infty)$-module π^j_∞ contains δ; the set of equivalence classes $\{\pi^j_\infty\}$ is infinite.

We remark that an optimal choice of L_v may, in future, be obtained along the lines of [BK98, MP96], but one can give an easy description of L_v that works here. More precisely, if (M_v, ρ_v) is the local Bernstein data, then we can take any L_v as long as it has an Iwahori decomposition

$$L_v = L_v^- L_v^0 L_v^+$$

and ρ_v has a non-zero $L_v^0 = M_v \cap L_v$-fixed vector. This follows from [Cas, Proposition 3.3.6].

The assumption (1.2) is satisfied if for at least one element $v \in T$ the Bernstein class $\mathfrak{M}_v$ satisfies $M_v = G(k_v)$, or if $G(k)\backslash G(\mathbb{A})$ is compact. It is also satisfied in a significant number of other cases. The following example was proposed by the referee. Let G be the k-split Sp_{2n}. We select two places $v_1, v_2 \in V_f$ and let $M_{v_1} = \mathrm{GL}_n(k_v)$ and $M_{v_2} = \mathrm{GL}_1(k_v)^{n-1} \times \mathrm{SL}_2(k_v)$. Then $\mathfrak{P} = \{\mathrm{Sp}_{2n}\}$ since $M_{v_1} = \mathrm{GL}_n(k_v)$ does not contain a long root of SL_2 which is contained in M_{v_2}.

Theorem 1.1(ii) is a direct consequence of the spectral decomposition of adelic compactly supported Poincaré series (see Theorem 7.2) together with the cuspidality criterion given by Proposition 5.3. Theorem 1.1(i) follows from Theorem 1.1(ii) upon enlarging T by a place $w \in V_f$ and taking some Bernstein class $\mathfrak{M}_w$ of $G(k_w)$ determined by a pair of the form $(G(k_w), \rho_w)$, where ρ_w is an arbitrary supercuspidal representation of $G(k_w)$.

Now we outline the content of the paper. Let $f \in C_c^\infty(G(\mathbb{A}))$. Then the adelic compactly supported Poincaré series $P(f)$ is defined as follows:

$$P(f)(g) = \sum_{\gamma \in G(k)} f(\gamma \cdot g).$$

It is well-known [God66] that the right-regular representation of $G(\mathbb{A})$ on $L^2_{\text{cusp}}(G(k)\backslash G(\mathbb{A}))$ can be decomposed into a countable direct sum of irreducible $G(\mathbb{A})$-invariant subspaces:

$$L^2_{\text{cusp}}(G(k)\backslash G(\mathbb{A})) = \bigoplus_j \mathfrak{H}_j.$$

We then define *the cuspidal spectral decomposition of* $P(f)$ as follows:

$$\text{the orthogonal projection of } P(f) \text{ to } L^2_{\text{cusp}}(G(k)\backslash G(\mathbb{A})) = \sum_j \psi_j \quad \text{with } \psi_j \in \mathfrak{H}^j.$$

In order to make this concept useful, we employ the following approach. We fix an arbitrary function $\bigotimes_{v\in V_f} f_v \in C_c^\infty(G(\mathbb{A}_f))$ which does not vanish at 1, and we select an open-compact group $L \subset G(\mathbb{A}_f)$ such that this function is right-invariant under L. Then, in § 4 we study possible K_∞-types δ which appear in $L^2(K_\infty \cap \Gamma_L\backslash K_\infty)$ and $f_\infty \in C_c^\infty(G_\infty)$ such that the following hold.

(a) The Poincaré series $P(f)$ and its restriction to G_∞ are non-trivial, where $f \overset{\text{def}}{=} f_\infty \otimes_{v\in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$.

(b) $P(f)$ is right-invariant under L and transforms according to δ on the right.

(c) The support of $P(f)|_{G_\infty}$ is contained in a set of the form $\Gamma_L \cdot C$, where C is a compact set that is right-invariant under K_∞ and $\Gamma_L \cdot C$ is not the whole of G_∞.[1]

The precise description of the K_∞-types is given by Theorem 4.2. The requirement that δ belong to $L^2(K_\infty \cap \Gamma_L\backslash K_\infty)$ is explained in [Mui08, Theorem 3-1]. This is necessary in order to apply a non-vanishing criterion from [Mui08, § 3]. We remark that $P(f)$ has a fairly large support because of (b); hence, its non-vanishing is difficult to ensure. The condition (c) is fundamental in establishing that the number of cusp forms in Theorem 1.1 is infinite. This is done in the main result of § 7; see Theorem 7.2(iv). It is based on a principle explained in [Mui08, § 4].

To make the results of § 4 useful, in § 5 we apply Bernstein's theory to the right-regular smooth representation of $C_c^\infty(G(k_v))$, for each finite place $v \in V_f$. The main results of that section are the principle of local cuspidality along a parabolic subgroup (Lemma 5.1) and the non-triviality of Bernstein components for the right-regular smooth representation of $C_c^\infty(G(k_v))$ (Lemma 5.2). The global consequence of cuspidality of Poincaré series is discussed in Proposition 5.3. In § 6, we show that an analogous theory does not exist in the archimedean case (see Proposition 6.1). Finally, in § 7, Theorem 7.2, we explain the spectral decomposition of cuspidal Poincaré series constructed in Theorem 4.2.

We believe that, when combined with the p-adic theory of types (see [BK98, MP96]), the main results of this paper will be even more useful in the construction of cuspidal automorphic representations. Some of this work is pursued in [Mui].

We remark that completely different adelic Poincaré series were studied in [Mui09]; there we established their cuspidality and non-vanishing properties.

2. Preliminary results

In this section we fix the notation to be used in this paper. Let G be a semisimple algebraic group defined over a number field k. We write V_f (respectively, V_∞) for the set of finite

[1] We remind the reader that to any open-compact subgroup $L \subset G(\mathbb{A}_f)$ we can attach a congruence subgroup $\Gamma_L \subset G_\infty$ (see § 2).

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(respectively, archimedean) places. For $v \in V := V_\infty \cup V_f$, we write k_v for the completion of k at v. If $v \in V_f$, we let $\mathcal{O}_v$ denote the ring of integers of k_v. Let $\mathbb{A}$ be the ring of adeles of k. For almost all places of k, G is defined over $\mathcal{O}_v$. The group of adelic points $G(\mathbb{A}) = \prod'_v G(k_v)$ is a restricted product over all places of k of the groups $G(k_v)$: $g = (g_v)_{v \in V} \in G(\mathbb{A})$ if and only if $g_v \in G(\mathcal{O}_v)$ for almost all v. Note that $G(\mathbb{A})$ is a locally compact group and $G(k)$ is embedded diagonally as a discrete subgroup of $G(\mathbb{A})$.

For a finite subset $S \subset V$, we let

$$G_S = \prod_{v \in S} G(k_v).$$

If, in addition, S contains all archimedean places V_∞, we let $G^S = \prod'_{v \notin S} G(k_v)$. Then

$$G(\mathbb{A}) = G_S \times G^S. \tag{2.1}$$

We let $G_\infty = G_{V_\infty}$ and $G(\mathbb{A}_f) = G^{V_\infty}$.

The group G_∞ is a semisimple Lie group. It may not be connected, but it has a finite center. The group $G(\mathbb{A}_f)$ is a totally disconnected group. Let $K_\infty \subset G_\infty$ be a maximal compact subgroup. Let $\mathfrak{g}_\infty = \mathrm{Lie}(G_\infty)$ be the (real) Lie algebra of G_∞. Let $\mathcal{U}(\mathfrak{g}_\infty)$ be the universal enveloping algebra of the complexified Lie algebra $\mathfrak{g}_{\infty,\mathbb{C}} = \mathfrak{g}_\infty \otimes_{\mathbb{R}} \mathbb{C}$. Let $\mathcal{Z}(\mathfrak{g}_\infty)$ be the center of $\mathcal{U}(\mathfrak{g}_\infty)$. The maximal compact subgroup K_∞ comes as a fixed-point set of a Cartan involution Θ of G_∞. The differential θ of Θ gives the following decomposition of $\mathfrak{g}_\infty$:

$$\mathfrak{g}_\infty = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ and $\mathfrak{p}$ are the $+1$ and -1 eigenspaces of θ, respectively. We have $\mathfrak{k} = \mathrm{Lie}(K_\infty)$. Let $\mathfrak{a}_\infty$ be a maximal abelian subalgebra of $\mathfrak{p}$. We choose some ordering of the roots $\Sigma(\mathfrak{a}_\infty, \mathfrak{g}_\infty)$ so that the positive roots $\Sigma^+(\mathfrak{a}_\infty, \mathfrak{g}_\infty)$ are determined. Let N_∞ be the corresponding unipotent radical. This determines a minimal parabolic subgroup $P_\infty = M_\infty A_\infty N_\infty$ of G_∞, where $A_\infty = \exp(\mathfrak{a}_\infty)$ and $M_\infty = Z_{K_\infty}(A_\infty)$. We have the diffeomorphism

$$N_\infty \times A_\infty \times K_\infty \xrightarrow{(n,a,k) \mapsto n \cdot a \cdot k} G_\infty = N_\infty A_\infty K_\infty.$$

The Iwasawa decomposition implies that there exist unique C^∞-functions $a : G_\infty \to A_\infty$, $n : G_\infty \to N_\infty$, and $k : G_\infty \to K_\infty$ such that

$$g = n(g) \cdot a(g) \cdot k(g) \quad \text{for } g \in G_\infty. \tag{2.2}$$

Let $\hat{K}_\infty$ be the set of equivalence classes of irreducible representations of K_∞. Let $\delta \in \hat{K}_\infty$, then we write $d(\delta)$ and ξ_δ for the degree and character of δ, respectively. We fix the normalized Haar measure dk on K_∞. Let π be a Banach representation of G_∞ on a Banach space $\mathcal{B}$. Then, for $b \in \mathcal{B}$ and $\delta \in \hat{K}_\infty$, we let

$$E_\delta(b) = \int_{K_\infty} d(\delta)\overline{\xi_\delta(k)}\pi(k)b \, dk.$$

This belongs to the δ-isotypic component $\mathcal{B}(\delta)$ of $\mathcal{B}$.

We say that a continuous function $f : G(\mathbb{A}) \to \mathbb{C}$ is smooth if $f(\cdot, g_f) \in C^\infty(G_\infty)$ for all $g_f \in G(\mathbb{A}_f)$ and there exists an open-compact subgroup $L \subset G(\mathbb{A}_f)$ such that $f(g_\infty, g_f \cdot l) = f(g_\infty, g_f)$ for all $(g_\infty, g_f) \in G_\infty \times G(\mathbb{A}_f)$ and $l \in L$. Here we consider f as a function of two variables $f(g) = f(g_\infty, g_f)$, where $g = (g_\infty, g_f)$. We write $C^\infty(G(\mathbb{A}))$ for the vector space of all smooth functions on $G(\mathbb{A})$. We let $C_c^\infty(G(\mathbb{A}))$ be the space of all smooth compactly supported

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functions on $G(\mathbb{A})$. It is easy to show that $C_c^\infty(G(\mathbb{A}))$ is a span of the functions $f_\infty \otimes_{v\in V_f} f_v$ where $f_\infty \in C_c^\infty(G_\infty)$, $f_v \in C_c^\infty(G(k_v))$ for all $v \in V_f$, and $f_v = \mathrm{char}_{G(\mathcal{O}_v)}$ for almost all v.

By definition, we let $C^\infty(G(k)\backslash G(\mathbb{A})) \subset C^\infty(G(\mathbb{A}))$ be the subspace consisting of all functions $f \in C^\infty(G(\mathbb{A}))$ such that $f(\gamma \cdot g) = f(g)$ for all $\gamma \in G(k)$ and $g \in G(\mathbb{A})$.

Let $X \in \mathfrak{g}_\infty$. Let $f \in C^\infty(G(\mathbb{A}))$. Then we set

$$X.f(g_\infty, g_f) = d/dt|_{t=0} f(g_\infty \exp(tX), g_f).$$

This gives the structure of a $\mathcal{U}(\mathfrak{g}_\infty)$-module on $C^\infty(G(\mathbb{A}))$. The subspace $C^\infty(G(k)\backslash G(\mathbb{A}))$ is a $\mathcal{U}(\mathfrak{g}_\infty)$-submodule. In fact, both spaces are invariant under the action of $G(\mathbb{A})$ by right-translation.

A function $f \in C^\infty(G(\mathbb{A}))$ is K_∞-finite (on the right) if

$$\mathrm{span}_{\mathbb{C}}\{(g_\infty, g_f) \to f(g_\infty k_\infty, g_f) : k_\infty \in K_\infty\}$$

is finite-dimensional. Similarly, $f \in C^\infty(G(\mathbb{A}))$ is $\mathcal{Z}(\mathfrak{g}_\infty)$-finite if the space spanned by $z.f$, $z \in \mathcal{Z}(\mathfrak{g}_\infty)$, is finite-dimensional; in other words, the annhilator of f in $\mathcal{Z}(\mathfrak{g}_\infty)$ has finite codimension. By a well-known result, if $f \in C^\infty(G(\mathbb{A}))$ is K_∞-finite and $\mathcal{Z}(\mathfrak{g}_\infty)$-finite, then it is real-analytic in g_∞. We write $C^\infty(G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)\text{-finite}}$ for the space of all $f \in C^\infty(G(\mathbb{A}))$ which are K_∞-finite and $\mathcal{Z}(\mathfrak{g}_\infty)$-finite on the right. In a similar way, we can define $C^\infty(G(k)\backslash G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)\text{-finite}}$. The space $C^\infty(G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)\text{-finite}}$ is no longer $G(\mathbb{A})$-invariant, but it is a $((\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f))$-module, and the space $C^\infty(G(k)\backslash G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)\text{-finite}}$ is its submodule.

An automorphic form is a function $f \in C^\infty(G(k)\backslash G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)\text{-finite}}$ which satisfies a certain growth condition (see [BJ79, 4.2]). We denote the space of all automorphic forms by $\mathcal{A}(G(k)\backslash G(\mathbb{A}))$. It is a $((\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f))$-submodule of $C^\infty(G(k)\backslash G(\mathbb{A}))_{K_\infty, Z(\mathfrak{g}_\infty)\text{-finite}}$. We denote the subspace of cuspidal automorphic forms by $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$. By definition, $f \in \mathcal{A}(G(k)\backslash G(\mathbb{A}))$ is a cuspidal automorphic form if

$$\int_{U_P(k)\backslash U_P(\mathbb{A})} f(ng)\, dn = 0 \quad \text{for all } g \in G(\mathbb{A}), \tag{2.3}$$

for all proper k-parabolic subgroups P of G. In this paper we write U_P for the unipotent radical of a k-parabolic subgroup P of G. In general, we say that a locally integrable function $f : G(k)\backslash G(\mathbb{A}) \to \mathbb{C}$ is a cuspidal function if it satisfies (2.3) for almost all $g \in G(\mathbb{A})$.

The space of cuspidal automorphic forms $\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ is a $((\mathfrak{g}_\infty, K_\infty) \times G(\mathbb{A}_f))$-submodule of $\mathcal{A}(G(k)\backslash G(\mathbb{A}))$.

The topological space $G(k)\backslash G(\mathbb{A})$ has a finite-volume $G(\mathbb{A})$-invariant measure,

$$\int_{G(k)\backslash G(\mathbb{A})} P(f)(g)\, dg = \int_{G(\mathbb{A})} f(g)\, dg \quad \text{for } f \in C_c^\infty(G(\mathbb{A})), \tag{2.4}$$

where *the adelic compactly supported Poincaré series* $P(f)$ is defined by

$$P(f)(g) = \sum_{\gamma \in G(k)} f(\gamma \cdot g) \in C_c^\infty(G(k)\backslash G(\mathbb{A})). \tag{2.5}$$

We say that $P(f)$ is a *an adelic compactly supported cuspidal Poincaré series* if the function $P(f)$ is a cuspidal function.

The measure introduced in (2.4) enables us to define the Hilbert space $L^2(G(k)\backslash G(\mathbb{A}))$ and its closed subspace $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ that consists of all cuspidal functions in $L^2(G(k)\backslash G(\mathbb{A}))$.

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Both spaces are unitary representations of $G(\mathbb{A})$. Moreover, we have the following result from representation theory (see [God66]).

Theorem 2.6. *The space* $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ *can be decomposed into a direct sum of irreducible unitary representations of* $G(\mathbb{A})$ *that each occur with a finite multiplicity.*

Let $L \subset G(\mathbb{A}_f)$ be an open-compact subgroup. Then the intersection

$$\Gamma = \Gamma_L = G(k) \cap L \subset G(\mathbb{A}_f), \tag{2.7}$$

which is taken in $G(\mathbb{A}_f)$, is a discrete subgroup of G_∞. It is called a congruence subgroup [BJ79]. It is well-known that we can fix a finite-volume G_∞-invariant measure on $\Gamma\backslash G_\infty$,

$$\int_{\Gamma\backslash G_\infty} P(f_\infty)(g)\, dg = \int_{G_\infty} f_\infty(g)\, dg \tag{2.8}$$

for $f_\infty \in C_c^\infty(G_\infty)$, where *the compactly supported Poincaré series* (for Γ) is defined as follows:

$$P(f_\infty)(g) \stackrel{\mathrm{def}}{=} \sum_{\gamma\in\Gamma} f_\infty(\gamma \cdot g). \tag{2.9}$$

The function $P(f_\infty)$ belongs to the space $C_c^\infty(\Gamma\backslash G_\infty)$ (the subspace of $C^\infty(G_\infty)$ consisting of all left-Γ-invariant functions compactly supported modulo Γ). We use the measure on $\Gamma\backslash G_\infty$ to define a Hilbert space $L^2(\Gamma\backslash G_\infty)$ which is a unitary representation of G_∞. Similarly to what we did before, we define the notion of cuspidality by letting $U_{P,\infty}$ be the product

$$U_{P,\infty} = \prod_{v\in V_\infty} U_P(k_v) \tag{2.10}$$

and integrating over $U_{P,\infty} \cap \Gamma\backslash U_{P,\infty}$, for any proper k-parabolic subgroup P of G. The analogue of Theorem 2.6 is valid (see [God66]).

3. Restriction of an adelic compactly supported Poincaré series to G_∞

In this section, we study the restriction of an adelic Poincaré series (2.5) to G_∞. As before, we write $g = (g_\infty, g_f) \in G(\mathbb{A}) = G_\infty \times G(\mathbb{A}_f)$. We have

$$P(f)(g_\infty, 1) = \sum_{\gamma\in G(k)} f(\gamma \cdot g_\infty, \gamma). \tag{3.1}$$

Now we show the following simple but important proposition.

Proposition 3.2. *Let* $f \in C_c^\infty(G(\mathbb{A}))$. *Assume that* L *is an open-compact subgroup of* $G(\mathbb{A}_f)$ *such that* f *is right-invariant under* L. *We define a congruence subgroup of* G_∞ *using (2.7). Then the function in (3.1) is a compactly supported Poincaré series attached to* G_∞ *for* Γ_L. *Moreover, if* $P(f)$ *is cuspidal, then the function in (3.1) is cuspidal for* Γ_L.

Proof. Since f is compactly supported, we can find $c_1, \ldots, c_l \in G(\mathbb{A}_f)$ and $f_{\infty,1}, \ldots, f_{\infty,l} \in C_c^\infty(G_\infty)$ such that $f = \sum_{i=1}^{l} f_{\infty,i} \otimes \mathrm{char}_{c_i\cdot L}$. Then (3.1) implies that

$$P(f)(g_\infty, 1) = \sum_{\gamma\in G(k)} f(\gamma \cdot g_\infty, \gamma) = \sum_{i=1}^{l} \sum_{\gamma\in G(k)\cap c_i\cdot L} f_{\infty,i}(\gamma \cdot g_\infty).$$

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It could happen that $G(k) \cap c_i \cdot L = \emptyset$, since $G(k)$ is not necessarily dense in $G(\mathbb{A}_f)$. Nevertheless, if $G(k) \cap c_i \cdot L \neq \emptyset$, we may assume that $c_i \in G(k)$. Hence $G(k) \cap c_i \cdot L = c_i \cdot \Gamma_L$. Thus

$$P(f)(g_\infty, 1) = \sum_{\substack{1 \leqslant i \leqslant l \\ G(k) \cap c_i \cdot L \neq \emptyset}} \sum_{\gamma \in \Gamma_L} f_{\infty,i}(c_i \cdot \gamma \cdot g_\infty).$$

This function belongs to $C_c^\infty(\Gamma_L \backslash G_\infty)$ and is a compactly supported Poincaré series for Γ_L.

Let P be a k-parabolic subgroup of G. Then, for a fixed $g_\infty \in G_\infty$, the function $u \mapsto P(f)(u \cdot (g_\infty, 1))$ is right-invariant under the open-compact subgroup $L_P \stackrel{\text{def}}{=} L \cap U_v(\mathbb{A}_f)$. Now, Lemma 3.3 below shows that the cuspidality of $P(f)$ implies the Γ_L-cuspidality of the function given by (3.1). □

It remains to state and prove Lemma 3.3. Let P be a k-parabolic subgroup of G. We remind the reader that $U_{P,\infty}$ is defined by (2.10). We fix Haar measures du_∞, du_f and du on $U_{P,\infty}$, $U_P(\mathbb{A}_f)$ and $U_P(\mathbb{A})$, respectively, such that

$$\int_{U_P(\mathbb{A})} \varphi(u)\, du = \int_{U_{P,\infty}} \int_{U_P(\mathbb{A}_f)} \varphi(u_\infty, u_f)\, du_\infty\, du_f \quad \text{for } \varphi \in C_c(U_P(\mathbb{A})).$$

Lemma 3.3. *Let $\psi : U_P(k) \backslash U_P(\mathbb{A}) \to \mathbb{C}$ be a continuous function which is right-invariant under an open-compact subgroup $L_P \subset U_P(\mathbb{A}_f)$. Then, if we let $\mathrm{vol}_{U_P(\mathbb{A}_f)}(L_P) = \int_{U_P(\mathbb{A}_f)} \mathrm{char}_{L_P}\, du_f$, we have the formula*

$$\int_{U_P(k) \backslash U_P(\mathbb{A})} \psi(u)\, du = \mathrm{vol}_{U_P(\mathbb{A}_f)}(L_P) \cdot \int_{\Gamma_{L_P} \backslash U_{P,\infty}} \psi(u_\infty)\, du_\infty,$$

where Γ_{L_P} is a discrete subgroup of $U_{P,\infty}$ defined as $\Gamma_{L_P} = U_P(k) \cap L_P$.

Proof. By the usual integration theory, we can find a compactly supported continuous function $\varphi : U_P(\mathbb{A}) \to \mathbb{C}$ such that $\psi = P(\varphi)$, where $P(\varphi)(u) \stackrel{\text{def}}{=} \sum_{\gamma \in U_P(k)} \varphi(\gamma \cdot u)$ for $u \in U_P(\mathbb{A})$. Since ψ is right-invariant under the open-compact subgroup L_P, we can assume that φ satisfies the same. Now, we can find $u_1, \ldots, u_l \in U_P(\mathbb{A}_f)$ and continuous compactly supported functions $\varphi_1, \ldots, \varphi_l$ on $U_{P,\infty}$ such that $\varphi = \sum_{i=1}^l \varphi_i \otimes \mathrm{char}_{u_i L_P}$, where we consider φ as a function of two variables

$$u = (u_\infty, u_f) \in U_P(\mathbb{A}) = U_{P,\infty} \times U_P(\mathbb{A}_f).$$

Next, the strong approximation implies that $U_P(\mathbb{A}) = U_P(k) U_{P,\infty} L_P$. Hence $U_P(\mathbb{A}_f) = U_P(k) L_P$, which implies that we can assume $u_1, \ldots, u_l \in U_P(k)$. This is used to determine the restriction of ψ to $U_{P,\infty}$. As in the proof of Proposition 3.2, we obtain that

$$\psi(u_\infty) = \sum_{i=1}^l \sum_{\gamma \in \Gamma_{L_P}} (l(u_i^{-1})\varphi_i)(\gamma \cdot u_\infty),$$

where l denotes the left-translation. Hence

$$\begin{aligned} \int_{\Gamma_{L_P} \backslash U_{P,\infty}} \psi(u_\infty)\, du_\infty &= \sum_{i=1}^l \int_{\Gamma_{L_P} \backslash U_{P,\infty}} \Big(\sum_{\gamma \in \Gamma_{L_P}} (l(u_i^{-1})\varphi_i)(\gamma \cdot u_\infty) \Big)\, du_\infty \\ &= \sum_{i=1}^l \int_{U_{P,\infty}} (l(u_i^{-1})\varphi_i)(u_\infty)\, du_\infty = \sum_{i=1}^l \int_{U_{P,\infty}} \varphi_i(u_\infty)\, du_\infty. \end{aligned}$$

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Again, by the definition we compute that

$$\int_{U_P(k)\backslash U_P(\mathbb{A})} \psi(u)\,du = \int_{U_P(k)\backslash U_P(\mathbb{A})} \left(\sum_{\gamma\in U_P(k)} \varphi(\gamma\cdot u)\right) du = \int_{U_P(\mathbb{A})} \varphi(u)\,du$$
$$= \int_{U_{P,\infty}} \left(\int_{U_P(\mathbb{A}_f)} \varphi(u_\infty, u_f)\,du_f\right) du_\infty$$
$$= \mathrm{vol}_{U_P(\mathbb{A}_f)}(L_P)\cdot \int_{U_{P,\infty}} \left(\sum_{i=1}^{l} \varphi_i(u_\infty)\right) du_\infty.$$

Combining the preceding two formulas gives the lemma. □

4. Non-vanishing of adelic compactly supported Poincaré series

In this section we develop a non-vanishing criterion for (2.5) which controls not only the non-vanishing of (2.5) but also the non-vanishing of the restriction to G_∞ (see § 3). The criterion is based on a non-vanishing criterion given by [Mui08, Lemma 4.2].

First we introduce some notation. Let S be a finite set of places which contains V_∞ and is large enough that G is defined over $\mathcal{O}_v$ for $v \notin S$. We use the decomposition of $G(\mathbb{A})$ given by (2.1). Let

$$\Gamma_S = \left(\prod_{v\notin S} G(\mathcal{O}_v)\right) \cap G(k) \quad \text{with the intersection taken in } G^S.$$

This can be considered as a subgroup of G_S by using the diagonal embedding of $G(k)$ into the product (2.1) and then the projection to the first component. Since $G(k)$ is a discrete subgroup of $G(\mathbb{A})$, it follows that Γ_S is a discrete subgroup of G_S.

For $v \in S - V_\infty$, we choose an open-compact subgroup L_v. We put

$$\Gamma = \left(\prod_{v\in S-V_\infty} L_v \times \prod_{v\notin S} G(\mathcal{O}_v)\right) \cap G(k) = \Gamma_S \cap \left(\prod_{v\in S-V_\infty} L_v\right). \tag{4.1}$$

This is a discrete subgroup of G_∞. Now we have the following non-vanishing criterion.

Theorem 4.2. *Let S be a finite set of places which contains V_∞ and is large enough that G is defined over $\mathcal{O}_v$ for $v \notin S$. Assume that for each $v \in V_f$, we have $f_v \in C_c^\infty(G(k_v))$ such that $f_v(1) \neq 0$ and $f_v = \mathrm{char}_{G(\mathcal{O}_v)}$ for all $v \notin S$. For $v \in S - V_\infty$, we choose an open-compact subgroup L_v such that f_v is right-invariant under L_v. Then the intersection*

$$\Gamma_S \cap \left[K_\infty \times \prod_{v\in S-V_\infty} \mathrm{supp}(f_v)\right]$$

is a finite set and can be written as

$$\bigcup_{j=1}^{l} \gamma_j\cdot(K_\infty\cap\Gamma). \tag{4.3}$$

Next, we let

$$c_j = \prod_{v\in S-V_\infty} f_v(\gamma_j).$$

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Then the K_∞-invariant map $C^\infty(K_\infty) \to C^\infty(K_\infty \cap \Gamma \backslash K_\infty)$ given by

$$\alpha \mapsto \left(k \mapsto \hat{\alpha}(k) \overset{\text{def}}{=} \sum_{j=1}^{l} \sum_{\gamma \in K_\infty \cap \Gamma} c_j \cdot \alpha(\gamma_j \gamma \cdot k)\right) \tag{4.4}$$

is non-trivial, and for every $\delta \in \hat{K}_\infty$ contributing to the decomposition of the closure of the image of (4.4) in $L^2(K_\infty \cap \Gamma \backslash K_\infty)$ we can find a non-trivial $f_\infty \in C_c^\infty(G_\infty)$ such that the following hold.

(i) $E_\delta(f_\infty) = f_\infty$.

(ii) *The Poincaré series $P(f)$ and its restriction to G_∞ (which is a Poincaré series for Γ_L) are non-trivial, where $f \overset{\text{def}}{=} f_\infty \otimes_{v \in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$.*

(iii) $E_\delta(P(f)) = P(f)$ *and $P(f)$ is right-invariant under L.*

(iv) *The support of $P(f)|_{G_\infty}$ is contained in a set of the form $\Gamma_L \cdot C$, where C is a compact set which is right-invariant under K_∞ and such that $\Gamma_L \cdot C$ is not the whole of G_∞.*

Proof. Arguing as in the proof of [Mui08, Lemma 4.1], we can find a neighborhood of $1 \in G_\infty$ of the form UVK_∞, where $U \subset N_\infty$ and $V \subset A_\infty$ are neighborhoods of identities, such that

$$\Gamma_S \cap \left[(UVK_\infty) \times \prod_{v \in S - V_\infty} \operatorname{supp}(f_v)\right] = \Gamma_S \cap \left[K_\infty \times \prod_{v \in S - V_\infty} \operatorname{supp}(f_v)\right]. \tag{4.5}$$

Obviously, the intersection in (4.5) is finite. It can be described as the set of all $\gamma \in \Gamma_S$ satisfying

$$\gamma \in K_\infty \quad \text{and} \quad \prod_{v \in S - V_\infty} f_v(\gamma) \neq 0. \tag{4.6}$$

The set of all $\gamma \in \Gamma_S$ satisfying $\prod_{v \in S - V_\infty} f_v(\gamma) \neq 0$ is clearly right-invariant under Γ. Hence, the characterization of the intersection in (4.5) given by (4.6) shows that the intersection in (4.5) is right-invariant under $K_\infty \cap \Gamma$ and can be written as a disjoint union of the form (4.3).

We now show that the map (4.4) is non-trivial. First of all, our assumption that $f_v(1) \neq 0$ for $v \in V_f$ and the characterization of the intersection (4.5) given by (4.6) enable us to assume that $\gamma_1 = 1$. Then $c_1 = \prod_{v \in S - V_\infty} f_v(1) \neq 0$.

Next, let W be a neighborhood of $\gamma_1 = 1 \in K_\infty$ such that W intersects the finite set (4.3) exactly in $\{\gamma_1\}$. Let $\alpha \in C^\infty(K_\infty)$ be such that it vanishes outside W and $\alpha(\gamma_1) \neq 0$. Then, for $k = 1$, the right-hand side of (4.4) becomes

$$\sum_{j=1}^{l} \sum_{\gamma \in K_\infty \cap \Gamma} c_j \cdot \alpha(\gamma_j \gamma) = c_1 \alpha(\gamma_1) \neq 0.$$

This shows the non-triviality of the map (4.4).

Let $\alpha \in C^\infty(K_\infty)$ be any function such that the right-hand side of (4.4) is non-trivial. We can write its spectral expansion in $L^2(K_\infty \cap \Gamma \backslash K_\infty)$ as

$$\hat{\alpha} = \sum_{\delta \in \hat{K}_\infty} E_\delta(\hat{\alpha}), \tag{4.7}$$

where

$$E_\delta(\hat{\alpha})(k) = \int_{K_\infty} d(\delta) \overline{\xi_\delta(k')} \hat{\alpha}(kk') \, dk'.$$

As we explain at the beginning of [Mui08, § 3], only those δ containing a non-trivial vector invariant under $K_\infty \cap \Gamma$ can contribute to the spectral expansion given by (4.7). For $\delta \in \hat{K}_\infty$ such

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that $E_\delta(\hat{\alpha}) \neq 0$, $E_\delta(\hat{\alpha})$ is a linear combination of matrix coefficients of the form [Mui08, (4)]. In particular, since $\widehat{E_\delta(\alpha)} = E_\delta(\hat{\alpha})$, this shows the existence of α such that $E_\delta(\alpha) = \alpha$ and $\hat{\alpha} \neq 0$, for every δ appearing in the decomposition of the closure of the image of the map $\alpha \mapsto \hat{\alpha}$ under K_∞.

Now, fix δ appearing in the decomposition under K_∞ of the closure in $L^2(K_\infty \cap \Gamma \backslash K_\infty)$ of the image of the map $\alpha \mapsto \hat{\alpha}$, and select an arbitrary $\xi \in C^\infty(K_\infty)$ such that $E_\delta(\xi) = \xi$ and $\hat{\xi} \neq 0$. We also take $\zeta \in C_c^\infty(U)$ and $\eta \in C_c^\infty(V)$ such that $\zeta(1) \neq 0$ and $\eta(1) \neq 0$. We define $f_\infty \in C_c^\infty(G_\infty)$ by

$$f_\infty(uvk) = \zeta(u)\eta(v)\xi(k).$$

Then, by a short calculation, we obtain $E_\delta(f_\infty) = f_\infty$. This proves (i). Also, it immediately implies that $E_\delta(P(f)) = P(f)$, which is the first claim in (iii). The right-invariance under L in (iii) is obvious.

By construction, we see that

$$\operatorname{supp}(f_\infty) \subset UVK_\infty. \tag{4.8}$$

This is used to prove the following observation.

Lemma 4.9. *Let $\gamma \in \Gamma_S$ be such that $\prod_{v \in S - V_\infty} f_v(\gamma) \neq 0$, and let $k \in K_\infty$. Then, $f_\infty(\gamma \cdot k) \neq 0$ implies $\gamma \in \Gamma_S \cap [K_\infty \times \prod_{v \in S - V_\infty} \operatorname{supp}(f_v)]$.*

Proof. Indeed, (4.8) implies that

$$\gamma \cdot k \in \Gamma_S \cap \left[(UVK_\infty) \times \prod_{v \in S - V_\infty} \operatorname{supp}(f_v) \right].$$

Hence

$$\gamma \in \Gamma_S \cap \left[(UVK_\infty \cdot k^{-1}) \times \prod_{v \in S - V_\infty} \operatorname{supp}(f_v) \right] = \Gamma_S \cap \left[(UVK_\infty) \times \prod_{v \in S - V_\infty} \operatorname{supp}(f_v) \right].$$

Now apply (4.5) and the result follows. □

For $k \in K_\infty$, by using Lemma 4.9 we compute that

$$P(f)(k, 1) = \sum_{\gamma \in \Gamma_S} \left(\prod_{v \in S - V_\infty} f_v(\gamma) \right) \cdot f_\infty(\gamma \cdot k) = \sum_{j=1}^{l} \sum_{\gamma \in K_\infty \cap \Gamma} c_j \cdot f_\infty(\gamma_j \gamma \cdot k) = \zeta(1)\eta(1)\hat{\xi}(k). \tag{4.10}$$

In particular, $P(f)$ is not identically zero on K_∞. This proves assertion (ii) of Theorem 4.2. Finally, let us prove (iv). Since f is factorizable, using the notation from the proof of Proposition 3.2 we see that $f_{\infty,1} = \cdots = f_{\infty,l} = f_\infty$ in the expression for f given at the beginning of the proof of Proposition 3.2. The same proof then gives the following expression for the restriction to G_∞:

$$P(f)(g_\infty, 1) = \sum_{\substack{1 \leqslant i \leqslant l \\ G(k) \cap c_i \cdot L \neq \emptyset}} \sum_{\gamma \in \Gamma_L} f_\infty(c_i \cdot \gamma \cdot g_\infty).$$

(We remind the reader that the c_i in the above formula are not those from the present theorem but are, rather, the ones from the proof of Proposition 3.2. In particular, if $G(k) \cap c_i \cdot L \neq \emptyset$, then we take $c_i \in G(k)$.) Since (4.8) holds, we see that the restriction has support contained in

$$\bigcup_{\substack{1 \leqslant i \leqslant l \\ G(k) \cap c_i \cdot L \neq \emptyset}} \Gamma_L \cdot c_i^{-1} \cdot UVK_\infty.$$

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One can easily show that this is different from G_∞, if we shrink U and V. (One can adjust the argument given in [Mui08, Lemma 4.2].) This completes the proof of Theorem 4.2(iv). □

We finish this section with the following remark.

Lemma 4.11. *Maintaining the assumptions of Theorem 4.2, there are infinitely many $\delta \in \hat{K}_\infty$ that contribute to the decomposition of the closure of the image of (4.4).*

Proof. Indeed, it is enough to show that given different elements $k_1, \dots, k_l \in K_\infty$ and non-zero $c_1, \dots, c_l \in \mathbb{C} - \{0\}$, the map $C^\infty(K_\infty) \to C^\infty(K_\infty)$ given by

$$\alpha \mapsto \left(k \mapsto \hat{\alpha}(k) \overset{\text{def}}{=} \sum_{i=1}^{l} c_i \cdot \alpha(k_i \cdot k) \right)$$

has no finite image. To accomplish this, we select a neighborhood U of $1 \in K_\infty$ such that $k_i k_j^{-1} U \cap U = \emptyset$ for all i, j with $i \neq j$. Then, if α is supported in U, we easily see that $\hat{\alpha} \neq 0$. □

5. Construction of cuspidal compactly supported adelic Poincaré series

In this section we use Bernstein's decomposition of the category of smooth complex representations of reductive p-adic groups [Ber92] to construct adelic cuspidal compactly supported Poincaré series on $G(\mathbb{A})$.

Let us fix a place $v \in V_f$. We introduce some notation following the standard references [BZ76, BZ77]. A parabolic subgroup of $G(k_v)$ is a group of k_v-points of a k_v-parabolic subgroup of G. We consider the category of smooth (or algebraic) representations of $G(k_v)$. Let P_v be a parabolic subgroup of $G(k_v)$ given by a Levi decomposition $P_v = M_v U_v$, where M_v is a Levi factor and U_v is the unipotent radical of P_v. If σ_v is a smooth representation of M_v that was extended trivially across U_v to a representation of P_v, then we denote the normalized induction by $\mathrm{Ind}_{P_v}^{G(k_v)}(\sigma_v)$. If π_v is a smooth representation of $G(k_v)$, then we denote by $\mathrm{Jacq}_{G(k_v)}^{P_v}(\pi_v)$ a normalized Jacquet module of π_v with respect to P_v. When restricted to U_v, $\mathrm{Jacq}_{G(k_v)}^{P_v}(\pi_v)$ is a direct sum of (possibly infinitely many) copies of a trivial representation. Therefore, when M_v is fixed, we write $\mathrm{Jacq}_{G(k_v)}^{M_v}(\pi_v) = \mathrm{Jacq}_{G(k_v)}^{P_v}(\pi_v)$. Let M_v^0 be the subgroup of M_v given by the intersection of the kernels of all characters $m_v \mapsto |\chi_v(m_v)|_v$, where χ_v ranges over the group of all k_v-rational algebraic characters $M_v \to k_v^\times$. We say that a character $\chi_v : M_v \to \mathbb{C}^\times$ is unramified if it is trivial on M_v^0. We say that an irreducible representation ρ_v of M_v is supercuspidal if $\mathrm{Jacq}_{M_v}^{Q_v}(\rho_v) = 0$ for all proper parabolic subgroups Q_v of M_v.

Now, following Bernstein [Ber92], on the set of pairs (M_v, ρ_v) where M_v is a Levi subgroup of $G(k_v)$ and ρ_v is a smooth irreducible supercuspidal representation of M_v we introduce an equivalence relation as follows: (M_v, ρ_v) and (M_v', ρ_v') are equivalent if we can find $g_v \in G(k_v)$ and an unramified character χ_v of M_v' such that $M_v' = g_v M_v g_v^{-1}$ and $\rho_v' \simeq \chi_v \rho_v^{g_v}$, that is,

$$\rho_v^{g_v}(m_v') = \chi_v(m_v') \rho_v(g_v^{-1} m_v' g_v) \quad \text{for } m_v' \in M_v'.$$

In the discussion below, we shall write $\mathfrak{M}_v$ for the Bernstein equivalence class of a pair (M_v, ρ_v).

Let V be a smooth complex representation of $G(k_v)$. Let $V(\mathfrak{M}_v)$ be the largest smooth submodule of V such that every irreducible subquotient of V is a subquotient of $\mathrm{Ind}_{P_v}^{G(k_v)}(\chi_v \rho_v)$ for some unramified character χ_v of M_v. Here P_v is an arbitrary parabolic subgroup of $G(k_v)$

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containing M_v as a Levi subgroup. The fundamental result of Bernstein is the decomposition

$$V = \bigoplus_{\mathfrak{M}_v} V(\mathfrak{M}_v).$$

Now, we prove the following lemma.

Lemma 5.1. *Fix a Bernstein equivalence class $\mathfrak{M}_v$ (of a pair (M_v, ρ_v)), and consider $C_c^\infty(G(k_v))$ as a smooth representation of $G(k_v)$ acting by right-translations. Let $f_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$. Let $P'_v = M'_v U'_{P_v}$ be a parabolic subgroup of $G(k_v)$ such that M'_v does not contain a conjugate of M_v. Then*

$$\int_{U'_{P_v}} f_v(g_v u_v g'_v)\, du_v = 0 \quad \text{for all } g_v, g'_v \in G(k_v).$$

Proof. Assume that we can find $g_v, g'_v \in G(k_v)$ such that

$$0 \neq \int_{U'_{P_v}} f_v(g_v u_v g'_v)\, du_v = \int_{(g'_v)^{-1} U'_{P_v} g'_v} f_v(g_v g'_v u_v)\, du_v = \int_{(g'_v)^{-1} U'_{P_v} g'_v} F_v(u_v)\, du_v,$$

where F_v is defined by $F_v(x) \stackrel{\text{def}}{=} f_v(g_v g'_v \cdot x)$ for $x \in G(k_v)$. Since the action of $G(k_v)$ by left-translations commutes with the action by right-translations, we obtain $F_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$. This enables us to assume that $g_v = g'_v = 1$. Let $X(f_v)$ be a subrepresentation of $C_c^\infty(G(k_v))(\mathfrak{M}_v)$ generated by f_v. Since $\int_{U'_{P_v}} f_v(u_v)\, du_v \neq 0$, we see that

$$\mathrm{Jacq}_{G(k_v)}^{P'_v}(X(f_v)) \neq 0.$$

The set of parabolic subgroups of $G(k_v)$ contained in P'_v is partially ordered by inclusion. Let P''_v be the minimal parabolic subgroup contained in P'_v such that $\mathrm{Jacq}_{G(k_v)}^{P''_v}(X(f_v)) \neq 0$. We write $P''_v = M''_v U''_v$ for some Levi decomposition of P''_v. By standard theory (see, e.g., [BZ76, 2.6]), there exists an irreducible smooth representation ρ''_v of M''_v which is a subquotient of $\mathrm{Jacq}_{G(k_v)}^{P''_v}(X(f_v))$. We claim that ρ''_v is supercuspidal. If ρ''_v is not supercuspidal, we can find a parabolic subgroup Q''_v of M''_v such that $\mathrm{Jacq}_{M''_v}^{Q''_v}(\rho''_v) \neq 0$. Then $R''_v = Q''_v U''_v$ is a proper parabolic subgroup of P''_v. The transitivity of Jacquet modules [BZ77, Proposition 2.3] implies that

$$\mathrm{Jacq}_{G(k_v)}^{R''_v}(X(f_v)) = \mathrm{Jacq}_{M''_v}^{Q''_v}(\mathrm{Jacq}_{G(k_v)}^{P''_v}(X(f_v))).$$

Now, the exactness of Jacquet functors implies that $\mathrm{Jacq}_{G(k_v)}^{R''_v}(X(f_v)) \neq 0$. But this is a contradiction; therefore ρ''_v is supercuspidal.

Now, since ρ''_v is supercuspidal, [BZ77, Theorem 2.4(c)] implies that

$$\mathrm{Hom}_{M''_v}(\mathrm{Jacq}_{G(k_v)}^{P''_v}(X(f_v)), \rho''_v) \neq 0.$$

Thus, Frobenius reciprocity gives

$$\mathrm{Hom}_{G(k_v)}(X(f_v), \mathrm{Ind}_{P''_v}^{G(k_v)}(\rho''_v)) \simeq \mathrm{Hom}_{M''_v}(\mathrm{Jacq}_{G(k_v)}^{P''_v}(X(f_v)), \rho''_v) \neq 0.$$

Since $X(f_v)$ is a subrepresentation of $C_c^\infty(G(k_v))(\mathfrak{M}_v)$, we have

$$(M''_v, \rho''_v) \in \mathfrak{M}_v.$$

In particular, M''_v is conjugate to M_v. But since $P''_v \subset P'_v$, by fixing some appropriate minimal parabolic subgroup of $G(k_v)$ contained in P'_v and the corresponding maximal split torus we see that M''_v is conjugate to a Levi subgroup in M'_v, which is a contradiction. □

The next lemma gives further information on the decomposition of $C_c^\infty(G(k_v))$.

LEMMA 5.2. *Let $\mathfrak{M}_v$ be a Bernstein equivalence class. Let (M_v, ρ_v) represent the class $\mathfrak{M}_v$. Then the following hold.*

(i) $C_c^\infty(G(k_v))(\mathfrak{M}_v) \neq 0$.

(ii) *Let π_v be a smooth irreducible representation of $G(k_v)$, and assume that $f_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$. If $\pi_v(f_v) \neq 0$, then the contragredient representation $\widetilde{\pi}_v$ belongs to the class $\mathfrak{M}_v$, i.e. there exist a parabolic subgroup P_v of $G(k_v)$ which has M_v as a Levi factor and an unramified character χ_v of M_v such that $\widetilde{\pi}_v$ is an irreducible subquotient of $\mathrm{Ind}_{P_v}^{G(k_v)}(\chi_v\rho_v)$. In other words, π_v belongs to the class of $(M_v, \widetilde{\rho}_v)$.*

Proof. As before, in this proof the group $G(k_v)$ acts on $C_c^\infty(G(k_v))$ by right-translations. We begin the proof with the following observation. Let (π_v, V_v) be a smooth (not necessarily irreducible) representation of $G(k_v)$. We write $(\widetilde{\pi}_v, \widetilde{V}_v)$ for the contragredient representation of π_v. We denote by $\langle\cdot,\cdot\rangle : V_v \times \widetilde{V}_v \to \mathbb{C}$ a canonical $G(k_v)$-invariant pairing. The functions $f_v \in C_c^\infty(G(k_v))$ act as follows:

$$\pi_v(f_v)v_v = \int_{G(k_v)} f_v(g_v)\pi_v(g_v)v_v \, dg_v \quad \text{for } v_v \in V_v.$$

For a fixed $\widetilde{v}_v \in \widetilde{V}_v$, this implies the following $G(k_v)$-invariant pairing:

$$(f_v, v_v) \mapsto \langle \pi_v(f_v)v_v, \widetilde{v}_v\rangle = \int_{G(k_v)} f_v(g_v)\langle \pi_v(g_v)v_v, \widetilde{v}_v\rangle \, dg_v.$$

If $\pi_v(f_v)$ is not trivial, then we can select $\widetilde{v}_v$ so that the pairing is non-trivial when restricted to $X(f_v) \times V_v$, where $X(f_v)$ is a $G(k_v)$-subrepresentation of $C_c^\infty(G(k_v))$ generated by f_v. Hence

$$\mathrm{Hom}_{G(k_v)}(X(f_v), \widetilde{\pi}_v) \neq 0.$$

This proves (ii) by the definition of $C_c^\infty(G(k_v))(\mathfrak{M}_v)$.

Let $\pi_v \overset{\text{def}}{=} \mathrm{Ind}_{P_v}^{G(k_v)}(\widetilde{\rho}_v)$. Then we can select some $f_v \in C_c^\infty(G(k_v))$ such that $\pi_v(f_v) \neq 0$. (For example, a characteristic function of a sufficiently small open-compact subgroup would do.) Then, the first part of the proof gives

$$\mathrm{Hom}_{G(k_v)}\big(X(f_v), \mathrm{Ind}_{P_v}^{G(k_v)}(\rho_v)\big) \neq 0.$$

If we make a decomposition

$$X(f_v) = \bigoplus_{\mathfrak{N}_v} X(f_v)(\mathfrak{N}_v)$$

according to the Bernstein classes and then apply (ii), we see that

$$\mathrm{Hom}_{G(k_v)}\big(X(f_v)(\mathfrak{M}_v), \mathrm{Ind}_{P_v}^{G(k_v)}(\rho_v)\big) \neq 0.$$

In particular, $X(f_v)(\mathfrak{M}_v) \neq 0$, and this implies (i). □

Now we go back to a global theory and prove the following proposition.

PROPOSITION 5.3. *Let $f = f_\infty \otimes_{v\in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$, and let P be a k-parabolic subgroup of G. Assume that there is a finite place w and an equivalence class $\mathfrak{M}_w$ (represented by (M_w, ρ_w)) such that a Levi subgroup of $P(k_w)$ does not contain a conjugate of M_w and $f_w \in C_c^\infty(G(k_w))(\mathfrak{M}_w)$. Then the constant term of $P(f)$ along P vanishes.*

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Proof. By definition, the constant term of $P(f)$ with respect to a k-parabolic subgroup P of G is

$$\int_{U_P(k)\backslash U_P(\mathbb{A})} P(f)(ug)\,du = \int_{U_P(k)\backslash U_P(\mathbb{A})} \bigg(\sum_{\gamma\in G(k)} \varphi(\gamma\cdot ug)\bigg)\,du$$
$$= \int_{U_P(\mathbb{A})} \bigg(\sum_{\gamma\in G(k)/U_P(k)} f(\gamma\cdot ug)\bigg)\,du$$
$$= \sum_{\gamma\in G(k)/U_P(k)} \int_{U_P(\mathbb{A})} f(\gamma\cdot ug)\,du. \tag{5.4}$$

Since f is factorizable, i.e. $f = f_\infty \otimes_{v\in V_f} f_v \in C_c^\infty(G(\mathbb{A}))$, every term on the right-hand side of the formula (5.4) is zero because of Lemma 5.1:

$$\int_{U_P(\mathbb{A})} f(\gamma\cdot ug)\,du = \bigg(\int_{U_{P,\infty}} f_\infty(\gamma\cdot u_\infty\cdot g_\infty)\,du_\infty\bigg)\cdot \prod_{v\in V_f}\int_{U_P(k_v)} f_v(\gamma\cdot u_v\cdot g_v)\,du_v = 0. \quad \square$$

6. A comment on the archimedean case

In this section we show that the analogue of the results of § 5 in the archimedean case does not give anything interesting.

Proposition 6.1. *Let P be a proper parabolic subgroup of a Lie group G_∞, and let $U_{P,\infty}$ be its unipotent radical. Let $\varphi \in C_c^\infty(G_\infty)$. If $\int_{U_{P,\infty}} \varphi(g_1\cdot u\cdot g_2)\,du = 0$ for all $g_1, g_2 \in G_\infty$, then $\varphi = 0$.*

Proof. We remind the reader that N_∞ is the unipotent radical of the minimal parabolic subgroup of G_∞ fixed in § 2. We show that the assumption in the lemma implies that

$$\int_{N_\infty} \varphi(g_1\cdot n\cdot g_2)\,dn = 0 \quad \text{for all } g_1, g_2\in G_\infty. \tag{6.2}$$

Indeed, after conjugation by an element of G_∞, we may assume that $U_{P,\infty}\subset N_\infty$. Now,

$$\int_{N_\infty} \varphi(g_1ng_2)\,dn = \int_{U_{P,\infty}\backslash N_\infty}\bigg(\int_{U_{P,\infty}} \varphi(g_1uu'g_2)\,du\bigg)\,du' = 0.$$

This proves (6.2).

Having established (6.2), let P now denote an arbitrary standard parabolic subgroup of G_∞ (i.e. it contains P_∞). We write the Langlands decomposition of P as $P = A_PM_P^1U_{P,\infty}$. The Haar measure is given by the formula

$$\int_{G_\infty} f(g)\,dg = \int_{U_{P,\infty}}\int_{A_P}\int_{M_P^1}\int_{K_\infty} f(uamk)\delta_P^{-1}(a)\,du\,da\,dm\,dk \tag{6.3}$$

with $f\in C_c^\infty(G_\infty)$, where we require that the Haar measure dk be normalized, i.e. $\int_{K_\infty} dk = 1$.

We assume that M_P^1 has representations in the discrete series. Let $\nu\in\mathfrak{a}_P^*\otimes_{\mathbb{R}}\mathbb{C}$ and $\sigma\in\hat{M_P^1}$ be a representation in the discrete series acting on a Hilbert space $\mathfrak{H}_\sigma$ with a M_P^1-invariant scalar product $(\cdot,\cdot)_\sigma$. We consider the induced representation $\mathrm{Ind}_P^{G_\infty}(\nu,\sigma)$ on the space of classes of measurable functions $F: G_\infty\to\mathfrak{H}_\sigma$ such that

$$F(uamg) = e^{\nu(\log a)}\delta_P^{1/2}(a)\sigma(m)F(g) \quad \text{for } a\in A_P, m\in M_P^1, u\in U_{P,\infty}, g\in G_\infty. \tag{6.4}$$

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The functions $f \in C_c^\infty(G_\infty)$ act on $\mathrm{Ind}_P^{G_\infty}(\nu,\sigma)$ as bounded operators:

$$\mathrm{Ind}_{P(\mathbb{R})}^{G_\infty}(\nu,\sigma)(f).F(g) = \int_{G_\infty} f(h)F(gh)\,dh.$$

The induced representation $\mathrm{Ind}_P^{G_\infty}(\nu,\sigma)$ is unitary under the usual scalar product

$$(F_1,F_2) = \int_{K_\infty} (F_1(k),F_2(k))_\sigma\,dk$$

if $\nu \in \sqrt{-1}\,\mathfrak{a}_P^*$.

For a minimal parabolic subgroup P_∞, the Langlands decomposition is $P_\infty = A_\infty M_\infty N_\infty$ (fixed in § 2). Now, letting $P = P_\infty$, (6.2) yields

$$\begin{aligned}\mathrm{Ind}_{P_\infty}^{G_\infty}(\nu,\sigma)(\varphi).F(g) &= \int_{G_\infty} F(gh)\varphi(h)\,dh = \int_{G_\infty} \varphi(g^{-1}h)F(h)\,dh\\ &= \int_{N_\infty}\int_{A_\infty}\int_{M_\infty}\int_{K_\infty} e^{\nu(\log a)}\delta_{P_\infty}^{-1/2}(a)\varphi(g^{-1}uamk)\sigma(m)F(k)\,du\,da\,dm\,dk\\ &= \int_{A_\infty}\int_{M_\infty}\int_{K_\infty}\Big\{e^{\nu(\log a)}\delta_{P_\infty}^{-1/2}(a)\Big(\int_{N_\infty(\mathbb{R})}\varphi(g^{-1}uamk)\,du\Big)\\ &\qquad\times\sigma(m)F(k)\Big\}\,da\,dm\,dk = 0.\end{aligned}$$

Hence

$$\mathrm{Ind}_{P_\infty}^{G_\infty}(\nu,\sigma)(\varphi) = 0 \quad \text{for } \nu \in \mathfrak{a}_\infty^* \otimes_{\mathbb{R}} \mathbb{C}. \tag{6.5}$$

Next, we show that $\mathbf{tr}(\pi(\varphi)) = 0$ for every irreducible admissible representation π of G_∞. Indeed, for an appropriate $\nu \in \mathfrak{a}_\infty^* \otimes_{\mathbb{R}} \mathbb{C}$ and $\sigma \in \hat{M}_\infty$, π is infinitesimally equivalent to a closed irreducible subquotient Π of $\mathrm{Ind}_{P_\infty}^{G_\infty}(\nu,\sigma)$. But (6.5) implies that $\Pi(\varphi) = 0$. Hence we obtain

$$\mathbf{tr}(\pi(\varphi)) = \mathbf{tr}(\Pi(\varphi)) = 0,$$

since irreducible infinitesimally equivalent representations have equal characters.

Now we apply the Plancherel theorem [Kna86]. Let $\mathcal{M}$ be the set of G_∞-classes of Levi subgroups M (including G_∞) such that M^1 has representations in the discrete series. We identify $\mathcal{M}$ with the set of representatives taken among Levi subgroups of standard parabolic subgroups. In other words, we identify $\mathcal{M} - \{G_\infty\}$ with the set $\mathcal{P}$ of representatives of the set of all standard parabolic subgroups of G_∞ under the association. If $\sigma \in \hat{M}_P^1$ is a representation in the discrete series, we write $d(\sigma)$ for its formal degree. Now we state the Plancherel theorem. We can fix measures on $\sqrt{-1}\,\mathfrak{a}_P^*$ and on the unitary dual $\hat{M}_P^1$ of M_P^1 such that

$$\varphi(1) = \sum_{\substack{\pi \text{ is in the discrete}\\ \text{series for } G_\infty}} d(\pi)\cdot\mathbf{tr}(\pi(\varphi)) + \sum_{P\in\mathcal{P}}\int_{\sqrt{-1}\,\mathfrak{a}_P^*}\int_{\hat{M}_P^1}\mathbf{tr}(\mathrm{Ind}_P^{G_\infty}(\nu,\sigma)(\varphi))\,d\nu\,d\sigma. \tag{6.6}$$

Since $\mathbf{tr}(\pi(\varphi)) = 0$ for every irreducible admissible representation π of G_∞, (6.6) implies that $\varphi(1) = 0$. Finally, we observe that for $g_0 \in G_\infty$ we can apply the above consideration to $r_{g_0}\varphi$, where $r_{g_0}\varphi(g) = \varphi(gg_0)$. Hence $r_{g_0}\varphi(1) = \varphi(g_0) = 0$ for all $g_0 \in G_\infty$. This proves the proposition. □

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7. Spectral decomposition of adelic Poincaré series

In this section we study the spectral decomposition of the Poincaré series defined by Theorem 4.2. We decompose $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ into irreducible subspaces:

$$L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A})) = \hat{\bigoplus_j} \mathfrak{H}^j. \tag{7.1}$$

Let $K = K_\infty \times \prod_{v\in V_f} K_v$ be a maximal compact subgroup of $G(\mathbb{A})$, where $K_v = G(\mathcal{O}_v)$ for almost all v. For each j, we find a unitary irreducible representation $(\hat{\pi}^j, \mathfrak{V}^j)$ of $G(\mathbb{A})$ which is unitary equivalent to $\mathfrak{H}^j$ and factorizable, with

$$\mathfrak{V}^j = \mathfrak{V}^j_\infty \; \hat{\otimes}_{v\in V_f} \; \mathfrak{V}^j_v,$$

into a restricted tensor product of local irreducible unitary representations $(\hat{\pi}^j_\infty, \mathfrak{V}^j_\infty)$ of G_∞ and $(\hat{\pi}^j_v, \mathfrak{V}^j_v)$ of $G(k_v)$, for $v \in V_f$.

The space of K-finite vectors $(\mathfrak{H}^j)_K$ in $\mathfrak{H}^j$ is isomorphic to the usual restricted tensor product $\pi^j = \pi^j_\infty \otimes_{v\in V_f} \pi^j_v$, where each π^j_v is a representation of $G(k_v)$ on the space of K_v-finite vectors $(\mathfrak{V}^j_v)_K$ in $\mathfrak{V}^j_v$ and π^j_∞ is a $(\mathfrak{g}_\infty, K_\infty)$-module on the space of K_∞-finite vectors $(\mathfrak{V}^j_\infty)_K$ in $\mathfrak{V}^j_\infty$. Let χ_j be the infinitesimal character of π^j_∞.

The main result of this section is the following theorem.

Theorem 7.2. *Let S be a finite set of places of k containing all infinite places such that G is defined over $\mathcal{O}_v$ for $v \notin S$. For each $v \in S - V_\infty$, let $\mathfrak{M}_v$ be a Bernstein equivalence class represented by $(M_v(k_v), \rho_v)$, where M_v is a Levi subgroup of G defined over k_v and ρ_v is a supercuspidal representation of $M_v(k_v)$. Further, for each $v \in S - V_\infty$, fix $f_v \in C_c^\infty(G(k_v))(\mathfrak{M}_v)$ such that $f_v(1) \neq 0$. We let $f_v = \mathrm{char}_{G(\mathcal{O}_v)}$ for $v \notin S$. For each $v \in S - V_f$, we choose an open-compact subgroup L_v such that f_v is right-invariant under L_v. We define the open-compact subgroup L of $G(\mathbb{A}_f)$ by $L = \prod_{v\in S-V_\infty} L_v \times \prod_{v\notin S} G(\mathcal{O}_v)$. Assume that $\delta \in \hat{K}_\infty$ appears in the closure of the image of the map (4.4) (see Theorem 4.2). Let $f_\infty \in C_c^\infty(G_\infty)$ be such that Theorem 4.2(i)–(iv) hold. Next, we make the decomposition*

$$\text{the orthogonal projection of } P(f) \text{ to } L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A})) = \sum_j \psi_j \quad \text{with } \psi_j \in \mathfrak{H}^j. \tag{7.3}$$

Then we have the following.

- (i) *For all j, $\psi_j \in \mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$ is right-invariant under L and transforms according to δ, i.e. $E_\delta(\psi_j) = \psi_j$.*[2]
- (ii) *Assume that $\psi_j \neq 0$; then π^j_v belongs to the Bernstein class of $(M_v(k_v), \rho_v)$ for all $v \in S - V_\infty$.*
- (iii) *Assume that $P(f)$ is cuspidal; then the number of indices j in (7.3) such that $\psi_j \neq 0$ is infinite. Moreover, let χ be an infinitesimal character; then there are only finitely many indices j such that $\psi_j \neq 0$ and $\chi_j = \chi$.*
- (iv) *Assume that $P(f)$ is cuspidal; then there exist infinitely many irreducible unitary representations of G_∞ which contain δ and belong to $L^2_{\mathrm{cusp}}(\Gamma_L\backslash G_\infty)$. Their $(\mathfrak{g}_\infty, K_\infty)$-modules are among the modules π^j_∞; more precisely, a $(\mathfrak{g}_\infty, K_\infty)$-module X is a K_∞-finite part of such a representation if and only if there exists j such that $\psi_j|_{G_\infty} \neq 0$ and $X \simeq \pi^j_\infty$.*

[2] Obviously, $z.\psi_j = \chi_j(z)\psi_j$ for all $z \in \mathcal{Z}(\mathfrak{g}_\infty)$.

Proof. First, Theorem 4.2(iii) implies that $E_\delta(P(f)) = P(f)$ and that $P(f)$ is right-invariant under L. Hence the same is true for the orthogonal projection of $P(f)$ to $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$. As it has a unique spectral decomposition, we obtain $E_\delta(\psi_j) = \psi_j$ and that ψ_j is right-invariant under L. It remains to show that $\psi_j \in \mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$. But ψ_j is K_∞-finite and L-invariant; hence it belongs to the K-finite part of $\mathfrak{V}_j$. In particular, the discussion before the statement of the theorem shows that ψ_j is also $\mathcal{Z}(\mathfrak{g}_\infty)$-finite. The claim now follows from [BJ79, 4.3(ii)].

We now prove (ii). One triviality is seen from (7.3), namely, for all j such that $\psi_j \neq 0$,

$$\int_{G(k)\backslash G(\mathbb{A})} P(f)(g)\overline{\psi_j(g)}\, dg = \int_{G(k)\backslash G(\mathbb{A})} \psi_j(g)\overline{\psi_j(g)}\, dg > 0. \tag{7.4}$$

Now, we unfold the integral on the left-hand-side of (7.4) to get

$$\begin{aligned} 0 < \int_{G(k)\backslash G(\mathbb{A})} P(f)(g)\overline{\psi_j(g)}\, dg &= \int_{G(k)\backslash G(\mathbb{A})} \bigg(\sum_{\gamma\in G(k)} f(\gamma\cdot g)\bigg)\overline{\psi_j(g)}\, dg \\ &= \int_{G(k)\backslash G(\mathbb{A})} \bigg(\sum_{\gamma\in G(k)} f(\gamma\cdot g)\overline{\psi_j(\gamma\cdot g)}\bigg)\, dg \\ &= \int_{G(\mathbb{A})} \overline{\psi_j(g)} f(g)\, dg. \end{aligned} \tag{7.5}$$

We remind the reader that $F \in C_c^\infty(G(\mathbb{A}))$ acts on a closed $G(\mathbb{A})$-invariant subspace $\mathfrak{H}$ of $L^2(G(k)\backslash G(\mathbb{A}))$ by the formula

$$F.\psi(g) = \int_{G(\mathbb{A})} \psi(gh)F(h)\, dh \quad \text{for } \psi \in \mathfrak{H}.$$

Also, the space $\overline{\mathfrak{H}}$ consisting of all $\overline{\psi}, \psi \in \mathfrak{H}$ is $G(\mathbb{A})$-invariant and closed. It is clear that $\mathfrak{H}$ is irreducible if and only if $\overline{\mathfrak{H}}$ is irreducible; it is a contragredient representation of $\mathfrak{H}$. Below, we denote by $\widetilde{\pi}$ the contragredient representation of π.

Next, we observe that $\overline{\psi_j} \in C^\infty(G(k)\backslash G(\mathbb{A}))$ since ψ_j is an automorphic form by (i). Hence $f.\overline{\psi_j}(g) = \int_{G(\mathbb{A})} \overline{\psi_j}(gh)f(h)\, dh$ again belongs to $C^\infty(G(k)\backslash G(\mathbb{A}))$. Therefore the inequality in (7.4) implies that $f.\overline{\psi_j}$ is not identically zero. Thus, using the notation introduced before the statement of the theorem, we obtain that

$$0 \neq \widetilde{\pi}^j(f) = \widetilde{\pi}^j_\infty(f_\infty)\ \hat{\otimes}_{v\in V_f}\ \widetilde{\pi}^j_v(f_v).$$

This implies that $\widetilde{\pi}^j_v(f_v) \neq 0$ for all $v \in V_f$. Hence $\widetilde{\pi}^j_v(f_v) \neq 0$ for all $v \in V_f$. Now (ii) follows from Lemma 5.2(ii).

To prove (iii), assume that $P(f)$ is cuspidal. Then it is equal to its orthogonal projection to $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$. If the sum in (7.3) is finite, we would obtain that $P(f) \in \mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))$. Hence, it is $\mathcal{Z}(\mathfrak{g}_\infty)$-finite and K_∞-finite. The same is true for its restriction to G_∞, which is a non-zero compactly supported Poincaré series for Γ_L (see (2.7) for a definition) by Theorem 4.2 and Proposition 3.2. Hence $P(f)|_{G_\infty}$ is real-analytic, but its support is contained in a set of the form described by Theorem 4.2(iv). It is easy to see that this is a contradiction by applying the argument from the proof given in the very last part of [Mui08, § 4]. Finally, by a theorem of Harish-Chandra [BJ79, 4.3(i)], the space of all automorphic forms on $G(\mathbb{A})$ which are right-invariant under L, transform according to δ and have infinitesimal character χ is finite-dimensional. Since non-zero functions among ψ_j are linearly independent, there must exist only finitely many indices j such that $\psi_j \neq 0$ and $\chi_j = \chi$. This completes the proof of (iii).

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Finally, we prove (iv). First, Proposition 3.2 shows that $P(f)|_{G_\infty}$ is Γ_L-cuspidal. Clearly, $E_\delta(P(f)|_{G_\infty}) = P(f)|_{G_\infty}$. Now, Theorem 4.2(iv) and the proof given in the very last part of [Mui08, § 4] imply that there exist infinitely many irreducible unitary representations of G_∞ which contain δ and belong to $L^2_{\mathrm{cusp}}(\Gamma_L\backslash G_\infty)$. Next, we describe a relation between the spectral decomposition of $P(f)$ when it is cuspidal and that of $P(f)|_{G_\infty} \in L^2_{\mathrm{cusp}}(\Gamma_L\backslash G_\infty)$. We first recall some statements that are contained in [BJ79] implicitly. Let $C \subset G(\mathbb{A}_f)$ be the minimal set such that $G(\mathbb{A}) = G(k)\cdot C\cdot G_\infty \cdot L$. Such a C always exists [Bor63]. We may assume that $1\in C$. The minimality of C implies that the classes $G(k)\cdot c\cdot G_\infty\cdot L$, $c\in C$, are disjoint. One can easily show that they are open and closed in $G(\mathbb{A})$. Let $\varphi \in C_c(G(k)\backslash G(\mathbb{A}))$ be supported in $G(k)\cdot c\cdot G_\infty\cdot L$ and right-invariant under L; then one can show the integration formula

$$\int_{G(k)\backslash G(\mathbb{A})} \varphi(g)\, dg = \mathrm{vol}_{G(\mathbb{A}_f)}(L)\cdot \int_{\Gamma_{cLc^{-1}}\backslash G_\infty} \varphi(g_\infty, c)\, dg_\infty$$

by arguing as in the proof of Lemma 3.3. We remind the reader that $\Gamma_{cLc^{-1}}$ is a congruence subgroup attached to the open-compact subgroup $cLc^{-1}\subset G(\mathbb{A}_f)$; see (2.7). This implies that the map

$$L^2(G(k)\backslash G(\mathbb{A}))^L \to \bigoplus_{c\in C} L^2(\Gamma_{cLc^{-1}}\backslash G_\infty)$$

defined by

$$\varphi \mapsto \bigoplus_{c\in C} \varphi|_{G_\infty\times\{c\}}$$

is a unitary equivalence of (unitary) representations of G_∞. In particular, the projection to a component $L^2(\Gamma_{cLc^{-1}}\backslash G_\infty)$ is a continuous G_∞-map. Next, it is indicated in [BJ79] (and easy to check) that we have the following isomorphism using the same map:

$$\mathcal{A}_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))^L \simeq \bigoplus_{c\in C} \mathcal{A}_{\mathrm{cusp}}(\Gamma_{cLc^{-1}}\backslash G_\infty),$$

which is now an equivalence of $(\mathfrak{g}_\infty, K_\infty)$-modules. In the same way, we obtain the unitary equivalence

$$L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))^L \simeq \bigoplus_{c\in C} L^2_{\mathrm{cusp}}(\Gamma_{cLc^{-1}}\backslash G_\infty). \tag{7.6}$$

(Actually, the cuspidality in both cases can be treated using the methods of Lemma 3.3. We leave the details to the reader.)

Since $P(f)$ is cuspidal, $P(f) = \sum_j \psi_j$ is a decomposition in $L^2_{\mathrm{cusp}}(G(k)\backslash G(\mathbb{A}))^L$. Thus, the corresponding decomposition in $L^2_{\mathrm{cusp}}(\Gamma_{cLc^{-1}}\backslash G_\infty)$ is the following one: $P(f)|_{G_\infty\times\{c\}} = \sum_j \psi_j|_{G_\infty\times\{c\}}$ for all $c\in C$. The above discussion shows that $\psi_j|_{G_\infty\times\{c\}} \in \mathcal{A}_{\mathrm{cusp}}(\Gamma_{cLc^{-1}}\backslash G_\infty)$. In particular, we have

$$P(f)|_{G_\infty} = \sum_j \psi_j|_{G_\infty} \quad \text{with } \psi_j|_{G_\infty} \in \mathcal{A}_{\mathrm{cusp}}(\Gamma_L\backslash G_\infty). \tag{7.7}$$

Finally, assume $\psi_j|_{G_\infty}\neq 0$. Then the closed G_∞-invariant subspace of $L^2(G(k)\backslash G(\mathbb{A}))^L$ generated by ψ_j is a direct sum of copies of $\hat{\pi}^j_\infty$ (see the beginning of this section for the notation). Note that the number of copies is finite, since it must be finite in each $L^2_{\mathrm{cusp}}(\Gamma_{cLc^{-1}}\backslash G_\infty)$; see (7.6). Since the projection to $L^2_{\mathrm{cusp}}(\Gamma_L\backslash G_\infty)$ in (7.6) is a bounded G_∞-map which is the restriction to G_∞, it follows that $\psi_j|_{G_\infty}$ generates a closed G_∞-invariant subspace of $L^2_{\mathrm{cusp}}(\Gamma_L\backslash G_\infty)$ which is isomorphic to the direct sum of finitely many copies of $\mathfrak{V}^j_\infty$. Because of (7.7), only such

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unitary representations of G_∞ contribute to the spectral decomposition of $P(f)|_{G_\infty}$. Now, a well-known equivalence between irreducible unitary representations of G_∞ and unitarizable $(\mathfrak{g}_\infty, K_\infty)$-modules proves (iv). This completes the proof of the theorem. □

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